

# THE TRUE CRAMER-RAO BOUND FOR CARRIER AND SYMBOL SYNCHRONIZATION

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## ABSTRACT

This contribution considers the Cramer-Rao bound (CRB) related to estimating the synchronization parameters (carrier phase, carrier frequency and time delay) of a noisy linearly modulated signal with random data symbols. We explore various scenarios, involving the estimation of a subset of the parameters while the other parameters are either considered as nuisance parameters or a priori known to the receiver. In addition, some results related to the CRB for coded transmission will be presented.

## 1. INTRODUCTION

The Cramer-Rao bound (CRB) is a lower bound on the error variance of any unbiased estimate, and as such serves as a useful benchmark for practical estimators [1]. In many cases, the statistics of the observation depend not only on the vector parameter to be estimated, but also on a *nuisance* vector parameter we do not want to estimate. The presence of this nuisance parameter makes the analytical computation of the CRB very hard, if not impossible.

In order to avoid the computational complexity caused by the nuisance parameters, a modified CRB (MCRB) has been derived in [2,3]. In [4] the high SNR limit of the CRB for the estimation of a scalar parameter has been evaluated analytically.

The true CRB for estimating the time delay and its low-SNR limit have been derived for PAM, PSK and QAM constellations in [5] and [6,7], respectively. The CRB related to carrier phase/frequency estimation, from matched filter output samples taken at correct decision instants, has been investigated in [8,9] (true CRB for BPSK, QPSK and QAM) and in [7,10] (low-SNR limit for PSK, QAM and PAM).

In this paper we further investigate the true CRBs for the estimation of the synchronization parameters (carrier phase  $\theta$ , carrier frequency  $F$  and time delay  $\tau$ ) from a noisy PSK, PAM or QAM signal. Section 3 deals with the estimation of  $F$  (with known  $\tau$ ) or  $\tau$  (with known  $F$ ), either jointly with  $\theta$  (i.e., scenario (i)) or irrespective of  $\theta$  (i.e., scenario (ii)), assuming uncoded transmission. In section 4 we consider the estimation of  $\theta$  (with known  $F$  and  $\tau$ ), assuming convolutional encoding.

## 2. PROBLEM FORMULATION

Let us consider the complex baseband representation  $r(t)$  of a noisy linearly modulated signal :

$$r(t) = \sum_{k=-K}^K a_k h(t - kT - \tau) \exp(j(2\pi F t + \theta)) + w(t) \quad (1)$$

where  $\mathbf{a} = (a_{-K}, \dots, a_K)$  is a vector of  $L = 2K+1$  symbols ( $E[|a_k|^2] = 1$ );  $h(t)$  is a real-valued unit-energy square-root Nyquist pulse;  $F$  is the carrier frequency offset;  $\theta$  is the carrier phase at  $t = 0$ ;  $\tau$  is the time delay;  $T$  is the symbol interval;  $w(t)$  is complex-valued zero-mean Gaussian noise with independent real and imaginary parts, each having a normalized power spectral density of  $N_0/2E_s$ , with  $E_s$  and  $N_0$  denoting the symbol energy and the noise power spectral density, respectively. The data symbols are either uncoded (section 3) or convolutionally encoded (section 4).

Suppose that one is able to produce from an observation vector  $\mathbf{r}$  an unbiased estimate  $\hat{\mathbf{u}}$  of a deterministic vector parameter  $\mathbf{u}$ . Then the estimation error variance is lower bounded by the Cramer-Rao bound (CRB) [1]:

$E_{\mathbf{r}}[(\hat{u}_i - u_i)^2] \geq CRB_i(\mathbf{u})$ , where  $CRB_i(\mathbf{u})$  is the  $i$ -th diagonal element of the inverse of the Fisher information matrix  $\mathbf{J}(\mathbf{u})$ . The  $(i,j)$ -th element of  $\mathbf{J}(\mathbf{u})$  is given by

$$\mathbf{J}_{ij}(\mathbf{u}) = E_{\mathbf{r}} \left[ \frac{\partial}{\partial u_i} \ln(p(\mathbf{r}; \mathbf{u})) \frac{\partial}{\partial u_j} \ln(p(\mathbf{r}; \mathbf{u})) \right] \quad (2)$$

