B. Goossens, A. Pizurica and W. Philips, "A Filter Design Technique for Improving the Directional Selectivity of the First Scale of the Dual-Tree Complex Wavelet Transform," in Proc. of the IEEE Int. Conf. Image Processing (ICIP2009), Nov. 7-11, Cairo, Egypt, p. 3805-3808

A FILTER DESIGN TECHNIQUE FOR IMPROVING THE DIRECTIONAL SELECTIVITY OF THE FIRST SCALE OF THE DUAL-TREE COMPLEX WAVELET TRANSFORM

Bart Goossens, Aleksandra Pižurica and Wilfried Philips

Ghent University - TELIN - IPI - IBBT
Sint-Pietersnieuwstraat 41, B-9000 Ghent, Belgium
bart.goossens@telin.ugent.be

ABSTRACT

The Dual-Tree complex wavelet transform (DT-CWT) uses approximate Hilbert transform pairs of wavelet, which requires that the filters of each tree of the dual-tree structure should be delayed approximately one half sample from each other. However, the filters of the first (finest) scale of the transform do not obey this condition, resulting in a poor directional selectivity for the first scale. In this paper, we describe a design technique for first-scale infinite impulse response (IIR) wavelet filters, that solves this problem. Results demonstrate that a much better directional selectivity is obtained, which indicates a performance improvement for many applications that use the DT-CWT where the preservation of high-frequency information is important.

Index Terms— wavelets, Dual-Tree Complex Wavelet Transform, directional selectivity

I. INTRODUCTION

Recently, the Dual-Tree complex wavelet transform (DT-CWT) [1]–[3] has been introduced to overcome certain disadvantages of the Discrete Wavelet Transform (DWT): its shift variance and its inability to distinguish features orientated at $+45°$ from features oriented at $-45°$. In [4] it is proposed to achieve shift invariance by removing the decimation operations of the DWT, using the *algorithme à trous* [4] (often called undecimated DWT). However the drawback is a huge redundancy factor (for S scales the redundancy factor is $(2^n-1)S$ in n-d). The DT-CWT offers approximate shift invariance at a much lower redundancy factor (2^n for n-d). The shift invariance is obtained by using two DWTs (called dual trees) in parallel and by designing the wavelet filters in each tree such that the corresponding wavelets form Hilbert transform pairs [2]. This requires that the filters of each tree of the dual-tree structure should be delayed approximately one half sample from each other (known as the *half-sample delay condition*) [2], [5], and which also leads to a better directional selectivity: the DT-CWT has 6 orientation bands instead of 3 and has separate orientation subbands for $+45°$ and $-45°$. Shift invariance and good directional selectivity are important in many applications, such as texture analysis and synthesis, noise suppression, segmentation, watermarking and even in compression.

However, the *half-sample delay condition* only holds starting from the second scale [1]–[3], while for the first scale the wavelet filters in each tree are delayed one sample (such as in a redundant DWT implemented with cycle-spinning [6]). As a consequence, the directional selectivity of the first scale of the DT-CWT is not better than that of the DWT or undecimated DWT. As far as the authors are aware of, this deficiency has not been addressed in literature so far. For many practical applications, such as image reconstruction, the first scale is by far the most important, because the performance is usually determined by the amount of fine details (high frequency content) that are preserved or reconstructed.

In this paper, we develop a design technique for wavelet filters for the *first scale* of the DT-CWT, that fulfill the *half-sample delay condition*. To obtain perfect reconstruction (PR) of the analysis and synthesis filterbanks, the wavelet filters being constructed are IIR filters. Fortunately, the IIR filters do not pose any problems in practice, as they can have a relatively low order and because they are stable.

The remainder of this paper is as follows: in Section II we introduce concepts that are used in the remainder of this paper. The problem to solve is explained in Section III. In Section IV we describe the proposed filter design method. Results and a discussion are given in Section V. Finally, Section VI concludes this paper.

