Low-SNR Limit of the Cramer–Rao Bound for Estimating the Carrier Phase and Frequency of a PAM, PSK, or QAM Waveform

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***Abstract*—In this letter we consider the Cramer–Rao bound (CRB) for the estimation of the carrier phase and frequency of a noisy linearly modulated signal with random data symbols. The observation vector consists of the matched filter output samples taken at the symbol rate, assuming known symbol timing. Because of the presence of the random data, the evaluation of this CRB is quite tedious. Instead, here we derive a simple closed-form expression for the limit of the CRB at low signal-to-noise ratio (SNR), which holds for arbitrary PAM, PSK, and QAM constellations.**

***Index Terms*—Carrier synchronization, Cramer–Rao bound, frequency and phase estimation.**

I. Introduction

THE CRAMER–RAO bound (CRB) is a lower bound on the error variance of any unbiased estimate, and as such serves as a useful benchmark for practical estimators [1]. In many cases, the statistics of the observation depend not only on the vector parameter to be estimated, but also on a nuisance vector parameter we do not want to estimate. The presence of this nuisance parameter makes the computation of the CRB very hard.

A typical example where a nuisance vector parameter occurs is the observation of a noisy linearly modulated waveform, that is a function of a time delay, a carrier frequency offset, a carrier phase and a data symbol sequence. In [2], the CRB's for estimating the frequency offset and the carrier phase have been computed for BPSK and QPSK, assuming the timing to be known; different constellations yield different expressions for these CRB's.

In order to avoid the computational complexity caused by the nuisance parameters, a modified CRB (MCRB) has been derived in [3]. The MCRB is much simpler to evaluate than the CRB, but is in general looser than the CRB. In [4] the high-SNR limit of the CRB has been evaluated analytically, and has been shown to coincide with the MCRB when estimating the delay, the frequency offset or the carrier phase of the linearly modulated waveform.

In the presence of coding, synchronization algorithms must operate at low SNR, so that the high-SNR limit of the CRB might no longer be a relevant benchmark. In [5], we have presented the low-SNR limit of the CRB related to timing recovery. In this contribution we derive a simple expression for the low-SNR limit of the CRB for carrier phase and frequency estimation. As in [2], timing is assumed to be known. The resulting expression is valid for arbitrary PAM, PSK, and QAM constellations.

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II. Problem Formulation

Consider a linearly modulated signal, obtained by applying a data symbol sequence to a square-root Nyquist transmit filter, that is transmitted over an additive white Gaussian noise (AWGN) channel. The resulting noisy signal is applied to a receiver filter, matched to the transmit filter. The receiver filter output signal is sampled at the decision instants, which yields the observation vector $\mathbf{r} = (r_{-K}, \ldots, r_K)$, with

$$r_k = \varepsilon a_k \exp(j(2\pi k\nu + \theta)) + n_k \tag{1}$$

for $k = -K, \ldots, K$. In (1), $\{a_k\}$ is a sequence of independent identically distributed (i.i.d.) data symbols that are taken uniformly from a PAM, PSK or QAM constellation with $E[|a_k|^2] = 1$. The sequence $\{n_k\}$ consists of i.i.d. zero-mean complex Gaussian noise variables, with independent real and imaginary parts each having a variance of 1/2. The deterministic unknown parameters θ and ν represent the carrier phase corresponding to $k = 0$, and the carrier frequency offset normalized to the symbol rate. Finally, $\varepsilon = (E_s/N_0)^{1/2}$, with E_s and N_0 denoting the symbol energy and the noise power spectral density. It has been tacitly assumed that $|\nu| \ll 1$, so that the intersymbol interference in r_k is negligible.

Suppose that one is able to produce from an observation vector $\mathbf{r}$ an *unbiased* estimate $\hat{\mathbf{u}}$ of a deterministic vector parameter $\mathbf{u}$. Then the estimation error variance is lower bounded by the Cramer–Rao bound (CRB) [1]: $E_{\mathbf{r}}[(\hat{u}_i - u_i)^2] \geq \mathrm{CRB}_i(\mathbf{u})$, where $\mathrm{CRB}_i(\mathbf{u})$ is the ith diagonal element of the inverse of the *Fisher information matrix* $\mathbf{J}(\mathbf{u})$. The (i, j)th element of $\mathbf{J}(\mathbf{u})$ is given by

$$\mathbf{J}(\mathbf{u}) = E_{\mathbf{r}}\left[-\frac{\partial^2}{\partial u_i \partial u_j}\ln(p(\mathbf{r}|\mathbf{u}))\right]. \tag{2}$$

The probability density $p(\mathbf{r}|\mathbf{u})$ of $\mathbf{r}$, corresponding to a given value of $\mathbf{u}$, is called the *likelihood function* of $\mathbf{u}$. The expectation $E_r[.]$ is with respect to $p(\mathbf{r}|\mathbf{u})$.

