Performance bounds in Synchronization for low Signal-to-Noise Ratios

Heidi Steendam, Marc Moeneclaey

Telecommunications and Information Processing (TELIN) department, Ghent University,
St-Pietersnieuwstraat 41, B-9000 Gent, Belgium
{Heidi.Steendam, Marc.Moeneclaey}@telin.rug.ac.be

ABSTRACT

In this contribution we consider the Cramer-Rao bound (CRB) for the estimation of the synchronization parameters of a noisy linearly modulated signal with random data symbols. We explore three scenarios, i.e., (i) joint estimation of carrier phase, carrier frequency and time delay, irrespective of the data; (ii) joint estimation of carrier frequency and time delay, irrespective of the data and the carrier phase; and (iii) estimation of carrier frequency, irrespective of the data, the carrier phase and the timing. Because of the presence of the random data (and, in scenarios (ii) and (iii), also of random synchronization parameters), the exact computation of the corresponding CRBs is extremely difficult. Instead, here we derive a simple closed-form expression for the limit of these CRBs at *low* signal-to-noise ratio (SNR), which holds for arbitrary PAM, PSK and QAM constellations.

INTRODUCTION

The Cramer-Rao bound (CRB) is a lower bound on the error variance of any unbiased estimate, and as such serves as a useful benchmark for practical estimators [1]. In many cases, the statistics of the observation depend not only on the vector parameter to be estimated, but also on a *nuisance* vector parameter we do not want to estimate. The presence of this nuisance parameter makes the computation of the CRB very hard, if not impossible.

A typical example where a nuisance vector parameter occurs is the observation of a noisy linearly modulated waveform, that is a function of a time delay, a carrier frequency offset, a carrier phase and a data symbol sequence. In [2], the CRBs for estimating the frequency offset and the carrier phase from matched filter output samples have been computed for BPSK and QPSK, considering the data symbols as nuisance parameters and assuming the timing to be known; different constellations yield different expressions for these CRBs.

In order to avoid the computational complexity caused by the nuisance parameters, a modified CRB (MCRB) has been derived in [3, 4]. The MCRB is much simpler to evaluate than the CRB, but is in general looser than the CRB. In [5] the high-SNR limit of the CRB has been evaluated analytically, and has been shown to coincide with the MCRB when estimating the delay, the frequency offset or the carrier phase of the linearly modulated waveform.

In the presence of coding, synchronization algorithms must operate at low SNR, so that the high-SNR limit of the CRB might no longer be a relevant benchmark. In [6], the low-SNR limit of the CRB related to timing recovery has been presented, assuming a slowly varying carrier phase. The low-SNR limit of the CRB for estimating the carrier phase and frequency, from matched filter output samples taken at the correct decision instants, has been investigated in [7]. In this contribution we derive a simple expression for the low-SNR limit of the CRBs for (i) joint phase, frequency and timing estimation, (ii) joint frequency and timing estimation, and (iii) frequency estimation. The resulting expressions are valid for arbitrary PAM, PSK, and QAM constellations, and for an arbitrary square-root Nyquist transmit pulse. The adopted signal model and the considered scenarios are different from those investigated in [6, 7]. Finally, from this low-SNR limit of the CRB and the known high-SNR limit of the CRB, we derive an approximate expression of the true CRB.

PROBLEM FORMULATION

Let us consider the complex baseband representation r(t) of a noisy linearly modulated signal :

$$r(t) = \varepsilon \sum_{k=-K}^{K} a_k h(t - kT - \tau) \exp\big(j(2\pi F t + \theta)\big) + w(t) \qquad (1)$$

where $\mathbf{a} = (a_{-K}, \dots a_K)$ is a vector of L = 2K+1 zero-mean statistically independent data symbols with $E[|a_k|^2] = 1$; h(t) is a real-valued even-symmetrical unit-energy square-root Nyquist pulse; τ and F are the time delay and the carrier frequency offset, respectively; θ is the carrier phase at t = 0; T is the symbol interval; n(t) is complex-valued zero-mean Gaussian noise with independent real and imaginary parts, each having a normalized power spectral density of 1/2; ε =

$(E_s/N_0)^{1/2}$, with E_s and N_0 denoting the symbol energy and the noise power spectral density, respectively. The distribution of the data symbols is not a function of the synchronization parameters (θ, F, τ).