II. BACKGROUND INFORMATION

In the DT-CWT framework [1], [2], a complex wavelet $\psi_c(t)$ is formed by a Hilbert transform pair of wavelets: a real-valued wavelet $\psi(t)$ and its Hilbert transform $\mathcal{H}\{\psi(t)\}$:

$$\psi_c(t) = \psi(t) + j\mathcal{H}\{\psi(t)\} \quad (1)$$

B. Goossens is supported in part by Special Research Fund (BOF) of Ghent University, Belgium. A. Pižurica is a postdoctoral researcher of the Fund for the Scientific Research in Flanders (FWO) Belgium.

with $\mathcal{H}\{\cdot\}$ the Hilbert transform of a function. In the frequency domain, we have the equivalent expression:

$$\Psi_c(\omega) = \Psi(\omega)\,(1 + jH_{\rm hil}(\omega)) \qquad (2)$$

with $H(\omega)$ the frequency response of the Hilbert transform:

$$H_{\rm hil}(\omega) = \begin{cases} j & \omega < 0 \\ 0 & \omega = 0 \\ -j & \omega > 0 \end{cases} \qquad (3)$$

The primary reason for this complex wavelet construction is that $\Psi_c(\omega)$ is analytic, i.e. the magnitude response $|\Psi_c(\omega)|$ completely suppresses negative frequencies: $|\Psi_c(\omega)| = 0$ for $\omega < 0$, while positive frequencies are completely passed: $|\Psi_c(\omega)| = |\Psi(\omega)|$ for $\omega > 0$. Higher dimensional (n-d) complex wavelets can be formed as tensor products of 1-d complex wavelets (or their complex conjugates). Because every 1-d complex wavelet is analytic in one dimension, the corresponding (n-d) complex wavelet passes all frequencies in one hyperoctant of the n-d frequency domain. This property is exploited in the DT-CWT framework to design oriented filters using separable wavelet filters (for further reading, see [2]).

In the following we will concentrate on wavelet analysis in 1-d. Extensions to higher dimensions are trivial because the n-d complex wavelets are formed using tensor products. Let the filters $G_k(z)$, $H_k(z)$, $\tilde{G}_k(z)$, $\tilde{H}_k(z)$, $k = 1, 2$ represent two real-valued quadrature mirroring filter (QMF) pairs with z-transforms. The wavelet filters are denoted as $G_k(z)$, $\tilde{G}_k(z)$, and the scaling filters as $H_k(z)$, $\tilde{H}_k(z)$ and k represents the tree index of the dual tree. Perfect reconstruction (PR) is possible if the following equations hold [7]:

$$G_k(z)\tilde{G}_k(z^{-1}) + H_k(z)\tilde{H}_k(z^{-1}) = 2 \qquad (4)$$
$$G_k(z)\tilde{G}_k(-z^{-1}) + H_k(z)\tilde{H}_k(-z^{-1}) = 0 \qquad (5)$$

In order for $\psi_c(t)$ to be analytic, it was shown in [5] that the following condition is sufficient:

$$H_2(z) = H_1(z)z^{-1/2} \qquad (6)$$

or informally, $h_2(n) \approx h_1(n - 0.5)$, which means that there must be a half sample delay between both scaling filters. Given that the filters form two QMF pairs, an equivalent condition for the wavelet filters is given by [8]:

$$G_2(e^{j\omega}) = G_1(e^{j\omega})e^{j\omega/2}H_{\rm hil}(\omega) \qquad (7)$$

which means that $G_2(z)$ is the Hilbert transform of $G_1(z)z^{1/2}$. In practice, a half sample delay and a discrete Hilbert transform cannot be satisfied *exactly*, because the wavelet filters are restricted to finite supports (resulting in only *approximate* analyticity). In literature, a number of methods have been described for designing (approximate) Hilbert pairs of wavelet bases, e.g. [9]–[12].

