The true Cramer-Rao bound for estimating the time delay of a linearly modulated waveform

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***Abstract* - In this contribution we consider the Cramer-Rao bound (CRB) for the estimation of the time delay of a noisy linearly modulated signal with random data symbols. In spite of the presence of the nuisance parameters (i.e., the random data symbols), we obtain a closed-form expression of this CRB for arbitrary PSK, QAM or PAM constellations and a bandlimited square-root Nyquist transmit pulse.**

I. INTRODUCTION

The Cramer-Rao bound (CRB) is a lower bound on the error variance of any unbiased estimate, and as such serves as a useful benchmark for practical estimators [1]. The CRB is formulated in terms of the likelihood function of the scalar parameter to be estimated. In many cases, the statistics of the observed vector depend not only on the parameter to be estimated, but also on a number of nuisance parameters we do not want to estimate.

A typical example where nuisance parameters occur is the observation of a noisy linearly modulated waveform, that is a function of a time delay, a carrier frequency offset, a carrier phase and a data symbol sequence. The presence of the nuisance parameters makes the computation of the likelihood function and the corresponding CRB very hard. Therefore, the closed-form expression of the CRB in the presence of random data has been presented only for very few cases. As far as we know, only the CRBs for estimating the frequency offset and the carrier phase, assuming the timing to be known; are available in the open literature [2].

In order to avoid the computational complexity caused by the nuisance parameters, a modified CRB (MCRB) has been derived in [3]. The MCRB is much simpler to evaluate than the CRB, but is in general looser than the CRB. In [4] the high-SNR limit of the CRB has been evaluated analytically, and has been shown to coincide with the MCRB when estimating the delay, the frequency offset or the carrier phase of the linearly modulated waveform. In [5, 6] the low-SNR limits of the CRBs related to carrier and timing recovery have been presented.

In this contribution, we tackle the problem of computing the CRB related to estimating the time delay of a linearly modulated waveform. The transmit pulse is an arbitrary square-root Nyquist pulse, and results are presented for PAM, PSK and QAM constellations.

II. PROBLEM FORMULATION

Let us consider the complex baseband representation r(t) of a noisy linearly modulated signal :

$$r(t) = \sum_{k=0}^{K-1} a(k)h(t-kT-\tau) + w(t) \tag{1}$$

where $\mathbf{a}$ = (a(0), ... a(K-1)) is a vector of zero-mean statistically independent equiprobable data symbols with $E[|a(m)|^2]=1$, h(t) is a real-valued unit-energy square-root Nyquist pulse, τ is a *deterministic* time delay, T is the symbol interval, and w(t) is complex-valued zero-mean Gaussian noise with independent real and imaginary parts, each having a power spectral density of $N_0/(2E_s)$. The probability density of the vector $\mathbf{a}$ is not a function of τ.

Suppose that one is able to produce from r(t) an *unbiased* estimate $\hat{\tau}$ of the delay τ. Then the estimation error variance is lower bounded by the Cramer-Rao bound (CRB) [1] : $E_{\mathbf{r}}[(\hat{\tau}-\tau)^2] \geq T^2\text{CRB}$, where

$$\text{CRB} = \left(E_{\mathbf{r}}\left[\left(T\frac{d}{d\tau}\ln p(\mathbf{r}\mid\tau) \right)^2 \right] \right)^{-1} \tag{2}$$

In (2), $\mathbf{r}$ is a vector representation of the signal r(t). The probability density $p(\mathbf{r};\tau)$ of $\mathbf{r}$, corresponding to a given value of τ, is called the *likelihood function* of τ. The expectation $E_{\mathbf{r}}[.]$ is with respect to the probability density $p(\mathbf{r};\tau)$.