Suppose that one is able to produce from an observation vector **r** an *unbiased* estimate $\hat{\mathbf{u}}$ of a deterministic vector parameter **u**. Then the estimation error variance is lower bounded by the Cramer-Rao bound (CRB) [1] : $E_{\mathbf{r}}[(\hat{u}_i - u_i)^2] \geq CRB_i(\mathbf{u})$, where $CRB_i(\mathbf{u})$ is the i-th diagonal element of the inverse of the *Fisher information matrix* **J**(**u**). The (i,j)-th element of **J**(**u**) is given by

$$\mathbf{J}(\mathbf{u}) = E_{\mathbf{r}}\left[-\frac{\partial^2}{\partial u_i \partial u_j}\ln(p(\mathbf{r};\mathbf{u}))\right] \tag{2}$$

Note that **J**(**u**) is a symmetrical matrix. The probability density p(**r**;**u**) of **r**, corresponding to a given value of **u**, is called the *likelihood function* of **u**, while ln(p(**r**;**u**)) is the *log-likelihood function* of **u**. The expectation $E_{\mathbf{r}}[.]$ in (2) is with respect to p(**r**;**u**).

Let us denote by $\tilde{\mathbf{u}}$ a trial value of the deterministic parameter **u**, and consider $E_{\mathbf{r}}[\ln(p(\mathbf{r};\tilde{\mathbf{u}})]$ as a function of $\tilde{\mathbf{u}}$. It has been shown in [1] that approximating $E_{\mathbf{r}}[\ln(p(\mathbf{r};\tilde{\mathbf{u}})]$ by a trucated Taylor series about $\tilde{\mathbf{u}} = \mathbf{u}$ yields

$$E_{\mathbf{r}}[\ln(p(\mathbf{r};\tilde{\mathbf{u}})] \cong E_{\mathbf{r}}[\ln(p(\mathbf{r};\mathbf{u})] \; - \; \tfrac{1}{2}(\tilde{\mathbf{u}} - \mathbf{u})^T \mathbf{J}(\mathbf{u})(\tilde{\mathbf{u}} - \mathbf{u}) \tag{3}$$

This indicates that $E_{\mathbf{r}}[\ln(p(\mathbf{r};\tilde{\mathbf{u}})]$, the average of the log-likelihood function, takes its maximum value at $\tilde{\mathbf{u}} = \mathbf{u}$, and that **J**(**u**) determines the behavior of $E_{\mathbf{r}}[\ln(p(\mathbf{r};\tilde{\mathbf{u}})]$ in the close vicinity of its maximum.

The maximum-likelihood (ML) estimate $\hat{\mathbf{u}}_{ML}$ of the parameter **u** maximizes the log-likelihood function ln(p(**r**;**u**)) over **u** for given **r**. The resulting mean-square estimation error is known to converge to the CRB when the number of observations increases [1]. We will make use of this property in section 5, when interpreting our results.

When the observation **r** depends not only on the parameter **u** to be estimated but also on a nuisance vector parameter **v**, the likelihood function of **u** is obtained by averaging the *joint* likelihood function p(**r**|**v**;**u**) of the vector (**u**,**v**) over the a priori distribution of the nuisance parameter : $p(\mathbf{r};\mathbf{u}) = E_{\mathbf{v}}[p(\mathbf{r} \mid \mathbf{v};\mathbf{u})]$.

In the following we will consider the following three scenarios :
(i) The joint estimation of (θ,F,τ) from r(t). This implies that the useful parameter and the nuisance parameter are given by **u** = (θ,F,τ) and **v** = **a**, respectively.
(ii) The joint estimation of (F, τ) from r(t). This implies that the useful parameter and the nuisance parameter are given by **u** = (F,τ) and **v** = (**a**, θ), respectively. In this scenario, θ is considered as uniformly distributed in (-π, π).
(iii) The estimation of F from r(t). This implies that the useful parameter and the nuisance parameter are given by **u** = F and **v** = (**a**, θ, τ), respectively. In this scenario, θ and τ are considered as uniformly distributed in (-π, π) and (-T/2, T/2), respectively.

For all three scenarios, the joint likelihood function p(**r**|**v**;**u**) is, within a factor not depending on (**u**,**v**), given by

$$p(\mathbf{r} \mid \mathbf{v};\mathbf{u}) = \prod_{k=-K}^{K} \exp\left(\varepsilon a_k \breve{z}_k^* + \varepsilon a_k^* \breve{z}_k - \varepsilon^2 \mid a_k \mid^2\right) \tag{4}$$

with $\breve{z}_k = z_k(F,\tau)\exp(-j\theta)$, and

$$z_k(F,\tau) = \int r(t)\exp(-j2\pi Ft)h(t-kT-\tau)dt \tag{5}$$

As indicated in Fig. 1, the quantity $z_k(F,\tau)$ is obtained by first applying to r(t) a constant-speed rotation of -2πF rad/s, feeding the result to a filter matched to the transmit pulse h(t), and sampling the matched filter output at instant kT+τ. The quantity $\breve{z}_k$ is obtained by applying to $z_k(F,\tau)$ a rotation of -θ rad. The resulting log-likelihood function ln(p(**r**;**u**)) is given by