III. PROBLEM STATEMENT

The above filter design techniques all provide PR and try to satisfy the half-sample delay condition as closely as possible. However, wavelet filters designed using these techniques cannot be used on the *first* scale, because a *half-sample delay condition* does not exist on the input level, i.e. the same input signal (or image) is used for every tree. To work around this issue, it is proposed in [1], [2] to use wavelet filters for the first scale that are delayed with one sample (or $H_2(z) = H_1(z)z^{-1}$), such that starting from the second scale, the *half-sample delay condition* is fulfilled. This is in fact equivalent to using a redundant DWT on the first scale implemented with cycle spinning, but this will give a poor directional selectivity. One may also consider invertible (allpass) fractional delay filtering of the first scale [13] and using the half-sample-delay pairs of filters starting from the first scale, but this would have the disadvantage that fractional delay filters have a relatively long support and this would reduce the localization properties of the transform on every scale. Hence we conclude that a specific design technique for the first scale that yields wavelet filters with the desired property (7) is needed. This is the topic of the next Section.

IV. THE WAVELET FILTER DESIGN METHOD

We start from a one sample delay between the scaling filters and subsequently impose the half-sample delay condition (7) to the wavelet filters. This gives the following relationship between the scaling filters and between the wavelet filters *for the first scale*:

$$H_2(z) = H_1(z)z^{-1} \qquad (8)$$
$$G_2(z) = G_1(z)z^{-1}Q(z) \qquad (9)$$

where $Q(z)$ is a filter that we will design to satisfy (7). First, we write the PR conditions for the second QMF pair:

$$G_1(z)\tilde{G}_1(z^{-1})Q(z)\tilde{Q}(z^{-1}) + H_1(z)\tilde{H}_1(z^{-1}) = 2$$
$$G_1(z)\tilde{G}_1(-z^{-1})Q(z)\tilde{Q}(-z^{-1}) + H_1(z)\tilde{H}_1(-z^{-1}) = 0 \qquad (10)$$

Identification with the PR conditions for the first QMF pair (4), (5) gives the following design constraints for $Q(z)$:

$$Q(z)\tilde{Q}(z^{-1}) = 1 \quad \text{and} \quad Q(-z)\tilde{Q}(z^{-1}) = 1 \qquad (11)$$

which is satisfied if $Q(z) = Q(-z)$ and $\tilde{Q}(z) = Q^{-1}(z^{-1})$. The former means that $Q(z)$ only has terms in even powers of z, the latter implies that either $Q(z)$ or $\tilde{Q}(z)$ cannot be a finite impulse response filter (FIR), because the inverse of a FIR filter is not a FIR filter in general. If we take $\tilde{Q}(z^{-1}) = Q^{-1}(z)$, we can entirely concentrate on the design of $Q(z)$. To establish the relationship between the wavelet filters from equation (7), we must have that:

$$Q(e^{j\omega}) \approx e^{3j\omega/2}H_{\rm hil}(\omega) \qquad (12)$$

Table I. Matlab program for designing the filter $Q(z)$.

```
b=-1./(1:4:-3+4*M)';
C=hankel(1./(5:4:-3+8*M));
C=C(1:M,1:M)+pi/2*eye(M);
B=upsample([1; C\b],2);
```

Table II. Expressions of $B(z)$ for different orders M, obtained using the program in Table I. Use $Q(z) = B(z)/B(1/z)$.

M	$B(z)$
1	$1 - 0.56472z^2$
2	$1 - 0.55947z^2 - 0.08365z^4$
3	$1 - 0.55785z^2 - 0.08236z^4 - 0.03915z^6$
4	$1 - 0.55711z^2 - 0.081737z^4 - 0.03861z^6 - 0.02412z^8$