Note that  $\mathbf{J}(\mathbf{u})$  is a symmetrical matrix. The probability density  $p(\mathbf{r}; \mathbf{u})$  of  $\mathbf{r}$ , corresponding to a given value of  $\mathbf{u}$ , is called the likelihood function of  $\mathbf{u}$ , while  $\ln(p(\mathbf{r}; \mathbf{u}))$  is the log-likelihood function of  $\mathbf{u}$ . The expectation  $E_{\mathbf{r}}[\cdot]$  in (3) is with respect to  $p(\mathbf{r}; \mathbf{u})$ .

When the observation  $\mathbf{r}$  depends not only on the parameter  $\mathbf{u}$  to be estimated but also on a nuisance vector parameter  $\mathbf{v}$ , the likelihood function of  $\mathbf{u}$  is obtained by averaging the joint likelihood function  $p(\mathbf{r}; \mathbf{v}; \mathbf{u})$  of the vector  $(\mathbf{u}, \mathbf{v})$  over the a priori distribution of the nuisance parameter :  $p(\mathbf{r}; \mathbf{u}) = E_{\mathbf{v}} [p(\mathbf{r}; \mathbf{v}; \mathbf{u})]$ .

For all scenarios considered, the joint likelihood function  $p(\mathbf{r}; \mathbf{v}; \mathbf{u})$  is, within a factor not depending on  $(\mathbf{u}, \mathbf{v})$ , given by

$$p(\mathbf{r} | \mathbf{v}; \mathbf{u}) = \prod_{k=-K}^K \exp\left(\frac{E_s}{N_0} \left(2 \operatorname{Re}(a_k^* x_k e^{-j\theta}) - |a_k|^2\right)\right) \quad (3)$$

with  $x_k = \int r(t) \exp(-j2\mathbf{p}Ft) h(t - kT - \mathbf{t}) dt$ . The log-likelihood function  $\ln(p(\mathbf{r}; \mathbf{u}))$  resulting from (3) is given by

$$\ln p(\mathbf{r}; \mathbf{u}) = \ln \left[ E_{\mathbf{v}} \left[ \exp\left(\frac{E_s}{N_0} \left(2 \operatorname{Re}(a_k^* x_k e^{-j\theta}) - |a_k|^2\right)\right) \right] \right] \quad (4)$$

Computation of the CRB requires the substitution of (4) into (2), and the evaluation of the various expectations included in (4) and (2).

As the evaluation of the expectations involved in  $\mathbf{J}(\mathbf{u})$  and  $p(\mathbf{r}; \mathbf{u})$  is quite tedious, a simpler lower bound, called the modified CRB (MCRB), has been derived in [2,3], i.e.,  $E_{\mathbf{r}}[(\hat{u}_i - u_i)^2] \geq \operatorname{CRB}_i(\mathbf{u}) \geq \operatorname{MCRB}_i(\mathbf{u})$ . The MCRB for phase, frequency and timing estimation, is given by [2,3]

$$\operatorname{MCRB}_{\mathbf{q}} = \frac{N_0}{2LE_s} \quad \operatorname{MCRB}_{\mathbf{t}} = \frac{N_0}{2L(-\ddot{g}(0))E_s}$$

$$\operatorname{MCRB}_F = \frac{3N_0}{2\mathbf{p}^2 E_s L(L^2 - 1)} \cdot \frac{1}{T^2} \quad (5)$$

where  $\ddot{g}(t)$  represents twice the differentiation of the transmit pulse  $g(t)$  with  $g(t) = \int_{-\infty}^{+\infty} h(w) h(t+w) dw$ . The MCRBs (5) are valid for the joint estimation of an arbitrary subset of the synchronization parameters (ranging from only one parameter to all three parameters), with the remaining unknown synchronization parameters considered as nuisance parameters (assuming large  $L$ ); this holds for uncoded as well as for coded transmission. For large  $L$ , it can also be shown that for high SNR ( $E_s/N_0 \rightarrow \infty$ ) the CRBs related to all scenarios considered converge to the MCRBs (5) [4].

