Analysis of an ML data-aided phase ambiguity resolution algorithm for M-PSK

Henk Wymeersch, Heidi Steendam and Marc Moeneclaey
DIGCOM research group, TELIN Dept., Ghent University
Sint-Pietersnieuwstraat 41, 9000 GENT, BELGIUM
E-mail: {hwymeers,hs,mm}@telin.ugent.be
Tel.: +32-9-264.89.00
Fax.: +32-9-264.42.95

***Abstract*— This contribution considers the data-aided (DA) maximum-likelihood carrier phase ambiguity resolution algorithm for coded M-PSK signaling. We derive analytical performance results of the algorithm when operating in the presence of phase estimation errors. As the use of a pilot sequence reduces the spectral efficiency of the system, we determine analytically the minimal pilot sequence length for coded and uncoded systems such that the overall BER degradation due to phase ambiguity resolution errors is limited to an acceptable value. We show that powerful error-correcting codes require pilot sequences that are substantially longer than for uncoded transmission; however, these longer pilot sequences usually account for only a small fraction of the overall frame length.**

I. Introduction

In packet transmission, frames arrive at the receiver with an unknown carrier phase. This phase can be estimated with a non data-aided algorithm (such as [1]). However, due to the rotational symmetries in the signaling constellation, the resulting estimate exhibits a phase ambiguity that needs to be resolved in order to perform coherent detection. The need for phase ambiguity resolution (PAR) can be removed by using differential encoding [2], which unfortunately results in a BER degradation, and requires significant changes to the decoder in case of iterative demodulation/decoding [3]. Another possibility is the use of rotationally invariant codes [4], [5], whereby rotated versions of codewords are decoded to the same information sequence. However, this technique requires specially tailored codes. In the current paper we do not assume rotational invariance, nor differential encoding.

As a PAR error results in the loss of an entire frame, its probability of occurrence should be sufficiently small. A well-known method to perform PAR is the insertion of a known word (pilot sequence) at some point in the frame, resulting in a purely data-aided (DA) PAR algorithm [6]. However, the presence of this word results in a reduction of spectral efficiency. It is therefore important to carefully select the length of the pilot sequence for a given coded or uncoded system.

In this contribution we consider the maximum likelihood (ML) DA PAR algorithm for M-PSK signaling. We analytically determine both its performance and its impact on the overall BER. From these results, we derive the minimum length of the pilot sequence such that the resulting BER degradation is limited to an acceptable amount. This paper is organized as follows: the system set-up is described in section II. In section III, we provide analytical performance results of the DA PAR algorithm. Numerical performance results are presented in section IV. Finally, conclusions are drawn in section V. Our main conclusion is that considerably longer pilot sequences are required for state of the art error-correcting codes than for uncoded transmission, but these longer sequences usually represent only a small fraction of the total frame length.

II. System description

We consider a transmitted (coded or uncoded) symbol sequence $\mathbf{s}$ consisting of a pilot sequence ($\mathbf{p}$) of L known symbols and an unknown data sequence ($\mathbf{a}$) of length N, so that $\mathbf{s} = [\mathbf{p}\,\mathbf{a}]$. The received vector is given by:

$$\mathbf{y} = \mathbf{s}e^{j\theta} + \mathbf{n} \tag{1}$$

where $\mathbf{n}$ is a row vector consisting of $L+N$ independent complex AWGN samples with $n_k \sim \mathcal{N}\left(0, 2\sigma^2\right)$. Here, $\sigma^2 = N_0/\left(2E_s\right)$, with E_s denoting the energy per transmitted symbol. The pilot and data symbols are taken from an M-PSK constellation with $|p_m| = |a_n| = 1$, for $m = 0, 1, \ldots, L-1$ and $n = 0, 1, \ldots, N-1$. The unknown carrier phase θ is in the interval $(-\pi, \pi)$. Detection of the data symbols $\mathbf{a}$ is based upon the rotated vector $\mathbf{y}e^{-j\hat{\theta}}$, with $\hat{\theta}$ denoting an estimate of the carrier phase θ. We introduce the integer part, $2\pi k/M$, and the fractional part, e_θ, of the phase θ, defined by

$$\theta = \frac{2\pi}{M}k + e_\theta \tag{2}$$

where $|e_\theta| < \pi/M$ and $k \in \{0, 1, \ldots, M-1\}$ for M-PSK signalling. A Non Data-Aided (NDA[1]) phase estimation algorithm (such as the Viterbi&Viterbi algorithm [1]) involves the estimation of the continuous parameter e_θ, whereas PAR refers to the estimation of the discrete parameter k. As the phase estimate $\hat{e}_\theta$ is not perfect, we