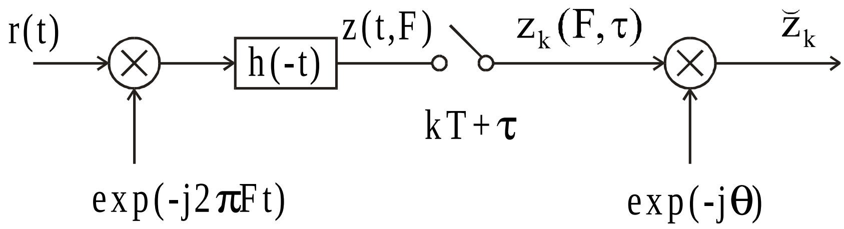

Fig. 1 : Computation of $\breve{z}_k$

$$\ln p(\mathbf{r};\mathbf{u}) = \ln\left(E_{\mathbf{v}}\left[\prod_{k=-K}^{K} \exp\left(\varepsilon a_k \breve{z}_k^* + \varepsilon a_k^* \breve{z}_k - \varepsilon^2 |a_k|^2\right)\right]\right) \tag{6}$$

Computation of the CRBs requires the substitution of (6) into (2) and the evaluation of the various expectations included in (6) and (2)

As the evaluation of the expectations involved in $\mathbf{J}(\mathbf{u})$ and $p(\mathbf{r};\mathbf{u})$ is quite tedious, no general closed-form expressions for the CRBs are available. To avoid these complications, a simpler lower bound, called the modified CRB (MCRB), has been derived in [3, 4], i.e., $E_{\mathbf{r}}[(\hat{u}_i - u_i)^2] \geq CRB_i(\mathbf{u}) \geq MCRB_i(\mathbf{u})$. The MCRBs for phase, frequency and timing estimation, corresponding to r(t) from (1), are given by [3, 4]

$$MCRB_\theta = \frac{N_0}{2E_s}\cdot\frac{1}{L} \qquad MCRB_F = \frac{N_0}{2E_s}\cdot\frac{3}{\pi^2 L(L^2-1)}\cdot\frac{1}{T^2} \qquad MCRB_\tau = \frac{N_0}{2E_s}\cdot\frac{1}{L}\cdot\frac{1}{(-\ddot{g}(0))} \tag{7a,7b,7c}$$

where L = 2K+1, and $\ddot{g}(t)$ denotes twice differentiation with respect to t of the Nyquist pulse g(t), which is defined as

$$g(t) = \int_{-\infty}^{+\infty} h(w)h(t+w)dw \tag{8}$$

The MCRBs (7) are valid for the joint estimation of an arbitrary subset of the synchronization parameters (ranging from only one parameter to all three parameters), with the random data and the remaining synchronization parameters considered as nuisance parameters. In [5] it has been shown that for *high* SNR (i.e., $E_s/N_0 \to \infty$) the CRBs for phase, frequency and timing estimation converge to the corresponding MCRBs (7). In the following, we derive a closed form expression for the *low*-SNR limit (i.e. $E_s/N_0 \to 0$) of the CRBs for the scenarios (i)-(iii) mentioned above. These low-SNR asymptotic CRBs (ACRBs) will be denoted as $ACRB_\theta$, $ACRB_F$ and $ACRB_\tau$.

LOW-SNR LIMIT OF THE LIKELIHOOD FUNCTIONS

For small E_s/N_0 (or equivalently, small ε), we obtain an approximation of $\ln(p(\mathbf{r};\mathbf{u}))$ by expanding the exponential functions in (6) into a Taylor series, averaging the resulting terms with respect to the nuisance parameter **v**, and keeping only the relevant terms that correspond to the smallest powers of ε. For scenario (i), the averaging is with respect to the data sequence **a**. For scenario (ii) an additional averaging over the carrier phase θ must be performed, while for scenario (iii) a further averaging over the timing parameter τ is required.