As r(t) from (1) depends not only on the delay τ to be estimated but also on the nuisance vector parameter $\mathbf{a}$, the likelihood function of τ is obtained by averaging the *joint* likelihood function $p(\mathbf{r}|\mathbf{a};\tau)$ of $(\mathbf{a},\tau)$ over the a priori distribution of the nuisance vector parameter : $p(\mathbf{r};\tau) = E_{\mathbf{a}}[p(\mathbf{r}\mid\mathbf{a};\tau)]$. From (1) it follows that (within a factor not depending on $(\mathbf{a}, \tau)$)

$$p(\mathbf{r}\mid\mathbf{a};\tau) = \prod_{k=0}^{K-1} F(a(k), z(k)) \tag{3}$$

where

$$F(a(k),z(k)) = \exp\left(\frac{E_s}{N_0}\left(2\,\text{Re}\left[a^*(k)z(k)\right] - |a(k)|^2\right) \right) \tag{4}$$

and

$$z(k) = \int_{-\infty}^{+\infty} r(t)h(t-kT-\tau)dt \tag{5}$$

As the expectations involved in CRB and $p(\mathbf{r};\tau)$ are hard to evaluate for an arbitrary PSK, QAM or PAM symbol constellation and for bandlimited h(t), a simpler lower bound, called the modified CRB (MCRB), has been derived in [3] : $E_{\mathbf{r}}[(\hat{\tau}-\tau)^2] \geq T^2\text{CRB} \geq T^2\text{MCRB}$. Defining the Nyquist pulse g(t) as

$$g(t) = \int_{-\infty}^{+\infty} h(v)h(t+v)dt \tag{6}$$

the MCRB for timing estimation, corresponding to r(t) from (1), is given by [3]

$$\mathrm{MCRB} = \frac{N_0}{2E_sK} \cdot \frac{1}{(-\ddot{g}(0)T^2)} \tag{7}$$

where $\ddot{g}(t)$ denotes twice derivation of g(t) with respect to t. In [4] it has been shown that for *high* SNR (i.e. $E_s/N_0 \to \infty$) the CRB (2) resulting from (1) converges to the MCRB (7).

In [6], a closed form expression for the *low*-SNR limit (i.e. $E_s/N_0 \to 0$) of the CRB that corresponds to (1) has been derived. This low-SNR asymptotic CRB is denoted $ACRB_0$, and is given by

$$\mathrm{ACRB}_0 = \frac{1}{K} \cdot \frac{1}{2}\left(\frac{N_0}{E_s}\right)^2 \frac{1}{(-\ddot{g}(0)T^2 - \sum_{n=-\infty}^{+\infty} \dot{g}^2(nT)T^2} \tag{8}$$

assuming complex-valued symbols with $E[a^2(k)] = 0$. It should be noted that (8) itself is not necessarily a lower bound on the timing error variance.

In this paper, we derive a closed-form expression for the true CRB (2) assuming a bandlimited square-root Nyquist transmit pulse and PAM, PSK and QAM constellations.

III. Evaluation of True CRB

In this section we first concentrate on rotationally symmetric constellations with $E[a^2(k)] = 0$, i.e. M-PSK with M>2 and M-QAM, but not M-PAM. The case of M-PAM is dealt with at the and of this section.

Taking (3) into account, the log-likelihood function $\ln p(\mathbf{r}|\tau)$ is given by

$$\ln p(\mathbf{r} \mid \tau) = \sum_{k=0}^{K-1} \ln\left(\frac{1}{M} \sum_{i=0}^{M-1} F(\alpha_i, z(k))\right) \tag{9}$$

where $\{\alpha_0, \alpha_1, ..., \alpha_{M-1}\}$ is the set of constellation points. Differentiation of (9) yields

$$T\frac{d}{d\tau} \ln p(\mathbf{r} \mid \tau) = \frac{E_s}{N_0} \sum_{k=0}^{K-1} \sum_{i=0}^{M-1} H(\alpha_i, z(k), z_\tau(k)) \tag{10}$$

where

$$H(\alpha_i, z(k), z_\tau(k)) = G(\alpha_i, z(k))(\alpha_i^* z_\tau(k) + \alpha_i z_\tau^*(k)) \tag{11}$$

$$G(\alpha_i, z(k)) = \frac{F(\alpha_i, z(k))}{\sum_{i=0}^{M-1} F(\alpha_i, z(k))} \tag{12}$$

and

$$z_\tau(k) = T\frac{dz(k)}{d\tau} = -T\int_{-\infty}^{+\infty} r(t)\dot{h}(t - kT - \tau)dt \tag{13}$$

where $\dot{}$ denotes differentiation with respect to t.