Also, a closed-form expression can be derived for the low-SNR limit (i.e.  $E_s/N_0 \rightarrow 0$ ) of the CRB, which we call the asymptotic CRB (ACRB) [6,7,10].

### 3. UNCODED TRANSMISSION

#### 3.1 CRB related to estimation of $\mathbf{F}$ or $\mathbf{t}$

Let us define  $u_1 = w_2 = F$  and  $u_2 = w_1 = \tau$ . Taking in (4)  $\mathbf{u}=(u_n, \theta)$  and  $\mathbf{v} = \mathbf{a}$ , the log-likelihood function related to the joint estimation of  $u_n$  and  $\theta$  (i.e., scenario (i)), is given by

$$\ln(p(\mathbf{r} | u_n, \theta)) = \ln \left( \prod_{k=-K}^K \sum_{m=0}^{M-1} \exp\left(\frac{E_s}{N_0} \left(2 \operatorname{Re}(\alpha_k^* x_k e^{-j\theta}) - |\alpha_k|^2\right)\right) \right) \quad (6)$$

where  $\{\alpha_m\}$  is the constellation alphabet and  $x_k$  is computed with  $w_n = 0$ . Taking in (4)  $\mathbf{u}=u_n$  and  $\mathbf{v} = (\mathbf{a}, \theta)$ , the log-likelihood function  $\ln(p(\mathbf{r}|u_n))$  related to the estimation of  $u_n$  irrespective of  $\theta$  (i.e., scenario (ii)) is given by

$$\ln(p(\mathbf{r} | u_n)) = \ln \left( \int_{-P}^P p(\mathbf{r} | u_n, \mathbf{q}) d\mathbf{q} \right) \quad (7)$$

Substituting (6) or (7) into (2) finally yields  $\operatorname{CRB}_F^{(i)}$  and  $\operatorname{CRB}_{\tau}^{(i)}$ , or  $\operatorname{CRB}_F^{(ii)}$  and  $\operatorname{CRB}_{\tau}^{(ii)}$ , where the superscript and subscript refer to the scenario and the parameter to be estimated, respectively.

For scenario (i) we obtain  $\mathbf{J}_{u_n, \theta} = 0$ , which indicates that there is no coupling between  $\theta$  and the parameter to be estimated. Consequently, the corresponding CRBs are the same as if  $\theta$  were known.

#### 3.2 Numerical results and discussion

We have obtained numerical results by a combination of an analytical approach and computer simulation.

In Figs. 1-2 and Figs. 3-4 the ratios  $\operatorname{CRB}/\operatorname{MCRB}$  are plotted for both the estimation of  $F$  or  $\tau$  jointly with and irrespective of  $\theta$  for different lengths  $L$  of the observation interval. Figs. 1 and 3 assume BPSK modulation, but are also representative for other real-valued symbols (PAM); Figs. 2 and 4 assume QPSK modulation, but are also representative for other complex-valued symbols (PSK, QAM). The transmit pulse is a square-root cosine rolloff pulse. The behavior of the various curves is as follows.