[1]To avoid confusion, we remind that a DA algorithm exploits the *exact* knowledge of the pilot symbols ($\mathbf{p}$), while a NDA algorithm exploits *statistical* properties of the entire sequence $\mathbf{s}$.

introduce the phase error, $\phi = e_\theta - \hat{e}_\theta$. This phase error is generally modeled as having a Gaussian distribution with zero-mean and variance σ_ϕ^2 [7]. The received vector $\mathbf{y}$ is rotated over an angle $-\hat{e}_\theta$ to compensate for the fractional part of the carrier phase, yielding

$$\mathbf{r} = \mathbf{y}\exp\left(-j\hat{e}_\theta\right) = \mathbf{s}\exp\left(j\frac{2\pi}{M}k\right)\exp\left(j\phi\right) + \tilde{\mathbf{n}} \quad (3)$$

where $\tilde{\mathbf{n}}$ is a noise vector with the same statistics as $\mathbf{n}$ in (1). The first L components of $\mathbf{r}$ (which we denote as $\mathbf{r}_p = [r_0, \ldots, r_{L-1}]$) correspond to the pilot sequence $\mathbf{p}$, and are used for PAR, i.e., for estimating the integer part $2\pi k/M$ of the carrier phase. After obtaining the estimate $\hat{k}$ of k, the last N components of $\mathbf{r}$ are rotated by an angle $-2\pi\hat{k}/M$ to compensate for the integer part of the carrier phase. Finally, these rotated components are fed to the decision device that detects the data symbol sequence $\mathbf{a}$.

III. ML DATA-AIDED PAR

The goal of the data-aided PAR algorithm is to determine from $\mathbf{r}_p$ the carrier phase shift $2\pi k/M$ in (3), based on the knowledge of the pilot sequence. The PAR algorithm assumes that the fractional part of θ has been perfectly compensated for, i.e., $\phi = 0$ in (3). Using the ML criterion, it can easily be shown that

$$\hat{k}_{ML} = \text{round}\left(\frac{M}{2\pi}\arg\left(C_p\right)\right) \quad (4)$$

where C_p denotes the time-correlation between the sequences $\mathbf{r}_p$ and $\mathbf{p}$, i.e., $C_p = \mathbf{r}_p\mathbf{p}^H$.

A. PAR Error probability

First, we consider the *conditional* error probability, $P_{par}(\phi) = P\left[\hat{k} \neq k \middle| \phi\right]$, which is the probability of a PAR when the fractional phase error equals ϕ. A PAR error occurs when the argument of $Z = C_p\exp\left(-j2\pi k/M\right)$ falls outside the range $(-\pi/M, \pi/M)$. For given k and ϕ, Z is a complex-valued Gaussian random variable with mean $L\exp\left(j\phi\right)$; the real and imaginary parts of Z are statistically independent, and both have a variance equal to $L\sigma^2$. Taking into account the statistics of $\tilde{\mathbf{n}}$, we obtain, for $M > 2$

$$P_{par}(\phi) \approx Q\left(\sqrt{2\gamma\sin^2\left(\frac{\pi}{M}+\phi\right)}\right) + Q\left(\sqrt{2\gamma\sin^2\left(\frac{\pi}{M}-\phi\right)}\right) \quad (5)$$

where $\gamma = L/\left(2\sigma^2\right)$ and $Q(x) = 1/\sqrt{2\pi}\int_x^{+\infty}\exp\left(-t^2/2\right)dt$. For $M = 2$, the conditional PAR error probability is given by $P_{par}(\phi) = Q\left(\sqrt{2\gamma\cos^2(\phi)}\right)$. Note that $P_{par}(\phi)$ depends on the pilot sequence only through its length, L, not through its constituent symbols, nor its location within the frame.

Secondly, the *unconditional* PAR error probability is then obtained as $P_{par} = E_\phi\left[P_{par}(\phi)\right]$, where $E_\phi[.]$ denotes the averaging with respect to the distribution of the phase error ϕ. Since ϕ is assumed to have a Gaussian distribution, P_{par} can be obtained through low-complexity numerical integration techniques.