When the observation $\mathbf{r}$ depends not only on the parameter $\mathbf{u}$ to be estimated but also on a nuisance vector parameter

$\mathbf{v}$, the likelihood function of $\mathbf{u}$ is obtained by averaging the *joint* likelihood function $p(\mathbf{r}|\mathbf{v};\,\mathbf{u})$ of the vector $(\mathbf{v},\,\mathbf{u})$ over the *a priori* distribution of the nuisance parameter: $p(\mathbf{r}|\mathbf{u}) = E_{\mathbf{v}}[p(\mathbf{r}|\mathbf{v};\,\mathbf{u})]$.

Now we apply (2) to the observation $\mathbf{r}$ consisting of the samples r_k from (1), with $\mathbf{u} = (\theta,\,\nu)$ and $\mathbf{v} = (a_{-K},\,\ldots,\,a_K)$. From (1) it follows that, within a constant term not depending on $(\mathbf{v},\,\mathbf{u})$,

$$\ln p(\mathbf{r}|\mathbf{u}) = \sum_{k=-K}^{K} \ln(E_{a_k}[\exp(2\varepsilon\,\mathrm{Re}[a_k^* \breve{r}_k] - \varepsilon^2|a_k|^2)]) \quad (3)$$

with $\breve{r}_k = r_k \exp(-j(2\pi k\nu + \theta))$. As the evaluation of the expectations involved in $\mathbf{J}(\mathbf{u})$ and $p(\mathbf{r}|\mathbf{u})$ is quite tedious for arbitrary PAM, PSK and QAM constellations, no general closed-form expressions for CRB_θ and CRB_ν are available (e.g., evaluation of the CRB's for BPSK and QPSK (see [2]) requires numerical integration). To avoid these complications, a simpler lower bound, called the modified CRB (MCRB), has been derived in [3], i.e., $E_{\mathbf{r}}[(\hat{u}_i - u_i)^2] \geq \mathrm{CRB}_i(\mathbf{u}) \geq \mathrm{MCRB}_i(\mathbf{u})$. The MCRB's for carrier phase and frequency estimation, corresponding to $\{r_k\}$ from (1), are given by [3]

$$\mathrm{MCRB}_\theta = \frac{N_0}{2E_s} \cdot \frac{1}{L} \quad (4)$$

$$\mathrm{MCRB}_\nu = \frac{N_0}{2E_s} \cdot \frac{3}{\pi^2 L(L^2-1)} \quad (5)$$

with $L = 2K + 1$. In [4] it has been shown that for high SNR (i.e., $E_s/N_0 \to \infty$) the CRB's for carrier phase and frequency estimation converge to the corresponding MCRB's (4) and (5). In the following, we derive a closed form expression for the low-SNR limit (i.e., $E_s/N_0 \to 0$) of the CRB's for carrier phase and frequency estimation. These low-SNR asymptotic CRB's will be denoted $\mathrm{ACRB}_\theta(\mathbf{u})$ and $\mathrm{ACRB}_\nu(\mathbf{u})$, respectively.

III. Low-SNR Limit of CRBs

For small E_s/N_0 (or equivalently, small ε), we obtain an approximation of $\ln(p(\mathbf{r}|\mathbf{u}))$ by expanding the exponential functions in (3) into a Taylor series, averaging each resulting term with respect to the data symbols, and keeping only the relevant terms that correspond to the smallest powers of ε. Expanding and averaging yields

$$\begin{aligned} E_{a_k}&\left[\exp\left(2\varepsilon\,\mathrm{Re}\left[a_k \breve{r}_k^*\right] - \varepsilon^2|a_k|^2\right)\right] \\ &= 1 + \sum_{p=1}^{\infty}\sum_{q=0}^{p}\sum_{r=0}^{p-q} F(p,\,q,\,r,\,\breve{r}_k)\varepsilon^{p+q} \\ &\quad \cdot E[(a_k^*)^{p-r} a_k^{q+r}] \end{aligned} \quad (6)$$

with

$$F(p,\,q,\,r,\,\breve{r}_k) = \frac{(-1)^q \breve{r}_k^{\,p-q-r}\left(\breve{r}_k^*\right)^r}{q!r!(p-q-r)!}. \quad (7)$$

When the symbol constellation is rotationally symmetric over $2\pi/N$ ($N = 2$ for M-PAM, $N = 4$ for M-QAM, $N = M$ for M-PSK), we obtain $E[(a_k^*)^{p-r} a_k^{q+r}] = 0$ for

