Low-SNR Limit of the Cramer–Rao Bound for Estimating the Time Delay of a PSK, QAM, or PAM Waveform

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***Abstract*—In this letter we consider the Cramer–Rao bound (CRB) for the estimation of the time delay of a noisy linearly modulated signal with random data symbols and random carrier phase. Because of the presence of the nuisance parameters (i.e., data symbols and carrier phase), a closed-form expression of this CRB is very hard to obtain for arbitrary PSK, QAM or PAM constellations and a band-limited transmit pulse. Instead, here we derive a simple expression for the limit of the CRB at low signal-to-noise ratio (SNR), which is a relevant benchmark for timing recovery algorithms operating at small E_s/N_0.**

I. INTRODUCTION

THE CRAMER–RAO bound (CRB) is a lower bound on the error variance of any unbiased estimate, and as such serves as a useful benchmark for practical estimators [1]. The CRB is formulated in terms of the likelihood function of the scalar parameter to be estimated. In many cases, the statistics of the observed vector depend not only on the parameter to be estimated, but also on a number of nuisance parameters we do not want to estimate. The presence of the nuisance parameters makes the computation of the likelihood function and the corresponding CRB very hard.

A typical example where nuisance parameters occur is the observation of a noisy linearly modulated waveform, that is a function of a time delay, a carrier frequency offset, a carrier phase and a data symbol sequence. In [2], the CRB's for estimating the frequency offset and the carrier phase have been computed for BPSK and QPSK, assuming the timing to be known; different constellations yield different expressions for the CRB's.

In order to avoid the computational complexity caused by the nuisance parameters, a modified CRB (MCRB) has been derived in [3]. The MCRB is much simpler to evaluate than the CRB, but is in general looser than the CRB. In [4] the high-SNR limit of the CRB has been evaluated analytically, and has been shown to coincide with the MCRB when estimating the delay, the frequency offset or the carrier phase of the linearly modulated waveform.

In the presence of coding, timing recovery algorithms must operate at low SNR, so that the high-SNR limit of the CRB is no longer a relevant benchmark. Therefore, in this contribution we derive a simple expression for the low-SNR limit of the CRB for timing estimation. The resulting expression is valid for arbitrary PSK, QAM and PAM constellations, and for an arbitrary square-root Nyquist transmit pulse.

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II. PROBLEM FORMULATION

Let us consider the complex baseband representation $r(t)$ of a noisy linearly modulated signal

$$r(t) = \sum_{k=0}^{K-1} a_k h(t - kT - \tau) e^{j\theta_k} + n(t) \tag{1}$$

where $\mathbf{a} = (a_0, \ldots, a_{K-1})$ is a vector of zero-mean *pairwise uncorrelated* data symbols with $E[a_k^* a_m] = \delta_{k-m}$; $h(t)$ is a real-valued unit-energy square-root Nyquist pulse; τ is a *deterministic* time delay; $\theta = (\theta_0, \ldots, \theta_{K-1})$ is a vector of carrier phases; T is the symbol interval; and $n(t)$ is complex-valued zero-mean Gaussian noise with independent real and imaginary parts, each having a power spectral density of $N_0/(2E_s)$. The vectors $\mathbf{a}$ and θ are statistically independent, and their probability density is not a function of τ. Note that pairwise uncorrelated data symbols occur not only for statistically independent $\{a_k\}$, but also for the large majority of practical codes [5]. The dependence of θ_k on the symbol index k allows modeling a carrier phase that is slowly varying with respect to the duration of $h(t)$.

Suppose that one is able to produce from $r(t)$ an *unbiased* estimate $\hat{\tau}$ of the delay τ. Then the estimation error variance is lower bounded by the CRB [1]: $E_{\mathbf{r}}[(\hat{\tau} - \tau)^2] \geq T^2 \mathrm{CRB}(\tau)$, where

$$\mathrm{CRB}(\tau) = \left(E_{\mathbf{r}} \left[-T^2 \frac{d^2}{d\tau^2} \ln(p(\mathbf{r}; \tau)) \right] \right)^{-1}. \tag{2}$$

In (2), $\mathbf{r}$ is a vector representation of the signal $r(t)$. The probability density $p(\mathbf{r}; \tau)$ of $\mathbf{r}$, corresponding to a given value of τ, is called the *likelihood function* of τ. The expectation $E_r[\cdot]$ is with respect to the probability density $p(\mathbf{r}; \tau)$.