In order to have identical magnitude responses of the wavelet basis functions in both trees, we take $|G_2(z)| = |G_1(z)|$, or equivalently, $|Q(z)| = 1$. This leads us to an allpass filter design: let $Q(z) = B(z)/B(1/z)$ with

$$B(z) \quad = \quad 1 + \sum_{m=1}^{M} b_m z^{2m} \tag{13}$$

with M the filter order. From equation (12) follows that $B(e^{j\omega}) \approx B(e^{-j\omega})e^{3j\omega/2}H_{\text{hil}}(\omega)$. Note that exactness of the approximation is not possible because an exact fractional delay filter $e^{3j\omega/2}$ and a Hilbert transform cannot be realized using IIR filters. Instead, we opt for an approximate solution, by defining the error function:

$$E_Q = \int_{-\pi}^{+\pi} \left| B(e^{j\omega}) - B(e^{-j\omega})e^{3j\omega/2}H_{\text{hil}}(\omega) \right|^2 \mathrm{d}\omega \tag{14}$$

Minimizing E_Q is a least squares problem, which corresponds to solving a linear system. In Table I an efficient Matlab program is given for computing the polynomial $B(z)$. In Table II expressions for $B(z)$ for different filter orders are given. We found that all filters obtained using this technique are stable.

Interestingly, for allpass filters the property $\tilde{Q}(z) = Q(z)$ holds, hence for QMF filters we have, apart from time reversal, identical analysis and reconstruction filters: $\tilde{G}_2(z) = G_2(z)$. Because the filter $z^{-2M}Q(z)$ is causal, implementing the anticausal filter $z^{2M}\tilde{Q}(z^{-1})$ can only be realized by filtering in the time-reversed direction. For images this does not pose any problems as we can filter from the right to the left or from the bottom to the top.

Further we remark that the proposed filter design only modifies the wavelet filters of the first scale (and not the scaling filters), hence the basis elements of the subsequent (coarser) scales are not affected.

V. RESULTS AND DISCUSSION

In Fig. 1, we check the accuracy of the approximation (12), for different filter orders. Fig. 1(a) shows the magnitude response of $Q(z) + jz$. Because of the Hilbert transform in (12), this filter should be analytic in the ideal case. It can be seen that there is a good suppression of the negative frequencies, of approximately 10dB-15dB. In Fig. 1(b), the group delay of $Q(z)$ is compared to the group delay in the ideal case: $-\frac{3}{2}$ (see (12)). Seemingly, the approximation is the best in a mid-frequency passband.

Next, we test the constructed wavelet filters in the task of estimation of the dominant orientation (DO) of features in an image. Here, the DO is estimated from the first-scale of the DT-CWT, based on same principles as used in [14]. In Fig. 2, the result is shown for a sinusoidal zoneplate image for both the original wavelet filters (Fig. 2(b)) and the proposed wavelet filters (Fig. 2(c)). Apparently, for the original wavelet filters, the orientation is misestimated in the corners of the zoneplate image. For the proposed solution, this problem is resolved and the transitions in orientation for the diagonal lines are much smoother.

Finally, we also analyze the improvement in directional selectivity in 3-d. In Fig. 3, frequency responses of the real parts of the complex wavelet basis functions are given and compared to the original wavelet filters (Fig. 3(d)). In Fig. 3(d) there is leakage of the filter energy to different orientations, which results in a poor directional selectivity. For the proposed filters, the frequency responses are well-localized and as expected.

VI. CONCLUSION

In this work, we proposed a design technique for IIR wavelet filters for the first scale of the DT-CWT, in order to improve the directional selectivity of the first scale. The main design criterion is that the wavelet filters in both trees must satisfy the half-sample delay condition while the corresponding scaling filters are one sample delayed. We have shown that perfect reconstruction is possible with IIR wavelet filters. The obtained filters are stable and even the lowest order filters perform well for this task. The results show a vast improvement in directional selectivity which is beneficial in many applications that use this transform.

VII. REFERENCES

[1] N. G. Kingsbury, "Complex Wavelets for Shift Invariant Analysis and Filtering of Signals," *Journal of Applied and Computational Harmonic Analysis*, vol. 10, no. 3, pp. 234–253, May 2001.