- For small (large) SNR, the CRBs converge to the corresponding ACRBs (to the MCRBs).
- For complex-valued symbols, the ACRBs are essentially the same irrespective of whether  $F$  or  $\tau$  is estimated jointly with or independently of  $\theta$ . However, for real-valued symbols, the ACRBs corresponding to scenarios (i) and (ii) are much different from each other.
- It follows from the numerical results that  $\operatorname{CRB}_u^{(ii)} > \operatorname{CRB}_u^{(i)}$ , for  $u=F$  as well as for  $u=\tau$ . This indicates that estimating  $u$  jointly with  $\theta$  (scenario (i)) is potentially more accurate than estimating  $u$  irrespective of  $\theta$  (scenario (ii)). Indeed, as  $u$  and  $\theta$  are uncoupled, the joint estimation of  $u$  and  $\theta$  yields the same CRB as estimating  $u$  when  $\theta$  is a priori known, and obviously this CRB is smaller than the one resulting from scenario (ii).
- For  $u = F$ , the shape of the transmit pulse has no effect on  $\operatorname{CRB}_F^{(i)}$  and  $\operatorname{CRB}_F^{(ii)}$  at moderate and high SNR. In Fig. 1 this is illustrated for BPSK. We have verified that this observation holds for other constellations as well. This is in contrast with  $\operatorname{CRB}_{\tau}^{(i)}$  and  $\operatorname{CRB}_{\tau}^{(ii)}$  which strongly depend on the shape of the transmit pulse (Figs. 3-4).
- The ratio  $\operatorname{CRB}_u^{(ii)}/\operatorname{MCRB}$  at SNR values of practical interest decreases with  $L$ , whereas the ratio  $\operatorname{CRB}_u^{(i)}/\operatorname{MCRB}$  does not depend on  $L$  (for  $\operatorname{CRB}_{\tau}$ : this holds only for  $LT$  much longer than the duration of  $\dot{g}(t)$ ). For given  $E_s/N_0$ , increasing  $L$  makes  $\operatorname{CRB}_u^{(ii)}$  approach  $\operatorname{CRB}_u^{(i)}$ , and hence reduces the penalty, caused by treating  $\theta$  as a nuisance parameter. The ratio  $\operatorname{CRB}^{(i)}/\operatorname{MCRB}$  being independent of  $L$  indicates that the penalty, caused by treating the data symbols as nuisance parameters, cannot be reduced by increasing the observation interval.
- The difference between  $\operatorname{CRB}_{\tau}^{(i)}$  and  $\operatorname{CRB}_{\tau}^{(ii)}$  is much smaller than the difference between  $\operatorname{CRB}_F^{(i)}$  and  $\operatorname{CRB}_F^{(ii)}$ .
- In Figs. 2 and 4 the dashed lines correspond to the CRB related to 8PSK, for  $L = 101$ . They illustrate that the CRBs at moderate and high  $E_s/N_0$  increase with increasing constellation size.

