Exact BER Analysis for Alamouti's Code on Arbitrary Fading Channels with Imperfect Channel Estimation

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***Abstract*— In this contribution, we investigate the effect of imperfect channel estimation on the bit error rate (BER) performance of multiple-input multiple-output (MIMO) systems employing Alamouti's code. The propagation channels from both transmit antennas to each of the N_r receive antennas are assumed to be affected by (possibly correlated) flat block fading with an arbitrary fading distribution. The transmitted symbols belong to a pulse amplitude modulation (PAM) or quadrature amplitude modulation (QAM) constellation. The mismatched maximum-likelihood (ML) receiver makes use of a least-squares or linear minimum mean-square error (MMSE) channel estimate, obtained from known pilot symbols sent among the data.**

The resulting average BER for QAM transmission can easily be written as an expectation over $8N_\mathrm{r}$ real-valued random variables, but the computing time needed for its numerical evaluation increases dramatically with N_r. We point out that the BER can be expressed in terms of the distribution of the Frobenius norm of the channel matrix, rather than the joint distribution of all channel coefficients. This allows to reduce the BER expression for any number of receive antennas to an expectation over only 6 random variables, which can easily be computed numerically. Moreover, we show that for real-valued constellations and/or real-valued channels, the BER expression reduces to an expectation over less than 6 variables. For BER levels of practical interest, the numerical evaluation of the BER is much less time-consuming than a straightforward computer simulation. The presented BER expression is useful not only when the fading distribution is given in closed form, but also when only experimental data (e.g. a histogram) on the fading are available.

I. Introduction

The performance of wireless communication systems is strongly affected by small-scale fading. To combat this detrimental phenomenon, several diversity techniques have been proposed to provide the receiver with independent replicas of the signal. Spatial diversity is achieved by using multiple antennas at the transmitter and/or receiver side. These so-called multiple-input multiple-output (MIMO) communication systems can achieve a maximum diversity order of $N_\mathrm{t}N_\mathrm{r}$ (with N_t and N_r denoting the number of transmit and receive antennas, respectively), provided that proper space-time coding is used. In 1998, Alamouti invented a simple coding scheme for data transmission using two transmit antennas [1], with the remarkable benefit that the maximum-likelihood (ML) decoding algorithm reduces to symbol-by-symbol detection, based only on linear processing at the receiver. Tarokh *et al*. generalized Alamouti's scheme to an arbitrary number of transmit antennas by introducing the concept of orthogonal space-time block coding [17], [18]. As the orthogonal space-time block codes (OSTBCs) achieve full spatial diversity, and require only linear processing at the receiver, these codes are a very attractive transmit diversity technique. Under the assumption of perfect channel knowledge (PCK), the bit error rate (BER) performance of OSTBCs has been studied extensively in e.g. [4], [9], [10] and [11].

In practical wireless applications, however, the receiver has to estimate the channel response, which inevitably results in a performance penalty. Most often, the investigation of the resulting performance has been carried out under the assumption of Rayleigh fading: an analytical expression for the BER of OSTBCs in case of minimum mean-square error (MMSE) channel estimation was derived in [8]; in [15], analytical BER expressions as well as the tight Chernoff bound were given for orthogonal space-time block coded systems employing M-ary phase-shift keying (M-PSK) modulation; high-SNR expressions for the pairwise error probability (PEP) were derived in [2] for quite general STBCs with coherent and noncoherent receivers, using an eigenvalue approach; in [5], an exact closed-form expression for the PEP of both orthogonal and non-orthogonal space-time codes in the case of least-squares channel estimation was obtained by means of characteristic functions. However, from the PEP one can compute only an upper bound on the BER, which in a fading environment does not converge to the true BER at high SNR. In [6], expressions for the exact decoding error probability (DEP) were presented for the case of square OSTBCs on arbitrary fading channels, but the analysis is restricted to PSK constellations.

In this contribution, we provide an exact BER analysis for Alamouti's code with pulse amplitude modulation (PAM) or quadrature amplitude modulation (QAM) constellations on arbitrary flat-fading channels with imperfect channel estimation (ICE). In section II we describe the observation model which includes the $2N_\mathrm{r}$ arbitrary fading channels. Section III presents a linear pilot-based channel estimation method (with the well-known least-squares estimation and linear MMSE estimation as special cases), and derives the statistical properties of the channel estimate. The mismatched ML receiver is briefly outlined in section IV. The main part of our contribution is in section V, where the exact BER expression is reduced to an expectation over (at most) 6 variables, that allows numerical evaluation with a computing time that is independent of the number (N_r) of receive antennas. In section VI the results obtained from the numerical evaluation of our BER expression are confirmed by computer simulations, assuming i.i.d. Nakagami-m fading channels. Finally, conclusions are drawn in section VII. The main conclusion is that the presented BER expression can be used when the fading distribution is available in closed form or as an experimentally obtained histogram, and allows for a faster evaluation than by means of straightforward computer simulation, for practical values of the operating BER.