Scenario (i) : Joint Estimation of θ, F and τ

Taking in (6) **v** = **a** yields the following intermediate result :

$$E_{a_k}\left[\exp\left(\varepsilon a_k \breve{z}_k^* + \varepsilon a_k^* \breve{z}_k - \varepsilon^2 |a_k|^2\right)\right] = 1 + \sum_{p=1}^{\infty}\sum_{q=0}^{p}\sum_{r=0}^{p-q} F(p,q,r,\breve{z}_k)\varepsilon^{p+q} E\left[\left(a_k^*\right)^{p-r} a_k^{q+r}\right] \tag{9}$$

with

$$F(p,q,r,\breve{z}_k) = \frac{(-1)^q \breve{z}_k^{p-q-r}\left(\breve{z}_k^*\right)^r}{q!r!(p-q-r)!} \tag{10}$$

When the symbol constellation is rotationally symmetrical over 2π/N (N=2 for M-PAM, N=4 for M-QAM, N=M for

M-PSK), we obtain $E\left[\left(a_k^*\right)^{p-r} a_k^{q+r}\right] = 0$ for 2r+q-p $\notin \{0, \pm N, \pm 2N, ...\}$. Hence, the relevant terms in the triple summation of (7) correspond to

- (p,q,r) = (2,0,1). This yields a term proportional to ε^2, depending on F and τ, but not on θ. Therefore, additional terms depending on θ must be included for scenario (i).
- (p,q,r) = (N,0,0) and (p,q,r) = (N,0,N). This yields terms proportional to ε^N, depending on θ, F and τ.

All other nonzero terms in the triple summation of (9) are either independent of the synchronization parameters (for small ε these terms can be neglected as compared to the term 1 in (9)), or depending on (F,τ) or (θ,F,τ) but containing a power of ε larger than 2 or N, respectively. Hence, keeping in (9) only the dominant terms yields

$$\begin{aligned} \ln p(\mathbf{r};\theta,F,\tau) &\cong \sum_{k=-K}^{K} \ln\left(1 + \varepsilon^2 |\breve{z}_k|^2 + \frac{1}{N!}\varepsilon^N A_N^* \breve{z}_k^N + \frac{1}{N!}\varepsilon^N A_N (\breve{z}_k^*)^N\right) \\ &\cong \sum_{k=-K}^{K} \left(\varepsilon^2 |\breve{z}_k|^2 + \frac{1}{N!}\varepsilon^N A_N^* \breve{z}_k^N + \frac{1}{N!}\varepsilon^N A_N (\breve{z}_k^*)^N\right) \end{aligned} \tag{11}$$

where $A_N = E\left[a_k^N\right]$, and we have made use of $E[|a_k|^2] = 1$.

Scenario (ii) : Joint Estimation of F and τ

Now we take in (6) $\mathbf{v} = (\mathbf{a}, \theta)$. Neglecting in (6) third-order and higher-order terms in ε, we obtain

$$\ln p(\mathbf{r};F,\tau) \cong \ln\left(1 + \varepsilon^2 \sum_{k=-K}^{K} |\breve{z}_k|^2\right) \cong \varepsilon^2 \sum_{k=-K}^{K} |\breve{z}_k|^2 \tag{12}$$

In order to obtain (12), it is sufficient to assume that the data symbols are *pairwise uncorrelated*, i.e., $E[a_k^* a_m] = \delta_{k-m}$. Note that pairwise uncorrelated data symbols occur not only for statistically independent $\{a_k\}$, but also for the large majority of practical codes [8].

Scenario (iii) : Estimation of F

Taking in (6) $\mathbf{v} = (\mathbf{a}, \theta, \tau)$ and neglecting third-order and higher-order terms in ε gives rise to

$$\ln p(\mathbf{r};F) \cong \ln\left(1 + \varepsilon^2 \frac{1}{T} \int_{-T/2}^{T/2} \sum_{k=-K}^{K} |z(kT+\tau,F)|^2 \, dt\right) \cong \frac{1}{T}\varepsilon^2 \int_{LT} |z(t,F)|^2 \, dt \tag{13}$$

In (13), z(t,F) denotes the continuous-time signal at the output of the matched filter, as indicated in Fig. 1. Here too, the assumption of pairwise uncorrelated (instead of statistically independent) data symbols is sufficient to arrive at (13).

LOW-SNR LIMITS OF THE CRAMER-RAO BOUNDS

The (dominant part of the) Fisher information matrix $\mathbf{J}$ is obtained by straightforward application of (2) to (11)-(13), for scenarios (i)-(iii). The diagonal elements of $\mathbf{J}^{-1}$ yield the ACRBs. Here we summarize the results of these computations.

Scenario (i) : Joint Estimation of θ, F and τ

When computing the Fischer information matrix $\mathbf{J}$, we must distinguish between N>2 and N=2.