[2] I. W. Selesnick, R. G. Baraniuk, and N. G. Kingsbury, "The Dual-Tree Complex Wavelet Transform," *IEEE Signal Processing Magazine*, vol. 22, no. 6, pp. 123–151, Nov. 2005.

[3] C. Chaux, L. Duval, and J.-C. Pesquet, "Image analysis using a dual-tree m-band wavelet transform," *IEEE Trans. Image Processing*, vol. 15, no. 8, pp. 2397–2412, Aug. 2006.

[4] S. Mallat, "Multifrequency channel decomposition of images and wavelet models," *IEEE Trans. Acoust., Speech, Signal Proc.*, vol. 37, no. 12, pp. 2091–2110, Dec. 1989.
[5] I. W. Selesnick, "Hilbert Transform Pairs of Wavelet Bases," *IEEE Sig. Proc. Letters*, vol. 8, no. 6, pp. 170–173, 2001.
[6] R. Coifman and D. Donoho, *Translation-invariant denoising*, A. Antoniadis and G. Oppenheim, Eds. New York: Springer Verlag, 1995.
[7] I. Daubechies, *Ten Lectures on Wavelets*. Philadelphia: Society for Industrial and Applied Mathematics, 1992.
[8] B. Dumitrescu and A. Bahrami Rad, "A method for designing the double-density dual-tree wavelet transform," in *Proc. of the 2008 Int. Workshop on Local and Non-Local Approx. in Image Processing, LNLA2008*, Lausanne, Switzerland, 2008.
[9] N. G. Kingsbury, "Design of q-shift complex wavelets for image processing using frequency domain energy minimalisation," in *Proc. IEEE Conf. on Image Process.*, Barcelona, Sept. 15-17 2003.
[10] I. W. Selesnick, "The design of approximate Hilbert transform pairs of wavelet bases," *IEEE Trans. on Signal Processing*, vol. 50, no. 5, pp. 1144–1152, May 2002.
[11] D. Tay, N. G. Kingsbury, and M. Palaniswami, "Orthonormal Hilbert-pair of wavelets with (almost) maximum vanishing moments," *IEEE Signal Processing Letters*, vol. 13, no. 9, pp. 553–536, 2006.
[12] B. Dumitrescu, "SDP Approximation of a Fractional Delay and the Design of Dual-Tree Complex Wavelet Transform," *IEEE Trans. Sig. Process.*, vol. 56, no. 9, p. 4255, 2009.
[13] T. I. Laakso, V. Välimäki, M. Karjalainen, and U. K. Laine, "Splitting the unit delay – tools for fractional delay filter design," *IEEE Sig. Process. Mag.*, vol. 13, pp. 30–60, 1996.
[14] W. T. Freeman and E. H. Adelson, "Design and use of steerable filters," *IEEE Transactions on Pattern analysis and Machine Intelligence*, vol. 13, no. 9, pp. 891–906, 1991.

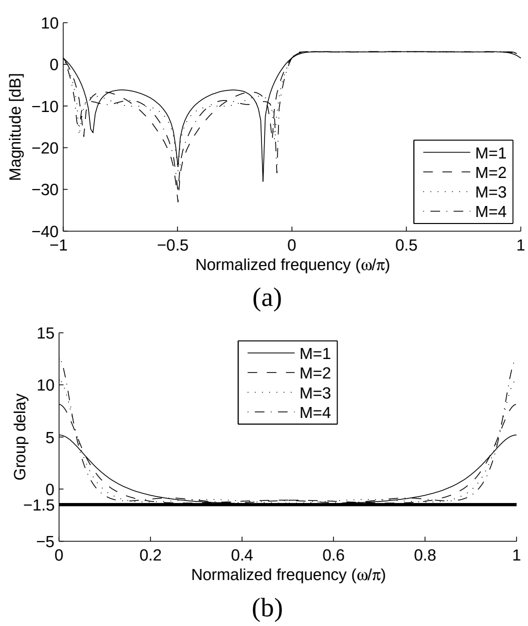

Fig. 1. (a) Magnitude response of $|Q(z) + jz|$, for different orders M, showing the analyticity of $Q(z)+jz$ (b) Group delay of $Q(z)$, for different orders M. The thick line signifies the ideal group delay ($-\frac{3}{2}$).