As $r(t)$ from (1) depends not only on the delay τ to be estimated but also on the nuisance vector parameter $\mathbf{u} = (\mathbf{a}, \theta)$, the likelihood function of τ is obtained by averaging the *joint* likelihood function $p(\mathbf{r}|\mathbf{u}; \tau)$ of $(\mathbf{u}, \tau)$ over the *a priori* distribution of the nuisance parameter: $p(\mathbf{r}; \tau) = E_{\mathbf{u}}[p(\mathbf{r}|\mathbf{u}; \tau)]$. From (1) it follows that $p(\mathbf{r}|\mathbf{u}; \tau) = C \exp(-\varepsilon L(\mathbf{u}, \tau))$, where C is a factor not depending on $(\mathbf{u}, \tau)$, $\varepsilon = E_s/N_0$ and

$$L(\mathbf{u}, \tau) = \sum_{k=0}^{K-1} \left(\left| a_k^2 \right| - 2\mathrm{Re}\left[a_k^* z_k(\tau) \exp(-j\theta_k) \right] \right) \tag{3}$$

with

$$z_k(\tau) = \int_{-\infty}^{+\infty} r(t)h(t - kT - \tau)\,dt. \tag{4}$$

As the expectations involved in CRB(τ) and $p(\mathbf{r};\,\tau)$ are hard to evaluate for an arbitrary PSK, QAM or PAM symbol constellation and for band limited $h(t)$, no closed-form expression for CRB(τ) is available. Therefore, a simpler lower bound, called the modified CRB (MCRB), has been derived in [3]: $E_{\mathbf{r}}[(\hat{\tau} - \tau)^2] \geq T^2\mathrm{CRB}(\tau) \geq T^2\mathrm{MCRB}(\tau)$. Defining the Nyquist pulse $g(t)$ as

$$g(t) = \int_{-\infty}^{+\infty} h(v)h(t+v)\,dt \tag{5}$$

the MCRB for timing estimation, corresponding to $r(t)$ from (1), is given by [3]

$$\mathrm{MCRB}(\tau) = \frac{N_0}{2E_sK} \cdot \frac{1}{(-\ddot{g}(0)T^2)} \tag{6}$$

where $\ddot{g}(t)$ denotes twice derivation of $g(t)$ with respect to t. In [4] it has been shown that for *high* SNR (i.e., $E_s/N_0 \to \infty$) the CRB (2) resulting from (1) converges to the MCRB (6). In the following, we derive a closed form expression for the *low*-SNR limit (i.e., $E_s/N_0 \to 0$) of the CRB that corresponds to (1). This low-SNR asymptotic CRB will be denoted $\mathrm{ACRB}_0(\tau)$.

III. LOW-SNR LIMIT OF CRB

For small E_s/N_0 (or equivalently, small ε), we approximate the joint likelihood function $p(\mathbf{r}|\mathbf{u};\,\tau)$ by a truncated Taylor series expansion in ε, and average over $\mathbf{u}$ to obtain an approximation of the likelihood function $p(\mathbf{r};\,\tau)$. Neglecting third-order and higher order terms of ε, one obtains $p(\mathbf{r};\,\tau) \cong C(1 - \varepsilon F_1(\tau) + (1/2)\varepsilon^2 F_2(\tau))$, where $F_i(\tau) = E_{\mathbf{u}}[L^i(\mathbf{u},\,\tau)]$. Note from (3) and $E[a_k] = 0$ that $F_1(\tau)$ is *not* a function of τ, implying $\dot{F}_1(\tau) = \ddot{F}_1(\tau) = 0$. Now we use

$$\frac{d^2}{d\tau^2}\ln(p(\mathbf{r};\,\tau)) = \frac{\ddot{p}(\mathbf{r};\,\tau)p(\mathbf{r};\,\tau) - \dot{p}^2(\mathbf{r};\,\tau)}{p^2(\mathbf{r};\,\tau)} \tag{7}$$

where $\dot{p}(\mathbf{r};\,\tau)$ and $\ddot{p}(\mathbf{r};\,\tau)$ denote once and twice differentiation of $p(\mathbf{r};\,\tau)$ with respect to τ. Keeping in (7) up to quadratic terms in ε, and taking the average $E_r[\cdot]$ yields

$$E_{\mathbf{r}}\left[\frac{d^2}{d\tau^2}\ln(p(\mathbf{r};\,\tau))\right] \cong \frac{1}{2}\varepsilon^2 E_{\mathbf{r}}\left[\ddot{F}_2(\tau)\right]. \tag{8}$$