- **N>2**. When computing J_{FF}, $J_{F\tau}$ and $J_{\tau\tau}$, the terms from (11) in ε^N can be neglected as compared to the term in ε^2, assuming small ε. For the computation of $J_{\theta\theta}$, $J_{F\theta}$ and $J_{\tau\theta}$, only the terms in ε^N should be considered, as the term in ε^2 is not a function of θ.
- **N=2**. The computation of $J_{\theta\theta}$, $J_{F\theta}$ and $J_{\tau\theta}$ involves the same terms from (11) as for N>2; hence these elements of $\mathbf{J}$ are simply obtained by replacing N by 2 in the corresponding expressions for N>2. For the computation of J_{FF}, $J_{F\tau}$ and $J_{\tau\tau}$, all terms from (11) should be considered. The log-likelihood function (11) for N=2 (assuming M-PAM) becomes

$$\ln p(\mathbf{r};\theta,F,\tau) \cong \sum_{k=-K}^{K}\left(\varepsilon^2\,|\breve{z}_k|^2 + \frac{1}{2}\varepsilon^2\breve{z}_k^2 + \frac{1}{2}\varepsilon^2(\breve{z}_k^*)^2\right) = 2\varepsilon^2\sum_{k=-K}^{K}\breve{z}_{I,k}^2 \qquad (14)$$

where $\breve{z}_{I,k} = \mathrm{Re}[\breve{z}_k]$. This indicates that for N=2 only the in-phase component of $\breve{z}_k$ is needed to estimate (θ,F,τ).

It turns out that **J** is block-diagonal : $J_{\tau\theta} = J_{\tau F} = 0$. This means that the vector parameter (θ,F) and the scalar parameter τ are *decoupled* : the ACRB for the joint estimation of both parameters is the same as the ACRB for the estimation of one parameter while the other parameter is known. In addition, for small ε (when N>2) or large L (when N=2), the coupling between θ and F can be ignored. The resulting ACRBs are given by

$$\mathrm{ACRB}_\theta \cong \left(\frac{N_0}{E_s}\right)^N \cdot \frac{1}{L}\cdot\frac{N!}{2N^2\,|A_N|^2} \qquad \mathrm{ACRB}_F \cong \begin{cases} \left(\frac{N_0}{E_s}\right)^2 \cdot \frac{1}{L}\cdot\frac{1}{8\pi^2\int t^2h^2(t)dt} & N>2 \\ \left(\frac{N_0}{E_s}\right)^2 \cdot \frac{1}{L(L^2-1)}\cdot\frac{3}{4\pi^2}\cdot\frac{1}{T^2} & N=2 \end{cases} \qquad (15a, 15b)$$

$$\mathrm{ACRB}_\tau \cong \begin{cases} \left(\frac{N_0}{E_s}\right)^2 \cdot \frac{1}{L}\cdot\frac{1}{2}\left(-\ddot{g}(0) - \sum_{m=-\infty}^{+\infty}\dot{g}^2(mT)\right)^{-1} & N>2 \\ \left(\frac{N_0}{E_s}\right)^2 \cdot \frac{1}{L}\cdot\frac{1}{4}\left(-\ddot{g}(0) - \sum_{m=-\infty}^{+\infty}\dot{g}^2(mT)\right)^{-1} & N=2 \end{cases} \qquad (15c)$$

where $\dot{g}(t)$ denotes differentiation with respect to t of the pulse g(t) from (8).

Scenario (ii) : Joint Estimation of F and τ

The results for scenario (ii) can be easily derived from those obtained in scenario (i), by observing that, as far as differentiation of the log-likelihood function with respect to F or τ is concerned, (11) with N>2 and (12) yield the same dominating terms. Hence, the parameters F and τ are decoupled, and the resulting ACRBs are given by

$$\mathrm{ACRB}_F \cong \left(\frac{N_0}{E_s}\right)^2 \cdot \frac{1}{L}\cdot\frac{1}{8\pi^2\int t^2h^2(t)dt} \qquad \mathrm{ACRB}_\tau \cong \left(\frac{N_0}{E_s}\right)^2 \cdot \frac{1}{L}\cdot\frac{1}{2}\left(-\ddot{g}(0) - \sum_{m=-\infty}^{+\infty}\dot{g}^2(mT)\right)^{-1} \qquad (16a, 16b)$$

Note that (16) holds for both N>2 and N=2.

Scenario (iii) : Estimation of F

The ACRB resulting from (13) is given by

$$\mathrm{ACRB}_F \cong \left(\frac{N_0}{E_s}\right)^2 \cdot \frac{1}{L}\cdot\frac{1}{\frac{4\pi^2}{T}\int t^2g^2(t)dt} \qquad (17)$$

DISCUSSION OF RESULTS

When SNR approches zero, maximum-likelihood (ML) synchronization corresponds to maximizing (11), (12) or (13), depending on the specific scenario. Hence, these maximizations yield essentially optimum estimates for small SNR when L is large, i.e., for small SNR the mean-square estimation error approaches the corresponding ACRB. Interpreting the ACRB as the low-SNR limit of the ML synchronizer performance explains the behavior of ACRB as a function of E_s/N_0. For estimating F or τ, the function to be maximized is quadratic in r(t); hence, for low SNR the estimation error is dominated by noise×noise terms, yielding a mean-square error inversely proportional to $(E_s/N_0)^2$. The function to be maximized when estimating θ is of order N in r(t), yielding a mean-square error inversely proportional to $(E_s/N_0)^N$.