The quantities z(k) and $z_\tau(k)$ can be decomposed as

$$z(k) = a(k) + N(k) \tag{14}$$

$$\begin{aligned} z_\tau(k) &= s_\tau(k) + N_\tau(k) \\ &= \sum_{m=0}^{K-1} a_m \dot{g}(kT - mT)T + N_\tau(k) \end{aligned} \tag{15}$$

where N(k) and $N_\tau(k)$ are complex Gaussian variables, with

$$\begin{aligned} E\left[N(m)N^*(n)\right] &= \frac{N_0}{E_s}\delta_{m-n} \\ E[N_\tau(m)N^*(n)] &= \frac{N_0}{E_s}\dot{g}(mT - nT)T \\ E[N_\tau(m)N_\tau^*(n)] &= \frac{N_0}{E_s}\left(-\ddot{g}(mT - nT)T^2\right) \end{aligned} \tag{16}$$

Note that $\dot{g}(t)$ has odd symmetry, and g(t) and $\ddot{g}(t)$ have even symmetry.

Taking (10) into account, the computation of the CRB (2) involves the evaluation of

$$M(k_1, k_2) = \sum_{i_1, i_2=0}^{M-1} E[H(\alpha_{i_1}, z(k_1), z_\tau(k_1)H((\alpha_{i_2}, z(k_2), z_\tau(k_2))] \tag{17}$$

for k_1, k_2 = 0, 1, ..., K-1, where E[.] denotes averaging over the data symbols and the noise. We separately consider the cases $k_1=k_2$ and $k_1 \neq k_2$.

For the case $k_1=k_2=k$ we obtain :

$$M(k,k) = 2E[|z_\tau(k)|^2]E\left[\left|\sum_{i=0}^{M-1} \alpha_i G(\alpha_i, z(k))\right|^2\right] \tag{18}$$

which follows from the statistical independence of z(k) and $z_\tau(k)$. The first expectation in (18) is easily obtained analytically :

$$E[|z_\tau(k)|^2] = \frac{N_0}{E_s}(-\ddot{g}(0)T^2) + \sum_{m=0}^{K-1} \dot{g}^2(kT - mT)T^2 \tag{19}$$

The second expectation in (18) should be evaluated numerically; note that this expectation does not depend on the shape of the Nyquist pulse g(t).

The case $k_1 \neq k_2$ is considerably more difficult, because the ISI caused by the bandlimited nature of g(t) gives rise to a mutual dependence of $z(k_1)$, $z(k_2)$, $z_\tau(k_1)$ and $z_\tau(k_2)$. Noting that $(z(k_1), z(k_2))$ depends only on $(a(k_1), a(k_2), N(k_1)\ N(k_2))$, we first consider some statistics of $(z_\tau(k_1), z_\tau(k_2))$, conditioned on $(a(k_1), a(k_2), N(k_1)\ N(k_2))$, that we will need later on.

The conditional expectations of $z_\tau(k_1)$ and $z_\tau(k_2)$ are given by

$$\begin{aligned} m_\tau(k_1) &= E\left[z_\tau(k_1) \mid a(k_1), a(k_2), N(k_1), N(k_2)\right] \\ &= a(k_2)\dot{g}(k_1T - k_2T) + N(k_2)\dot{g}(k_1T - k_2T) \\ &= z(k_2)\dot{g}(k_1T - k_2T) \end{aligned} \tag{20}$$

$$\begin{aligned} m_\tau(k_2) &= E\left[z_\tau(k_2) \mid a(k_1), a(k_2), N(k_1), N(k_2)\right] \\ &= z(k_1)\dot{g}(k_2T - k_1T) \end{aligned} \tag{21}$$