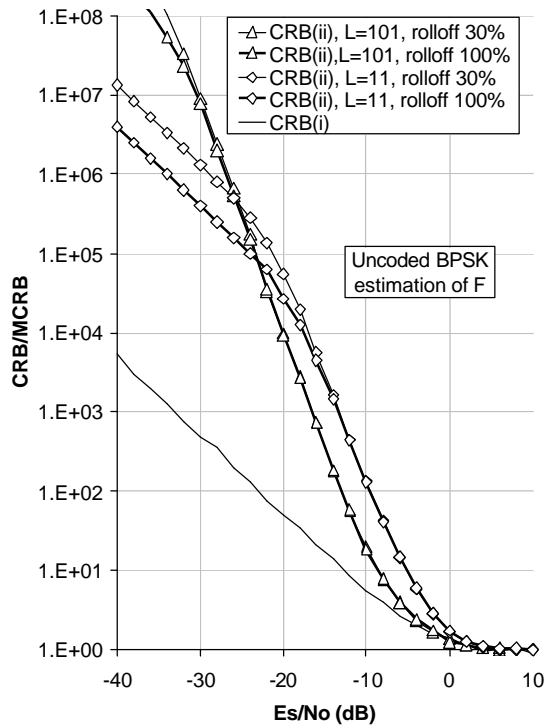


Fig. 1: CRB for frequency estimation from uncoded BPSK symbols

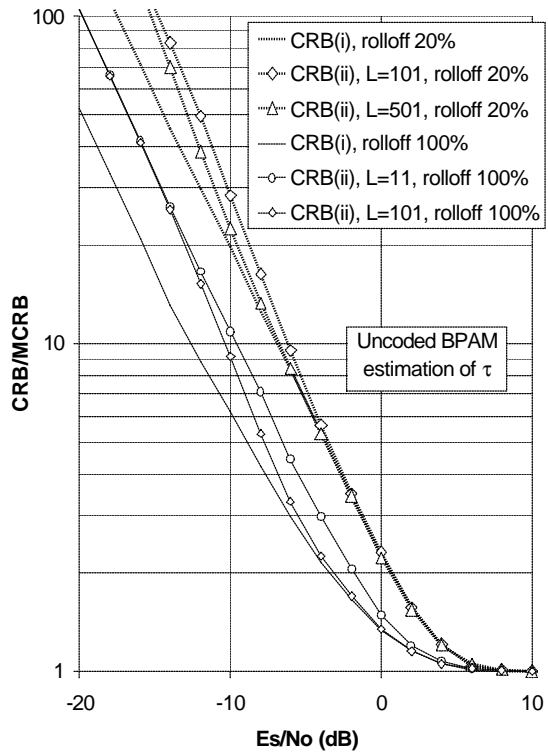


Fig. 3: CRB for timing estimation from uncoded BPSK symbols

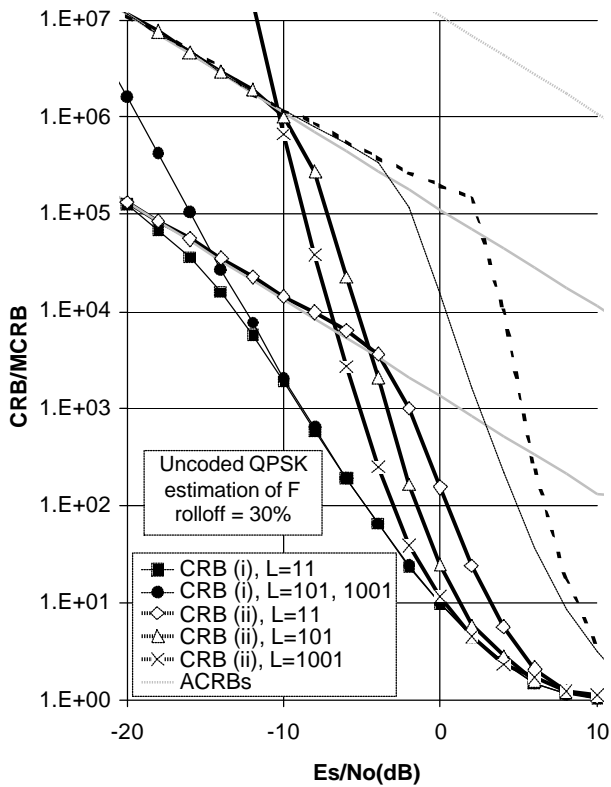


Fig. 2: CRB for frequency estimation from uncoded QPSK symbols

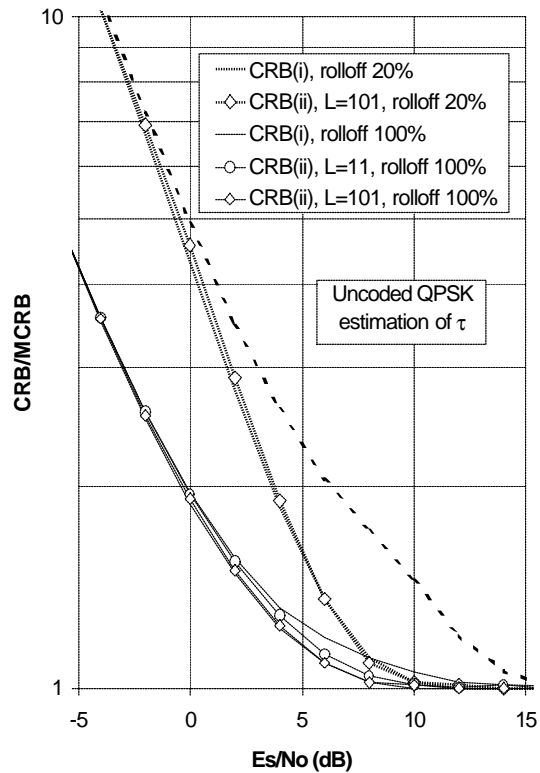


Fig. 4: CRB for timing estimation from uncoded QPSK symbols

## 4. CODED TRANSMISSION

### 4.1 CRB related to estimation of $\mathbf{q}$

Taking  $\mathbf{u} = \theta$ ,  $\mathbf{v} = \mathbf{a}$  and  $\mathbf{F} = \tau = 0$ , we have evaluated (4), assuming that  $\mathbf{a}$  is a sequence of convolutionally encoded symbols. Denoting by  $\xi$  the set of legitimate coded sequences of length  $L$ , we have used  $\Pr[\mathbf{a} = \mathbf{c}] = M^{-L}$  for  $\mathbf{c} \in \xi$  and  $\Pr[\mathbf{a} = \mathbf{c}] = 0$  otherwise, with  $r$  and  $M$  denoting the rate of the code and the constellation size, respectively.