B. BER performance

Of more practical interest than the PAR error rate itself is the BER degradation due to PAR errors. Let us denote by $BER_0(\phi)$ the bit error rate conditioned on the phase error ϕ, for frames that are not affected by a PAR error; similarly, we define $BER_1(\phi)$ as the conditional bit error rate for the frames that are affected by a PAR error. Then the overall conditional bit error rate $BER(\phi)$ is given by

$$\begin{aligned} &BER(\phi) \\ &= BER_0(\phi)\left(1 - P_{par}(\phi)\right) + BER_1(\phi)P_{par}(\phi) \\ &< BER_0(\phi) + P_{par}(\phi). \end{aligned} \quad (6)$$

From (6), we obtain the following bound:

$$BER < BER_0 + P_{par} = BER_0\left(1 + \frac{P_{par}}{BER_0}\right)$$

where BER and BER_0 are the unconditional bit error rates, obtained by averaging $BER(\phi)$ and $BER_0(\phi)$ over ϕ. Our goal is, for a given scenario (i.e., for given BER_0), to determine the length L of the pilot sequence such that the resulting BER degradation is negligible.

In order that occasional PAR errors cause a low BER degradation, the ratio P_{par}/BER_0 must be sufficiently small, say less than ε, so that BER/BER_0 is limited to $1+\varepsilon$. From curves of BER_0 as a function of E_b/N_0 $\left(= 1/\left(2r\sigma^2\log_2(M)\right)\right.$, with r denoting the rate of the coder[2]) it can be verified that taking $\varepsilon = 1/8$ for uncoded systems and $\varepsilon = 1/2$ for coded systems corresponds to a degradation in E_b/N_0 not exceeding about 0.1 dB. The condition $P_{par}/BER_0 < \varepsilon$ imposes a minimal value L_{min} on the length of the pilot sequence, so that phase ambiguity resolution errors cause no noticeable BER degradation. Note that L_{min} increases with the coding gain: a 3 dB increase in coding gain requires L_{min} to be doubled.

Finally, note that a similar argument can be applied to the frame error rate (FER). Since the FER is always greater than the BER, the minimal pilot sequence length that results in a negligible FER degradation will be smaller that the one that results in a negligible BER degradation.

IV. NUMERICAL PERFORMANCE RESULTS

We first illustrate the accuracy of the approximation (5) for $P_{par}(\phi)$. Assuming a pilot sequence of 5 symbols, we show in Fig. 1 (left part) $P_{par}(0)$ as a function of E_b/N_0 for uncoded M-PSK. We observe that $P_{par}(0)$ according to the approximation (5) and the simulated $P_{par}(0)$ are nearly

[2]This E_b/N_0 does not include the rate loss due to the pilot insertion. To include this loss, one should replace r with $r \times \left(N/\left(N+L\right)\right)$.

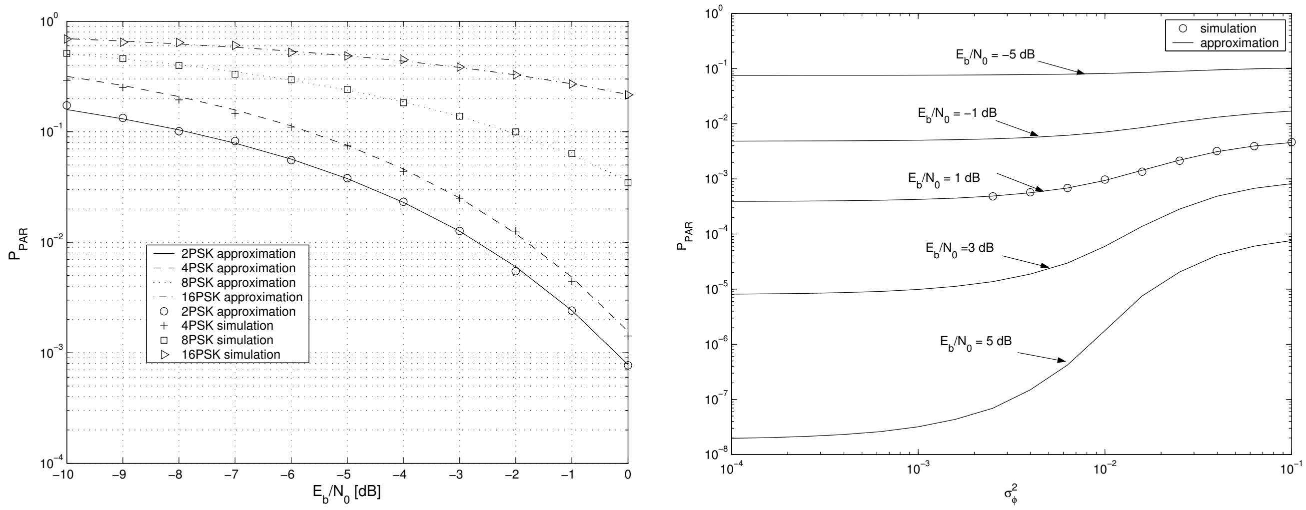

Fig. 1. P_{par} as a function of the SNR for a pilot sequence of 5 symbols for perfect phase estimation (left) and for Gaussian distributed phase errors (for 4PSK; right)