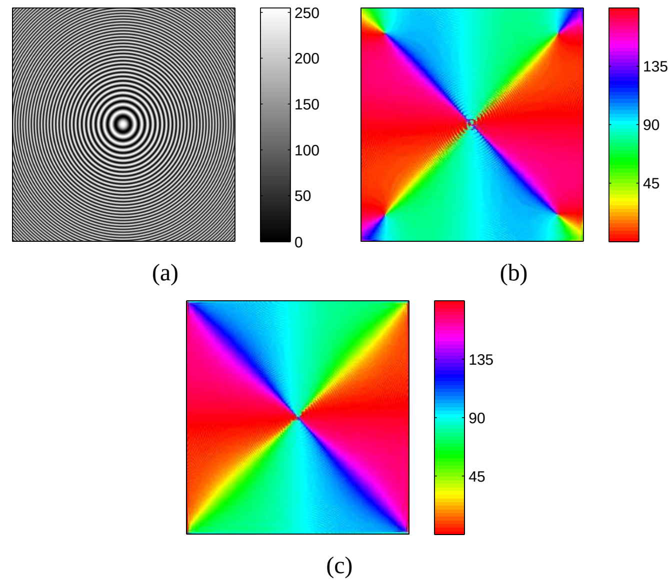

Fig. 2. Results for orientation estimation for a sinusoidal zone-plate image (a) The zone-plate image (image domain) (b) Dominant orientation using original wavelet filters (*Db2*) (c) Dominant orientation, estimated according using the proposed IIR wavelet filters (*Db2*, first order, $M = 1$).

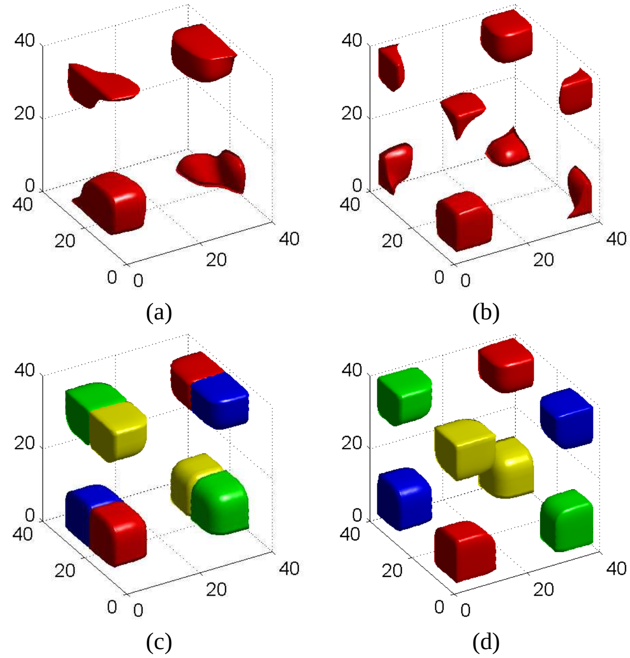

Fig. 3. Iso-energy plot of the frequency responses of the real parts of the basis functions for the first scale of the 3-d DT-CWT (different colors signify different orientations), for (a)-(b) The original wavelet filters (*Db8*), in *one orientation* only (corresponding to the red surfaces in (c)-(d)). (c)-(d) The proposed wavelet filters (*Db8*), for the first order $M = 1$, and in *four orientations*. For the original wavelet filters (top row), there is leakage of the basis function energy to different orientations, indicating a poor directional selectivity. With the proposed filters (bottom row), the leakage is well suppressed.