Throughout this paper, the superscripts T and H represent the vector (matrix) transpose and conjugate transpose, respectively. $\Re\{x\}$, $\Im\{x\}$ and $\mathbb{E}[x]$ denote the real part, the imaginary part and the expected value of x, respectively, while $\|\mathbf{X}\|$ refers to the Frobenius norm of $\mathbf{X}$.

II. Signal model

We consider a MIMO wireless communication system with 2 transmit antennas and N_r receive antennas. The propagation channels between both transmit antennas and each of the receive antennas are affected by flat fading with an arbitrary distribution. Transmission is organized in frames: in one frame, each transmit antenna sends K_p known pilot symbols and K coded data symbols; the pilot symbols enable channel estimation at the receiver. Within one frame of $N_\mathrm{fr} = K + K_\mathrm{p}$ symbols, the channel is assumed to be constant (block fading). The $N_\mathrm{r} \times N_\mathrm{fr}$ received signal matrix $\mathbf{R}_\mathrm{tot}$ is given by

$$\mathbf{R}_\mathrm{tot} = [\mathbf{R}_\mathrm{p}, \mathbf{R}] = \mathbf{H}[\mathbf{A}_\mathrm{p}, \mathbf{A}] + \mathbf{W}, \tag{1}$$

where the $2 \times K_\mathrm{p}$ pilot matrix $\mathbf{A}_\mathrm{p}$ and the $2 \times K$ data matrix $\mathbf{A}$ consist of the pilot symbols and the coded data symbols, respectively, transmitted at each transmit antenna. The propagation channel is represented by the $N_\mathrm{r} \times 2$ complex random matrix $\mathbf{H}$. We consider an arbitrary joint pdf $p(\mathbf{H})$ of the (possibly correlated) $2N_\mathrm{r}$ complex fading gains. The $N_\mathrm{r} \times N_\mathrm{fr}$ matrix $\mathbf{W}$ represents additive spatially and temporally white noise and consists of i.i.d. zero-mean circularly symmetric complex Gaussian (ZMCSCG) random variables with variance N_0.

Alamouti's space-time block code [1], that has been designed for two transmit antennas, transforms two information symbols $s_1(k)$ and $s_2(k)$ into a 2×2 coded symbol matrix $\mathbf{C}(k)$, given by

$$\mathbf{C}(k) = \begin{bmatrix} s_1(k) & -s_2^*(k) \\ s_2(k) & s_1^*(k) \end{bmatrix}. \tag{2}$$

Hence, assuming that an even number K of information symbols is sent, the transmitted data symbol matrix is given by $\mathbf{A} = \sqrt{E_\mathrm{s}}\,[\mathbf{C}(1), ..., \mathbf{C}(K/2)]$. Considering a normalized information symbol constellation ($\mathbb{E}[|s_i|^2] = 1$), it follows that the average energy of the transmitted coded symbols is given by E_s:

$$\frac{1}{2K}\mathbb{E}\left[\|\mathbf{A}\|^2\right] = E_\mathrm{s}. \tag{3}$$

Similarly, the average energy of the pilot symbols is E_p.

III. Pilot-based channel estimation

The receiver estimates the channel matrix $\mathbf{H}$ using $\mathbf{R}_\mathrm{p}$ and the known pilot matrix $\mathbf{A}_\mathrm{p}$. We assume that the rows of $\mathbf{A}_\mathrm{p}$ are orthogonal, i.e. $\mathbf{A}_\mathrm{p}\mathbf{A}_\mathrm{p}^H = K_\mathrm{p}E_\mathrm{p}\mathbf{I}_{N_\mathrm{t}}$, and consider linear channel estimates of the form

$$\hat{\mathbf{H}} = \frac{\alpha}{K_\mathrm{p}E_\mathrm{p}}\mathbf{R}_\mathrm{p}\mathbf{A}_\mathrm{p}^H, \tag{4}$$

with $\alpha \in \mathbb{R}$, such that $\hat{\mathbf{H}}$ can be decomposed into the sum of two statistically independent contributions:

$$\hat{\mathbf{H}} = \alpha\mathbf{H} + \mathbf{N}, \tag{5}$$

where the entries of $\mathbf{N} = (\alpha/(K_\mathrm{p}E_\mathrm{p}))\,\mathbf{W}_\mathrm{p}\mathbf{A}_\mathrm{p}^H$ are ZMCSCG random variables; the real and imaginary parts of the entries of $\mathbf{N}$ have a variance $\sigma_\mathbf{n}^2 = \alpha^2 N_0/(2K_\mathrm{p}E_\mathrm{p})$. Hence, when conditioned on $\mathbf{H}$, the channel estimate $\hat{\mathbf{H}}$ is a complex Gaussian random matrix with mean $\alpha\mathbf{H}$ and diagonal covariance matrix with diagonal elements $2\sigma_\mathbf{n}^2$. It is readily verified that both least-squares and linear MMSE estimation satisfy (4) with $\alpha = 1$ and $\alpha = K_\mathrm{p}E_\mathrm{p}/(K_\mathrm{p}E_\mathrm{p} + N_0)$, respectively [16].

Allocating a large total energy $K_\mathrm{p}E_\mathrm{p}$ to pilot symbols yields an accurate channel estimate, but on the other hand gives rise to a reduction of the symbol energy E_s. Denoting by E_b the energy per information bit and writing $E_\mathrm{p} = \gamma E_\mathrm{s}$, we have

$$E_\mathrm{s} = \frac{K}{K + \gamma K_\mathrm{p}}\rho\log_2(M)E_\mathrm{b}, \tag{6}$$

where $\rho = 1/2$ is the ratio of the number of information symbols to the total number of symbols in Alamouti's code matrix, and M denotes the number of constellation points. Hence, E_s decreases with increasing K_p.

IV. ML Detection

When the channel state information (CSI) is available at the receiver, ML detection is known to be the optimal detection algorithm for the transmitted data:

$$\hat{\mathbf{A}} = \arg\min_{\tilde{\mathbf{A}}}\left\|\mathbf{R} - \mathbf{H}\tilde{\mathbf{A}}\right\|^2. \tag{7}$$

Denoting by $[\mathbf{r}_1\,\mathbf{r}_2]$ the $N_r \times 2$ observation matrix corresponding to a coded symbol matrix $\mathbf{C}$ (we omit the time index for notational convenience) and writing $\mathbf{H} = [\mathbf{h}_1\,\mathbf{h}_2]$, the detection algorithm for the information symbols s_1 and s_2 reduces to symbol-by-symbol detection:

$$\hat{s}_i = \arg\min_{s_i}|u_i - s_i|\,,\ i \in \{1, 2\}\,, \tag{8}$$

with

$$u_1 = \frac{\mathbf{h}_1^H\mathbf{r}_1 + \mathbf{h}_2^T\mathbf{r}_2^*}{\sqrt{E_\mathrm{s}}\|\mathbf{H}\|^2}, \tag{9}$$

$$u_2 = \frac{\mathbf{h}_2^H\mathbf{r}_1 - \mathbf{h}_1^T\mathbf{r}_2^*}{\sqrt{E_\mathrm{s}}\|\mathbf{H}\|^2}. \tag{10}$$

Since the CSI is not known by the receiver, we assume a mismatched receiver which uses the estimated channel matrix $\hat{\mathbf{H}}$ in the same way an ML receiver with PCK would use the actual channel matrix $\mathbf{H}$. In this way, the decision variables become

$$u_1 = \frac{\hat{\mathbf{h}}_1^H \mathbf{r}_1 + \hat{\mathbf{h}}_2^T \mathbf{r}_2^*}{\sqrt{E_\mathrm{s}}\left\|\hat{\mathbf{H}}\right\|^2}, \quad (11)$$

$$u_2 = \frac{\hat{\mathbf{h}}_2^H \mathbf{r}_1 - \hat{\mathbf{h}}_1^T \mathbf{r}_2^*}{\sqrt{E_\mathrm{s}}\left\|\hat{\mathbf{H}}\right\|^2}. \quad (12)$$

In this contribution, we consider square M-QAM transmission with Gray mapping, which is equivalent to $\sqrt{M}$-PAM transmission for both the in-phase and quadrature information bits. The in-phase (quadrature) bits corresponding to the transmitted information symbol s_i, $i \in \{1,2\}$, are detected correctly when the real (imaginary) part of the decision variable u_i is located inside the projection of the decision area of s_i on the real (imaginary) axis. Hence, we can compute the BER of Alamouti's code as the average of the BERs for the in-phase and the quadrature bits of the symbols s_1 and s_2.