Now we consider the conditional covariances of $z_\tau(k_1)$ and $z_\tau(k_2)$ and of $z_\tau^*(k_1)$ and $z_\tau(k_2)$:

$$\begin{aligned}&\mathrm{Cov}[z_\tau(k_1), z_\tau(k_2)]\\&\quad= E\left[\Delta_\tau^*(k_1)\Delta_\tau(k_2)\,|\,a(k_1), a(k_2), N(k_1), N(k_1)\right]\end{aligned}\tag{22}$$

$$\begin{aligned}&\mathrm{Cov}[z_\tau^*(k_1), z_\tau(k_2)]\\&\quad= E\left[\Delta_\tau(k_1)\Delta_\tau(k_2)\,|\,a(k_1), a(k_2), N(k_1), N(k_1)\right]\end{aligned}\tag{23}$$

where

$$\begin{aligned}\Delta_\tau(k_1) &= z_\tau(k_1) - m_\tau(k_1)\\ \Delta_\tau(k_2) &= z_\tau(k_2) - m_\tau(k_2)\end{aligned}\tag{24}$$

It turns out that $\mathrm{Cov}[z_\tau(k_1), z_\tau(k_2)]$ does not depend on ($a(k_1)$, $a(k_2)$, $N(k_1)$ $N(k_2)$)), and that $\mathrm{Cov}[z_\tau^*(k_1), z_\tau(k_2)] = 0$. Finally,

$$\begin{aligned}&E\left[z_\tau^*(k_1)z_\tau(k_2)\,|\,a(k_1), a(k_2), N(k_1), N(k_2)\right]\\&\quad= m_\tau^*(k_1)m_\tau(k_2) + \mathrm{Cov}\left[z_\tau(k_1), z_\tau(k_2)\right]\\&E\left[z_\tau(k_1)z_\tau(k_2)\,|\,a(k_1), a(k_2), N(k_1), N(k_2)\right]\\&\quad= m_\tau(k_1)m_\tau(k_2)\end{aligned}\tag{25}$$

The rotational symmetry of the constellation (α_0, ..., α_{M-1}) and of the probability density function of z(k) gives rise to :

$$E\left[\sum_{i=0}^{M-1} G(\alpha_i, z(k))\alpha_i\right] = E\left[\sum_{i=0}^{M-1} G(\alpha_i, z(k))\alpha_i z(k)\right] = 0 \tag{26}$$

Taking (25-26) into account, we obtain

$$M(k_1, k_2) = 2\left(-\dot{g}^2(k_1T - k_2T)T^2\right)\left(E\left[\sum_{i=0}^{M-1} G(\alpha_i, z(k))\alpha_i^* z(k)\right]\right)^2 \tag{27}$$

Collecting the intermediate results (18, 27) yields

$$\begin{aligned}&E\left[\left(T\frac{d}{d\tau}\ln p(\mathbf{r}\,|\,\tau)\right)^2\right] = \left(\frac{E_s}{N_0}\right)^2 \sum_{k_1,k_2=0}^{K-1} M(k_1, k_2)\\&= 2\left(\frac{E_s}{N_0}K(-\ddot{g}(0)T^2)\right)A\left(\frac{E_s}{N_0}\right)\\&\qquad -2\left(\frac{E_s}{N_0}\right)^2\left(\sum_{k_1,k_2=0}^{K-1}\dot{g}^2(k_1T - k_2T)T^2\right)B\left(\frac{E_s}{N_0}\right)\end{aligned}\tag{28}$$

where

$$\begin{aligned}A\left(\frac{E_s}{N_0}\right) &= E\left[\left|\sum_{i=0}^{M-1} G(\alpha_i, z(k))\alpha_i\right|^2\right]\\ B\left(\frac{E_s}{N_0}\right) &= \left(E\left[\sum_{i=0}^{M-1} G(\alpha_i, z(k))z(k)\alpha_i^*\right]\right)^2 - A\left(\frac{E_s}{N_0}\right)\end{aligned}\tag{29}$$

Note that $A(E_s/N_0)$ and $B(E_s/N_0)$ depend on E_s/N_0 and on the type and size of the constellation, but not on the shape of the transmit pulse. The quantities $A(E_s/N_0)$ and $B(E_s/N_0)$ can easily be evaluated by means of numerical integration over a two-dimensional Gaussian probability density. The shape of the transmit pulse h(t) affects only the quantities involving $\ddot{g}(0)$ and $\dot{g}(k_1T - k_2T)$. For large K the following approximation is very accurate :

$$\sum_{k_1,k_2=0}^{K-1}\dot{g}^2(k_1T - k_2T)T^2 \approx K\sum_{m=-\infty}^{+\infty}\dot{g}^2(mT)T^2 \tag{30}$$

Substituting (30) in (28) yields a CRB that is inversely proportional to the sequence length K.