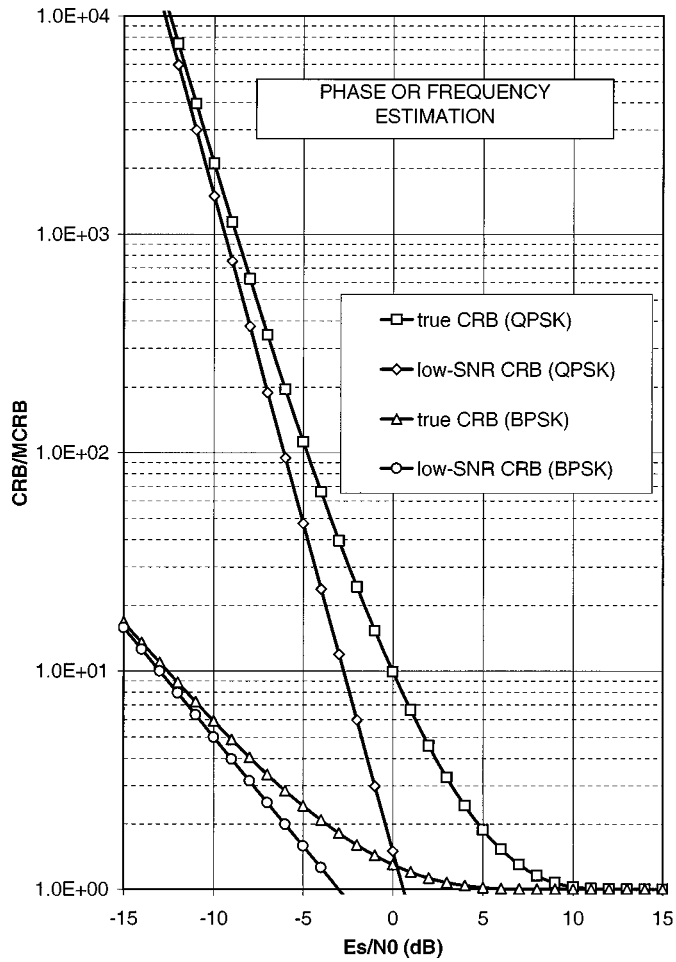

Fig. 1. Comparison of low-SNR CRB and true CRB.

$2r + q - p \notin \{0,\,\pm N,\,\pm 2N,\,\ldots\}$. Hence, the relevant terms in the triple summation of (6) correspond to $(p,\,q,\,r) = (N,\,0,\,0)$ and $(p,\,q,\,r) = (N,\,0,\,N)$, and contain a factor ε^N. All other nonzero terms in the triple summation are either independent of $(\theta,\,\nu)$ [for small ε these terms can be neglected as compared to the term 1 in (6)], or dependent of $(\theta,\,\nu)$ but containing a power of ε that is larger than N. Hence,

$$\begin{aligned} \ln p(\mathbf{r}|\mathbf{u}) &\cong \sum_{k=-K}^{K} \ln\left(1 + \frac{2}{N!}\,\varepsilon^N\,\mathrm{Re}\left[\breve{r}_k^{\,N} A_N^*\right]\right) \\ &\cong \frac{2}{N!}\,\varepsilon^N \sum_{k=-K}^{K} \mathrm{Re}\left[\breve{r}_k^{\,N} A_N^*\right] \end{aligned} \quad (8)$$

where $A_N = E[a_k^N]$. Straightforward application of (2) to (8) yields (the dominant part of) $\mathbf{J}$, which turns out to be diagonal. Inverting $\mathbf{J}$ then gives rise to the following low-SNR limit of the CRBs

$$\mathrm{ACRB}_\theta = \frac{N!}{2N^2|A_N|^2}\left(\frac{N_0}{E_s}\right)^N \cdot \frac{1}{L}$$

$$\mathrm{ACRB}_\nu = \frac{N!}{2N^2|A_N|^2}\left(\frac{N_0}{E_s}\right)^N \cdot \frac{3}{\pi^2 L(L^2-1)}. \quad (9)$$

Due to the fact that $\mathbf{J}$ is diagonal, the result (9) is also valid in the case where the value of θ or ν is known by the receiver, and only the other parameter is to be estimated. Note that the ACRB's are inversely proportional to the N*th power* of E_s/N_0. This is in contrast with the MCRB's, which are inversely proportional to E_s/N_0 [see (4) and (5)].

Let us verify the validity of the above derivation by comparing the asymptotic analytical result (4) and (5) to some true CRB's available from the literature. For $a_k = 1$ (i.e., the case of no modulation, or $N = 1$) the true CRB's equal the MCRB's (4) and (5), and substituting $N = 1$ into (9) also yields these MCRB's. For BPSK and QPSK modulation, Fig. 1 shows the ratios CRB/MCRB (from [2]) and ACRB/MCRB [from (9)] for phase estimation (identical results are obtained for frequency estimation). As it should, excellent agreement is found at low SNR.

IV. Conclusions and Remarks

In this contribution we have derived a simple closed-form analytical expression for the low-SNR limit of the CRB's pertaining to the estimation of the carrier phase and frequency of a linearly modulated waveform. When the symbol constellation has rotational symmetry over $2\pi/N$, this limit turns out to be inversely proportional to the Nth power of E_s/N_0.

The low-SNR limit of the tracking error variance of the nondata-aided Nth power carrier phase estimator [6, Secs. 5.10 and 6.3.6] equals ACRB$_\theta$ from (9), which indicates that this phase estimator is optimum at very small E_s/N_0, for PAM, PSK and QAM constellations. At low SNR the nondata-aided frequency estimators from [6, Sec. 8.4], resulting from joint phase and frequency estimation, turn out to be worse than ACRB$_\nu$, so that these frequency estimators are not optimum at very small E_s/N_0.

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