Let us now discuss separately the ACRBs pertaining to the estimation of timing, carrier frequency and carrier phase, respectively.

Timing Estimation

The ACRBs (15c, 16b) for timing estimation are proportional to $(E_s/N_0)^{-2}$. This is in contrast with the MCRB (7c), which is proportional to $(E_s/N_0)^{-1}$. Further, note that both the ACRBs (15c, 16b) and the MCRB (7c) are inversely proportional to the sequence length L.

For scenario (i), i.e., the estimation of $\mathbf{u} = (\theta,F,\tau)$, τ is decoupled from (θ,F). Hence, the corresponding $ACRB_\tau$ is the same as if τ were estimated while θ and/or F are known. We observe from (15c) that for given E_s/N_0, $ACRB_\tau$ for N=2 is half as large as for N>2. This can be verfied by comparing timing recovery for BPSK (N=2, $E_s = E_b$) and QPSK (N=4, $E_s = 2E_b$), assuming $\theta = F = 0$. The timing synchronizer for BPSK operates only on the in-phase component of the received signal, whereas for QPSK both the (uncorrelated) in-phase and quadrature components are used for timing estimation. Let us assume a fixed value of E_b/N_0, for both BPSK and QPSK. Under this assumption, the in-phase and quadrature components for QPSK have the same statistics as the in-phase component for BPSK, so that the timing error variance for QPSK is half as large as for BPSK. Substituting in (15c) $E_s = 2E_b$ (QPSK) and $E_s = E_b$ (BPSK), respectively, shows that the resulting $ACRB_\tau$ for QPSK is indeed half as large as for BPSK, when E_b/N_0 is the same for both modulations. Note that this reasoning also applies to the MCRB : substituting in (7c) $E_s = 2E_b$ (QPSK) and $E_s = E_b$ (BPSK) yields a $MCRB_\tau$ for QPSK that is half as large as for BPSK, when E_b/N_0 is fixed. Assuming F and θ known, Fig. 3 shows the ratios $CRB_\tau/MCRB_\tau$ (explanation about the computation of the true CRB for timing is beyond the scope of this paper) and $ACRB_\tau/MCRB_\tau$ (from (15c)). Note the excellent agreement between those ratios at low SNR.

For scenario (ii), i.e., the estimation of $\mathbf{u} = (F,\tau)$, τ is decoupled from F. Hence, the corresponding $ACRB_\tau$ is the same as if τ were estimated while F is known. We observe from (15c) and (16b) that $ACRB_\tau$ for (scenario (ii), any N) is the same as $ACRB_\tau$ for (scenario (i), N>2); this is because the dominating terms of the corresponding log-likelihood functions are the same. Hence, for N>2, the ACRB for the timing is not influenced by whether the carrier phase is known, is jointly estimated with the timing, or is considered as a nuisance parameter. Comparing (15c) and (16b) reveals that for N=2, treating the carrier phase as a nuisance parameter increases the ACRB by a factor of 2 as compared to the case where the carrier phase is known (or is jointly estimated with the timing). This is because the timing synchronizer that maximizes the relevant log-likelihood function (i.e, (14) for scenario (i) and (12) for scenario (ii)) is affected by both the in-phase and quadrature noise×noise terms in scenario (ii), but only by the in-phase noise×noise terms in scenario (i). This behavior is in contrast with $MCRB_\tau$ from (7c), which for given E_s/N_0 assumes the same value for any N, irrespective of whether the carrier phase is known, is estimated jointly with the timing, or is considered as a nuisance parameter.

Frequency Estimation

The ACRBs for frequency estimation (see (15b), (16a) and (17)) are proportional to $(E_s/N_0)^{-2}$. This is in contrast with $MCRB_F$ from (7b), which is proportional to $(E_s/N_0)^{-1}$.

For scenario (i), i.e., the estimation of $\mathbf{u} = (\theta,F,\tau)$, F is decoupled from τ, and the coupling with θ is very weak. Hence, $ACRB_F$ is essentially the same as if F were estimated while θ and/or τ are known. For N=2, $ACRB_F$ is inversely proportional to $L(L^2-1)$. However, for N>2, $ACRB_F$ is inversely proportional to L only. This is because the maximum of the low-SNR limit of $E[\ln(p(\mathbf{r} \mid \theta, \tilde{F}, \tau))]$ (with $\tilde{F}$ denoting a trial value) is much sharper for N=2 than for N>2 ,as the low-SNR limit (11) of the log-likelihood function for N>2 ignores the phase information contained in $\breve{z}_k$. Hence, according to (3), the case N=2 yields a much larger value of J_{FF}. Finally, in contrast with $ACRB_F$, $MCRB_F$ from (7b) is inversely proportional to $L(L^2-1)$, irrespective of N.