### 4.2 Numerical results and discussion

In Fig. 5 we consider rate  $r = 1/2$  and  $r = 1/4$  maximum free distance convolutional codes with  $n = 2, 4, 16$  and  $64$  states [11]. The 2- or 4-bit output of the encoder is mapped on to one or two QPSK symbols. The transmit pulse is a square-root Nyquist pulse. To make abstraction of any edge effect we assume that transmitted sequences are long and can start or end in any state with the same probability  $1/n$ . The result for uncoded transmission is also displayed.

Our results show that the ratio  $\text{CRB}/\text{MCRB}$  for SNR below/above a certain cross-over value  $\text{SNR}_x(r)$  increases/decreases when the number of states increases. Decreasing the code rate reduces  $\text{SNR}_x$ , and hence enlarges the SNR region in which the ratio  $\text{CRB}/\text{MCRB}$  decreases with  $n$ . For small SNR, the CRB is considerably smaller for coded transmission than for uncoded transmission. This indicates that estimating  $\theta$  from coded data is potentially more accurate than estimating  $\theta$  from non-coded data; the accuracy increases as the coding gain becomes larger.

For very large  $E_s/N_0$  the CRB converges to the MCRB. For very small  $E_s/N_0$  the CRB converges to the corresponding ACRB which can be computed in a similar way as for uncoded transmission.

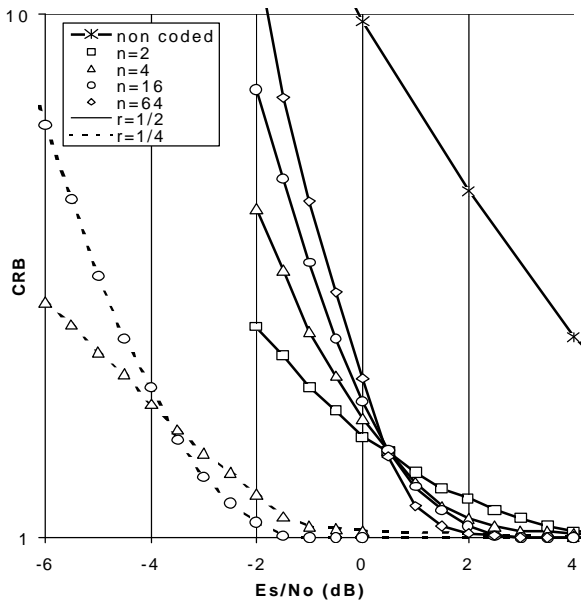


Fig. 5: CRB for phase estimation from symbols taken from a QPSK constellation according to a convolutional encoding rule

## 5. CONCLUSIONS AND REMARKS

In this contribution we have considered the CRB related to the estimation of the carrier phase, the carrier frequency and the time delay of a noisy linearly modulated waveform. Let us summarize the results for uncoded transmission. In contrast to  $\text{CRB}_\tau$ ,  $\text{CRB}_F$  is essentially unaffected by the pulse shape at practical values of SNR. For moderate SNR, the CRB increases with increasing constellation size. Frequency/timing estimation irrespective of the carrier phase yields a larger CRB than does joint frequency/timing and phase estimation. For given SNR, the penalty of the former strategy with respect to the latter decreases with increasing observation interval. For  $\text{CRB}_F$ , compared to  $\text{CRB}_\tau$ , considerably longer observation intervals are required to make the penalty very small. We also investigated the behavior of the true CRB for phase estimation in the presence of coding. Our results indicate that, for practical values of SNR, estimating the phase from coded data is potentially more accurate than estimating it from non-coded data. This effect is more pronounced as the coding gain gets larger.

## 6. ACKNOWLEDGEMENTS

The second author gratefully acknowledges the financial support from the Belgian National Fund for Scientific Research (FWO Flanders).

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