identical. The right part of Fig. 1 shows for 4-PSK the effect of a zero-mean Gaussian phase error ϕ with variance σ_ϕ^2. As expected, an increase in σ_ϕ^2 results in a higher PAR error probability; the effect of σ_ϕ^2 on P_{par} is more pronounced at larger E_b/N_0. Again, simulations confirm the accuracy of the approximation (5).

To illustrate the effect of PAR errors on the bit error rate, we have performed computer simulations for uncoded 4-PSK and for a turbo code [8] with bit-interleaved [9] 4-PSK signaling. The constituent convolutional codes of the turbo code are systematic and recursive with rate $1/2$, generator polynomials $(21, 37)_8$ and constraint length 5. The turbo code consists of the parallel concatenation of two unpunctured constituent encoders, which yields an overall code rate of 1/3. Codewords consist of 1002 bits (not including pilot bits), yielding 501 coded 4-PSK symbols.

Fig. 2 shows the value of L_{min} resulting from the condition $P_{par}/BER_0 < \varepsilon$. Results pertain to

- uncoded 4-PSK with perfect estimation of the fractional part of the carrier phase (denoted by 'perfect PE') for various values of ε
- turbo coded 4-PSK with $\varepsilon = 1/2$ taking two different assumptions regarding phase estimation, i.e., perfect estimation and Viterbi&Viterbi (V&V) estimation [1] using a fourth-power nonlinearity.

Obviously, reducing ε gives rise to a larger value of L_{min}. For uncoded 4-PSK transmission, L_{min} decreases with increasing E_b/N_0: taking into account that $P_{par}(0) \approx 2Q\left(\sqrt{2LE_b/N_0}\right)$ and $BER_0(0) \approx Q\left(\sqrt{2E_b/N_0}\right)$ it can be verified that L_{min} converges to 1 (irrespective of ε) when E_b/N_0 gets very large. Hence, for uncoded transmission a short pilot sequence of only 2 symbols should be sufficient at normal operating values of E_b/N_0. For the turbo-coded 4-PSK system a substantially larger pilot sequence is required, because of the large coding gain with respect to uncoded 4-PSK. In the case of perfect phase estimation, L_{min} steeply increases with E_b/N_0 when operating in the waterfall region (0 dB < E_b/N_0 < 2 dB) of the turbo code; in the error floor region, L_{min} decreases with E_b/N_0 because of the decreasing coding gain, and converges for a very large E_b/N_0 to a value that is determined by the asymptotic coding gain. In the case of V&V phase estimation, both the PAR performance and the BER performance degrade as compared to the case of perfect phase estimation. When the performance of the PAR is less (more) sensitive to phase estimation errors than the performance of the decoder, then L_{min} in the presence of phase estimation errors is smaller (larger) than in the case of perfect phase estimation; Fig. 2 indicates that the turbo decoding algorithm is more sensitive to phase errors than is the PAR algorithm, when operating in the waterfall region. Fig. 3 shows straightforward BER simulation results for turbo-coded 4-PSK with PAR, for various pilot sequence lengths L, assuming both perfect phase estimation and V&V phase estimation. For a given E_b/N_0, the value of L that yields a negligible bit error rate degradation (see Fig. 3) agrees well with the value of L that results from the condition $P_{par}/BER_0 < \varepsilon$ (see Fig. 2). Assuming V&V (perfect) phase estimation, a pilot sequence of 15 symbols (19 symbols) is sufficient to obtain a very small BER degradation for the considered turbo code; although this pilot sequence is much longer than what is needed for uncoded transmission, the resulting frame overhead is only about 3% (4%). This overhead causes an equivalent rate loss of less than 0.2 dB (not shown in Fig. 3). It should be noted that obtaining a suitable value of L by means of straightforward simulations of the PAR algorithm and the turbo decoding algorithm requires a new simulation of the PAR algorithm for each value of L considered. This is in contrast with evaluating P_{par} analytically and then determining L from the condition $P_{par}/BER_0 < \varepsilon$. The latter approach only requires a single simulation (viz., of the turbo decoder, to determine BER_0) .