For scenario (ii), i.e., the estimation of $\mathbf{u} = (F,\tau)$ irrespective of θ, F is decoupled from τ. Hence, the corresponding ACRB is the same as if F were estimated while τ is known. We observe from (15b) and (16a) that $ACRB_F$ for (scenario (ii), any N) is the same as $ACRB_F$ for (scenario (i), N>2); this is because the dominating terms of the corresponding log-likelihood functions are the same. Hence, for N>2, the ACRB for frequency estimation is not influenced by whether the carrier phase is known, is jointly estimated with the frequency, or is considered as a nuisance parameter. This is unlike the case N=2, where the joint estimation of F and θ and the estimation of F considering θ as a nuisance parameter yield a $ACRB_F$ that is inversely proportional to $L(L^2-1)$ and L, respectively. This behavior is in contrast with $MCRB_F$ from (7b), which for given E_s/N_0 assumes the same value for any N, irrespective of whether the carrier phase is known, is estimated jointly with the frequency, or is considered as a nuisance parameter.

It can be verified that $ACRB_F$ (17) resulting from scenario (iii) is larger than $ACRB_F$ (16a) resulting from scenario (ii). This indicates that considering τ as a nuisance parameter gives rise to a penalty as compared to scenario (ii). For a square-root cosine rolloff transmit pulse, it turns out that $ACRB_F$ from (17) is a factor of 2 larger than $ACRB_F$ from (16a). This is in contrast with $MCRB_F$ from (7b), which is not depending on whether the timing is known, is estimated jointly with the frequency or is considered as a nuisance parameter.

Phase Estimation

When estimating $\mathbf{u} = (\theta,F,\tau)$, the phase is decoupled from τ and is only weakly coupled with F. Hence, $ACRB_\theta$ resulting from this joint estimation is essentially the same as if θ were estimated while τ and/or F are known.

$ACRB_\theta$ from (15a) is proportional to $(E_s/N_0)^{-N}$. This is in contrast with $MCRB_\theta$ from (7a), which is proportional to $(E_s/N_0)^{-1}$. Further, note that both $ACRB_\theta$ and the $MCRB_\theta$ are inversely proportional to the sequence length L.

Assuming F and τ to be known, Fig. 3 shows the ratios $CRB_\theta/MCRB_\theta$ (from [2]) and $ACRB_\theta/MCRB_\theta$ (from (15a)). As it should, excellent agreement between those ratios is found at low SNR.

CONCLUSIONS AND REMARKS

In this contribution we have derived a simple closed-form analytical expression for the low-SNR asymptote of the CRBs pertaining to the estimation of the carrier phase θ, the carrier frequency F and the time delay τ from a linearly modulated waveform. We have considered three different scenarios, i.e., (i) the estimation of (θ,F,τ) with the data symbol sequence **a** considered as nuisance vector parameter, (ii) the estimation of (F,τ) with $(\mathbf{a}, \theta)$ considered as nuisance vector parameter, and (iii) the estimation of F with $(\mathbf{a}, \theta, \tau)$ considered as nuisance vector parameter. Considering a symbol constellation with rotational symmetry of $2\pi/N$, we have to distinguish between N>2 and N=2. For N>2, scenarios (i) and (ii) yield the same value of $ACRB_F$ and of $ACRB_\tau$, which indicates that without loss of performance at small SNR, the carrier phase can be considered as a nuisance parameter when estimating the carrier frequency and the timing. However, for N=2 it turns out that the values of $ACRB_F$ and $ACRB_\tau$ are smaller in the first scenario than in the second, which indicates that estimating θ jointly with F and τ yields the better performance at small SNR. $ACRB_F$ is larger in the third scenario than in the second scenario, which indicates that considering the timing as a nuisance parameter gives rise to a performance penalty. The ACRBs for estimating F and τ are inversely proportional to the square of E_s/N_0, whereas $ACRB_\theta$ is inversely proportional to the *N-th power* of E_s/N_0.

Based upon the maximization of (11) or (12), various nondata-aided (NDA) feedback and feedforward synchronization algorithms for recovering the carrier frequency, carrier phase and symbol timing have been derived and their performance analyzed. The interested reader is referred to [9 (sect. 5, 6.3)], [10 (sect. 3, 5, 8)] and the references therein.