Now let us compute $F_2(\tau)$. Taking (3) into account, we obtain

$$F_2(\tau) = E_{\mathbf{a},\,\theta}\left[\sum_{k=0}^{K-1}\left(a_k^{*2}e^{-2j\theta_k}z_k^2(\tau) + a_k^2 e^{2j\theta_k}z_k^{*2}(\tau) + 2\left|a_k^2\right|\left|z_k^2(\tau)\right|\right)\right] \tag{9}$$

where terms not depending on τ have been dropped. Assuming that the symbol constellation is rotationally symmetric over $2\pi/N$ with $N > 2$ ($N = M$ for M-PSK, $N = 4$ for QAM), it follows that $E[a_k^2] = 0$. In this case, (9) reduces to

$$F_2(\tau) = 2\sum_{k=0}^{K-1}\left|z_k^2(\tau)\right| \tag{10}$$

which holds *irrespective* of the *a priori* distribution of θ. Hence,

$$E_{\mathbf{r}}\left[\ddot{F}_2(\tau)\right] = 2\sum_{k=0}^{K-1} E_{\mathbf{r}}\left[\ddot{z}_k(\tau)z_k^*(\tau) + 2\dot{z}_k(\tau)\dot{z}_k^*(\tau) + \ddot{z}_k^*(\tau)z_k(\tau)\right]. \tag{11}$$

The averaging in (11) is equivalent to averaging over the noise and the nuisance parameters. It can be verified that the noise term $n(t)$ from (1) does not contribute to (11). Straightforward computation of the signal contribution to (11) yields

$$E_r\left[\ddot{F}_2(\tau)\right] = 4K\ddot{g}(0) + 4\sum_{k,\,m=0}^{K-1}\dot{g}^2(kT - mT). \tag{12}$$

From (2), (8) and (12), the low-SNR asymptote of the CRB is obtained as

$$\begin{aligned}\mathrm{ACRB}_0(\tau) &= 2\left(\frac{N_0}{2E_s}\right)^2 \cdot \left(K(-\ddot{g}(0)) - \sum_{k,\,m=0}^{K-1}\dot{g}^2(kT - mT)\right)^{-1} \\ &\cong \frac{2}{K}\cdot\left(\frac{N_0}{2E_s}\right)^2 \cdot \left((-\ddot{g}(0)T^2) - \sum_{m=-\infty}^{+\infty}\dot{g}^2(mT)T^2\right)^{-1}.\end{aligned} \tag{13}$$

The above approximation is accurate when KT is much longer than the effective duration of the pulse $\dot{g}(t)$. Note that $\mathrm{ACRB}_0(\tau)$ is inversely proportional to the *square* of E_s/N_0. This is in contrast with the high-SNR limit of CRB(τ), which is inversely proportional to E_s/N_0 [see (6)].

IV. CONCLUSIONS AND REMARKS

In this letter we have derived a closed-form analytical expression for the low-SNR limit of the CRB pertaining to the estimation of the time delay of a linearly modulated waveform. This limit $\mathrm{ACRB}_0(\tau)$ turns out to be inversely proportional to the *square* of E_s/N_0.

For M-PSK with $M > 2$ or QAM, $\mathrm{ACRB}_0(\tau)$ is independent of the *a priori* distribution of the carrier phase vector θ, which indicates that knowing θ does not reduce $\mathrm{ACRB}_0(\tau)$ as compared to the case where θ is unknown. It is easily verified from (9) that, for zero-mean and *real-valued* data symbols (such as M-PAM), the resulting $\mathrm{ACRB}_0(\tau)$ is still given by (13) when the K marginal *a priori* distributions $p(\theta_k)$ are constant over $(-\pi,\,\pi)$. For M-PAM with a priori known θ, we obtain from (9)

$$\begin{aligned}\mathrm{ACRB}_0(\tau) &= \left(\frac{N_0}{2E_s}\right)^2 \cdot \left(K\left(-\ddot{g}(0)T^2\right) - \sum_{k,\,m=0}^{K-1}\dot{g}^2(kT - mT)T^2 \cdot \cos^2(\theta_k - \theta_m)\right)^{-1} \\ &\leq \left(\frac{N_0}{2E_s}\right)^2 \cdot \left(K(-\ddot{g}(0)T^2) - \sum_{k,\,m=0}^{K-1}\dot{g}^2(kT - mT)T^2\right)^{-1}.\end{aligned} \tag{14}$$

Note that (14) is half as large as (13) when the *a priori* known carrier phase is *constant* ($\theta_k = \theta$ for all k).

The tracking error variance at low SNR of the popular non-data-aided noncarrier-aided filter and square timing recovery algorithm [6, Secs. 5.4 and 6.3.6] equals $\mathrm{ACRB}_0(\tau)$ from (13), which indicates that this algorithm is optimum at small E_s/N_0.

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