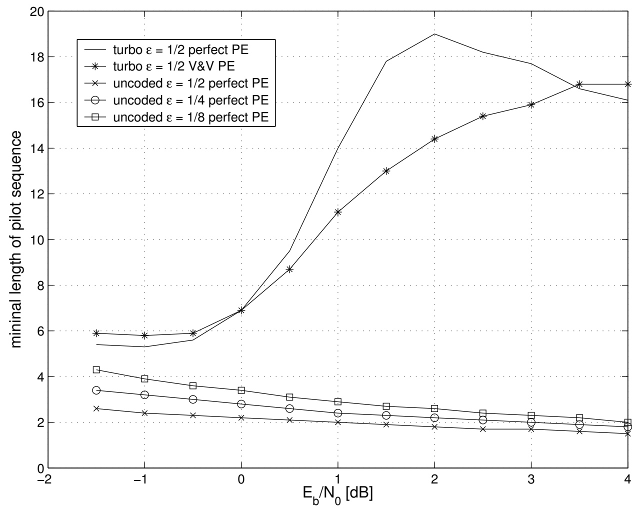

Fig. 2. Minimal length (L) of pilot sequence for uncoded and turbo coded QPSK (for perfect phase estimation (PE) and a NDA PE algorithm from [1]).

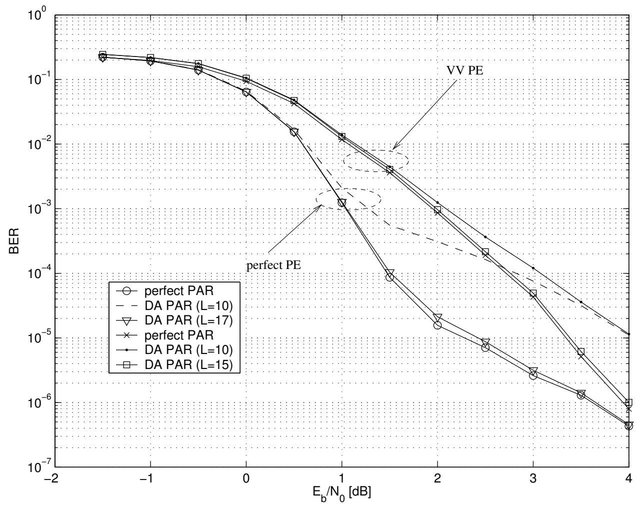

Fig. 3. BER performance for turbo coded QPSK for different values of L (for perfect phase estimation (PE) and a NDA PE algorithm from [1]).

V. Conclusions and Remarks

We have investigated the performance of the ML data-aided PAR algorithm for M-PSK signaling in the presence of residual carrier phase estimation errors. We have presented an accurate approximation for the PAR error probability, and determined a simple bound on the overall BER. From these results, we have computed the minimal length of the pilot sequence, such that the BER degradation caused by occasional PAR errors is limited to a very small value (say, about 0.1 dB in E_b/N_0). For uncoded transmission, a pilot sequence of only 1 or 2 pilot symbols is sufficient, but for coded transmission the required pilot sequence length is considerably larger, and increases with the coding gain. Our analytical results have been validated by means of computer simulations.

The ML PAR algorithm (4) makes use of the pilot symbols, but does not exploit the remaining data symbols in the frame. In the case of uncoded transmission, it can be verified that the remaining data symbols do not provide any information regarding the phase ambiguity, in which case the PAR algorithm (4) is optimum. In the case of coded transmission, the remaining symbols give an indication about the ambiguity, provided that rotating the symbols of a codeword by a multiple of $2\pi/M$ does not yield another codeword. Hence, for some codes the PAR performance can be improved by taking into account the coded symbols: examples can be found in [10], [11]. Although such *code-aided* PAR algorithms yield a gain in spectral efficiency (as compared to (4), they need less pilot symbols), they come with a significant cost: as they generally require many decoding operations, their computational complexity can be very high.

Finally, we should note that the pilot sequence is not only used for PAR: for instance, it is also commonly exploited during frame synchronization (FS) [12] and channel gain estimation. Especially frame synchronization puts additional requirements on the properties of the pilot sequence. Performance analysis of DA joint FS and PAR remains a topic of future research.

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