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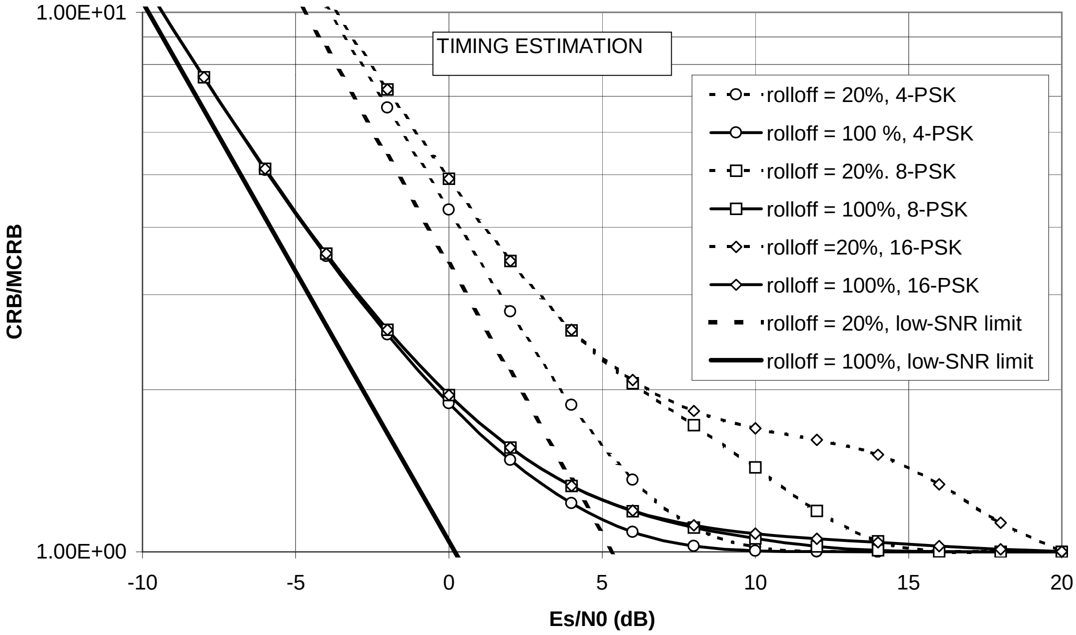

Fig. 2 : CRB for timing estimation

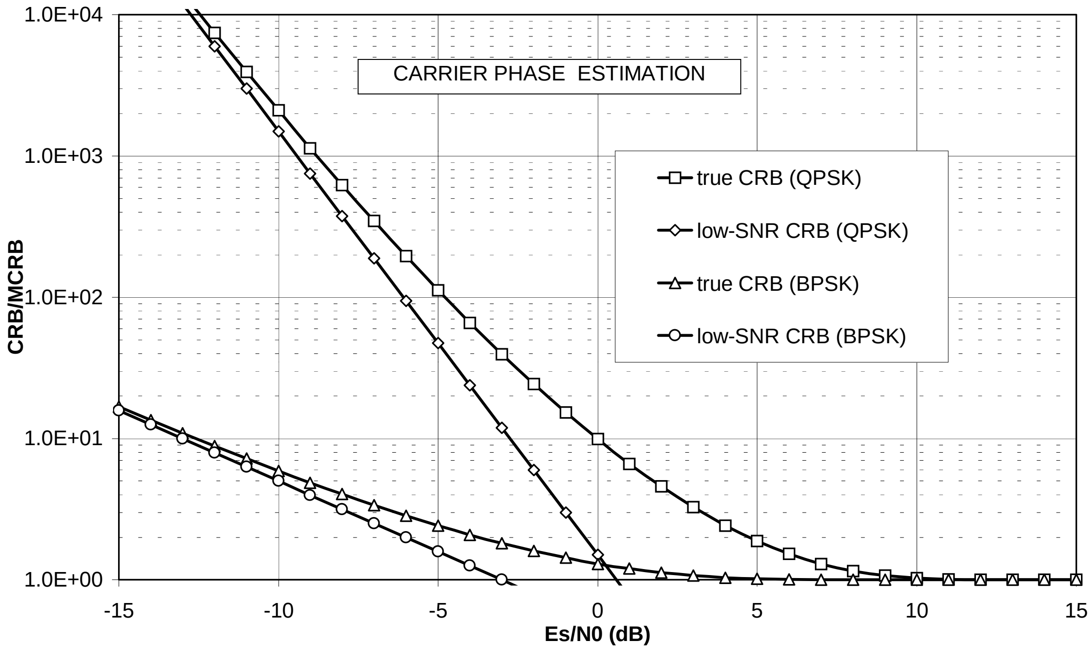

Fig. 3 : CRB for carrier phase estimation