Till now we have excluded from consideration the M-PAM constellation. The case of M-PAM can be handled in a similar way as above. The CRB for M-PAM is related to the CRB for M^2-QAM in the following way :

$$\begin{aligned}\mathrm{CRB}_{M-PAM}\left(\frac{E_s}{N_0}\right) &= 2\mathrm{CRB}_{M^2-QAM}\left(\frac{2E_s}{N_0}\right)\\ \mathrm{CRB}_{M^2-QAM}\left(\frac{E_s}{N_0}\right) &= \frac{1}{2}\mathrm{CRB}_{M-PAM}\left(\frac{E_s}{2N_0}\right)\end{aligned}\tag{31}$$

IV. Numerical Results and Discussion

Assuming a square-root cosine rolloff transmit pulse (with 20% and 100% rolloff), we have computed the ratios CRB/MCRB and $ACRB_0$/MCRB. Fig.1 and Fig.2 show the results for M-PSK and M-QAM, respectively.

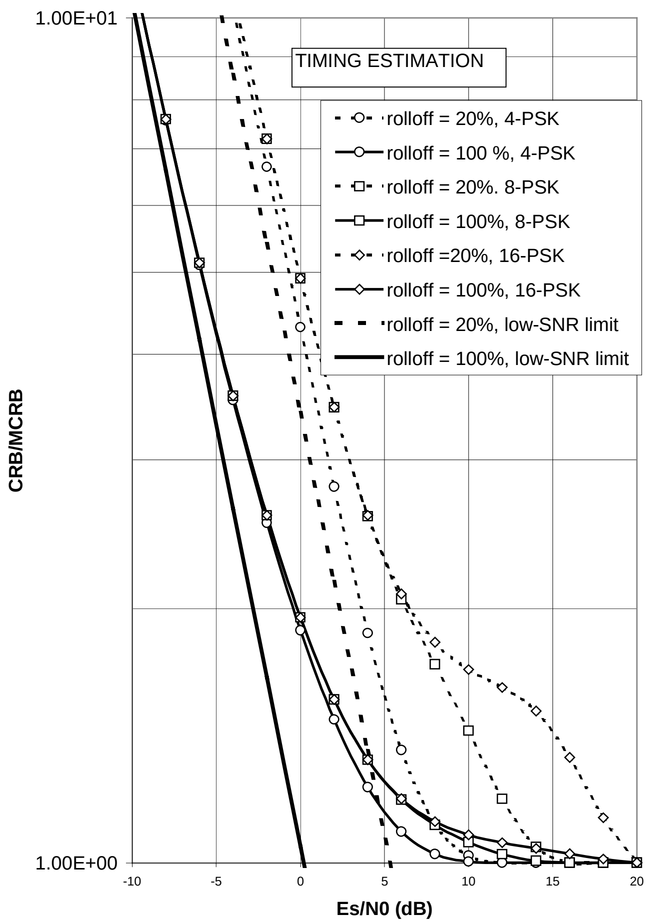

Fig. 1 : CRB for timing estimation of M-PSK

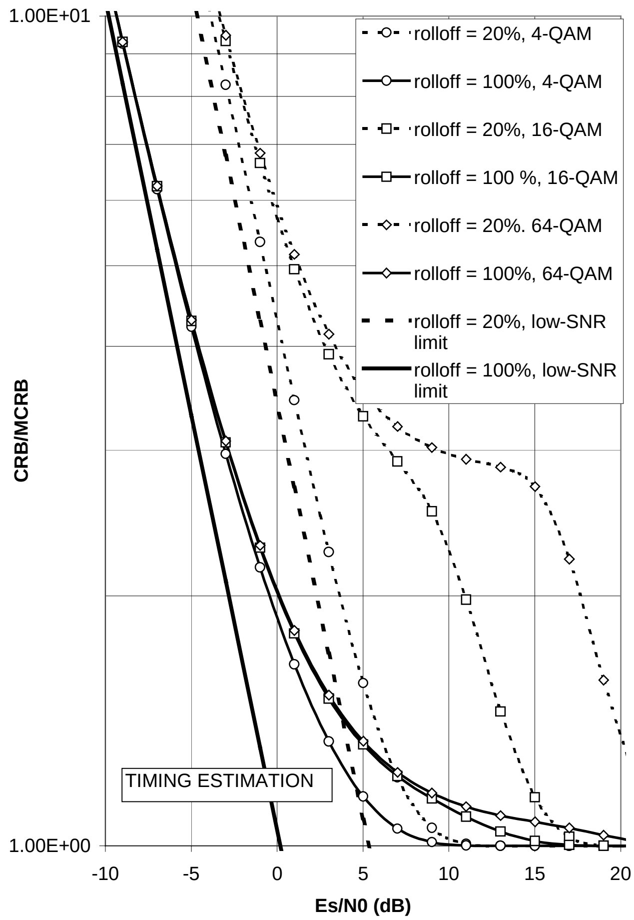

Fig. 2 : CRB for timing estimation of M-QAM

For both M-PSK and M-QAM, we observe that CRB/MCRB increases with M, which indicates that for the larger constellations timing recovery is inherently harder to accomplish. This effect is caused by ISI, and therefore is more pronounced for a smaller rolloff factor.

For small E_s/N_0, the effect of the size and type of the constellation on CRB/MCRB is small; the CRB converges to ACRB_0 (8).

For sufficiently large E_s/N_0, the CRB converges to the MCRB (7). However, the value of E_s/N_0, at which CRB is close to MCRB, considerably increases with increasing constellation size.

In the case of transmit pulses h(t) that are time-limited to one symbol interval T, we have $\dot{g}(mT) = 0$ for all m, so that only the first term of (28) contributes. This yields $\mathrm{CRB/MCRB} = 1/A(E_s/N_0)$. We have verified that the curves corresponding to a rolloff of 100% yield essentially the same result. Hence, the curves for a 100% rolloff are representative also for time-limited transmit pulses.