V. Bit error rate analysis

Focusing our attention to the in-phase information bits related to the transmitted symbol s_1, the BER is obtained by averaging the conditional BER (conditioned on the channel $\mathbf{H}$ and the channel estimate $\hat{\mathbf{H}}$):

$$\mathrm{BER}_{1,\mathrm{R}} = \mathbb{E}_{\mathbf{H},\hat{\mathbf{H}}}\left[\mathrm{BER}_{1,\mathrm{R}}(\mathbf{H},\hat{\mathbf{H}})\right], \quad (13)$$

with

$$\mathrm{BER}_{1,\mathrm{R}}(\mathbf{H},\hat{\mathbf{H}}) = \frac{1}{M^2}\sum_{s_1\in\Psi}\sum_{s_2\in\Psi}\mathrm{BER}_{1,\mathrm{R}}\left(s_1,s_2,\mathbf{H},\hat{\mathbf{H}}\right), \quad (14)$$

where s_1 and s_2 are the symbols actually transmitted, and Ψ denotes the normalized M-QAM constellation. An in-phase decision error occurs when the real part $u_{1,\mathrm{R}}$ of the decision variable u_1 is inside the projection (on the real axis) of the decision area of a QAM symbol b for which the in-phase component b_R is different from the in-phase component $s_{1,\mathrm{R}}$ of the transmitted symbol s_1; this projection will be referred to as the decision region of b_R. In this way, $\mathrm{BER}_{1,\mathrm{R}}(s_1,s_2,\mathbf{H},\hat{\mathbf{H}})$ is given by

$$\mathrm{BER}_{1,\mathrm{R}}\left(s_1,s_2,\mathbf{H},\hat{\mathbf{H}}\right) = \sum_{b_\mathrm{R}\in\Psi_\mathrm{R}}\frac{N(s_{1,\mathrm{R}},b_\mathrm{R})}{\frac{1}{2}\log_2 M}P(s_1,s_2,b_\mathrm{R},\mathbf{H},\hat{\mathbf{H}}), \quad (15)$$

Here, Ψ_R is the set consisting of the real parts of the constellation points, $N(s_{1,\mathrm{R}},b_\mathrm{R})$ represents the Hamming distance between the in-phase bits of the transmitted symbol s_1 and the in-phase bits of the detected symbol b, and $P(s_1,s_2,b_\mathrm{R},\mathbf{H},\hat{\mathbf{H}})$ is the probability that the real part of the decision variable u_1 is located inside the decision area of b_R (when the transmitted symbols s_1 and s_2, the channel $\mathbf{H}$ and the channel estimate $\hat{\mathbf{H}}$ are known):

$$P(s_1,s_2,b_\mathrm{R},\mathbf{H},\hat{\mathbf{H}}) = Pr\left[\hat{s}_{1,\mathrm{R}} = b_\mathrm{R}\middle| s_1,s_2,\mathbf{H},\hat{\mathbf{H}}\right]. \quad (16)$$

Expanding the real part $u_{1,\mathrm{R}}$ of the decision variable (11) yields:

$$u_{1,\mathrm{R}} = u'_{1,\mathrm{R}} + \frac{\Re\left\{\hat{\mathbf{h}}_1^H\mathbf{w}_1 + \hat{\mathbf{h}}_2^T\mathbf{w}_2^*\right\}}{\sqrt{E_\mathrm{s}}\left\|\hat{\mathbf{H}}\right\|^2}, \quad (17)$$

with

$$\begin{aligned} u'_{1,\mathrm{R}} = {} & s_{1,\mathrm{R}}\frac{\Re\left\{\hat{\mathbf{h}}_1^H\mathbf{h}_1 + \hat{\mathbf{h}}_2^T\mathbf{h}_2^*\right\}}{\left\|\hat{\mathbf{H}}\right\|^2} \\ & - s_{1,\mathrm{I}}\frac{\Im\left\{\hat{\mathbf{h}}_1^H\mathbf{h}_1 + \hat{\mathbf{h}}_2^T\mathbf{h}_2^*\right\}}{\left\|\hat{\mathbf{H}}\right\|^2} + s_{2,\mathrm{R}}\frac{\Re\left\{\hat{\mathbf{h}}_1^H\mathbf{h}_2 - \hat{\mathbf{h}}_2^T\mathbf{h}_1^*\right\}}{\left\|\hat{\mathbf{H}}\right\|^2} \\ & - s_{2,\mathrm{I}}\frac{\Im\left\{\hat{\mathbf{h}}_1^H\mathbf{h}_2 - \hat{\mathbf{h}}_2^T\mathbf{h}_1^*\right\}}{\left\|\hat{\mathbf{H}}\right\|^2}. \end{aligned} \quad (18)$$