As CRB is hard to evaluate, one might be tempted to use either MCRB or ACRB as a first approximation of CRB. Denoting by $(E_s/N_0)_c$ the value of E_s/N_0 for which CRB/MCRB = 1, this approximation would yield

$$\mathrm{CRB} \approx \begin{cases} \mathrm{ACRB}_0 & E_s/N_0 < (E_s/N_0)_c \\ \mathrm{MCRB} & E_s/N_0 > (E_s/N_0)_c \end{cases} \qquad (32)$$

For a rolloff of 100% (and also for time-limited transmit pulses), $(E_s/N_0)_c \approx 0$ dB, so that for E_s/N_0 values of practical interest the true CRB is closer to MCRB than to ACRB_0. For a rolloff of 20 %, $(E_s/N_0)_c \approx 5$ dB, so that only for systems with sufficient coding gain (i.e., operating reliably at $E_s/N_0 <$ 5 dB) the true CRB is closer to ACRB_0 than to MCRB.

Using (31), results for M-PAM are easily derived from Fig.2.

V. Conclusions and Remarks

In this contribution, we have considered the CRB related to the estimation of the time delay of a noisy linearly modulated signal with arbitrary square-root Nyquist transmit pulse and containing random PSK, QAM or PAM symbols. In spite of the presence of the random data symbols, we have been able to present a relatively simple expression of the CRB. The evaluation of this expression requires only two numerical integrations per considered value of E_s/N_0. The effect of the pulse shape is analytically accounted for.

The numerical results indicate that for small E_s/N_0 and very large E_s/N_0, the effect of the type and size of the constellation on the CRB is small. For moderate E_s/N_0, the CRB increases with increasing constellation size; this effect is more pronounced when the excess bandwidth is small. The CRB is a decreasing function of the excess bandwidth.

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