In (18), the subscripts R and I refer to the real and imaginary parts of the transmitted symbols, respectively. The second term in (17) represents zero-mean Gaussian noise with variance $N_0/(2E_\mathrm{s}\left\|\hat{\mathbf{H}}\right\|^2)$. The first term in (18) is the useful term, whereas the second term represents interference from the quadrature component $s_{1,\mathrm{I}}$. The third and fourth term represent interference from the in-phase and quadrature component of the transmitted symbol s_2, respectively. Note that $u'_{1,\mathrm{R}} = s_{1,\mathrm{R}}$ when $\hat{\mathbf{H}} = \mathbf{H}$ (i.e. for PCK). Let $g_1(b_\mathrm{R})$ and $g_2(b_\mathrm{R})$ denote the boundaries of the projection of the decision area of b on the real axis, with $g_1(b_\mathrm{R}) < g_2(b_\mathrm{R})$; we set $g_1(b_\mathrm{R}) = -\infty$ ($g_2(b_\mathrm{R}) = \infty$) if b is a left (right) outer constellation point. In this way, (16) reduces to

$$P(s_1,s_2,b_\mathrm{R},\mathbf{H},\hat{\mathbf{H}}) = Q_1 - Q_2, \quad (19)$$

where the quantities Q_k, $k \in \{1,2\}$, are given by

$$Q_k = Q\left(\sqrt{2\frac{E_\mathrm{s}}{N_0}}\left\|\hat{\mathbf{H}}\right\|\left(g_k(b_\mathrm{R}) - u'_{1,\mathrm{R}}\right)\right). \quad (20)$$

In (20), $Q(.)$ is the Gaussian Q-function, defined as

$$Q(x) \triangleq \frac{1}{\sqrt{2\pi}}\int_x^\infty \exp\left(-\frac{x^2}{2}\right)\mathrm{d}x. \quad (21)$$

Hence, (13) can be expressed as

$$\mathrm{BER}_{1,\mathrm{R}} = \int \mathrm{BER}_{1,\mathrm{R}}(\mathbf{H},\hat{\mathbf{H}})p(\hat{\mathbf{H}}\middle|\mathbf{H})p(\mathbf{H})\mathrm{d}\hat{\mathbf{H}}\mathrm{d}\mathbf{H}. \quad (22)$$

where $p(\hat{\mathbf{H}}|\mathbf{H})$ is the joint Gausian pdf of $2N_\mathrm{r}$ complex-valued random variables, with mean $\alpha\mathbf{H}$ and diagonal covariance matrix with diagonal elements $2\sigma_\mathbf{n}^2$, and $p(\mathbf{H})$ represents the arbitrary fading distribution. Note that this general description also allows correlation between the components of $\mathbf{H}$. Although (22) is conceptually simple, it is not well suited for numerical evaluation. Indeed, taking into account that the components of $\mathbf{H}$ and $\hat{\mathbf{H}}$ are complex-valued, the evaluation of (22) requires taking the expectation over $8N_\mathrm{r}$ real-valued variables; hence, the associated computing time increases dramatically with the number N_r of receive antennas.

In order to avoid the computational complexity corresponding to the evaluation of (22), we will manipulate (22) into an expectation over only 6 variables. To this end, we introduce the following real-valued vectors:

$$\begin{cases} \hat{\mathbf{h}}' = \left[\hat{\mathbf{h}}_{1,\mathrm{R}}^T, \hat{\mathbf{h}}_{1,\mathrm{I}}^T, \hat{\mathbf{h}}_{2,\mathrm{R}}^T, \hat{\mathbf{h}}_{2,\mathrm{I}}^T\right]^T \\ \mathbf{h}_1' = \left[\mathbf{h}_{1,\mathrm{R}}^T, \mathbf{h}_{1,\mathrm{I}}^T, \mathbf{h}_{2,\mathrm{R}}^T, \mathbf{h}_{2,\mathrm{I}}^T\right]^T \\ \mathbf{h}_2' = \left[\mathbf{h}_{1,\mathrm{I}}^T, -\mathbf{h}_{1,\mathrm{R}}^T, -\mathbf{h}_{2,\mathrm{I}}^T, \mathbf{h}_{2,\mathrm{R}}^T\right]^T \\ \mathbf{h}_3' = \left[\mathbf{h}_{2,\mathrm{R}}^T, \mathbf{h}_{2,\mathrm{I}}^T, -\mathbf{h}_{1,\mathrm{R}}^T, -\mathbf{h}_{1,\mathrm{I}}^T\right]^T \\ \mathbf{h}_4' = \left[\mathbf{h}_{2,\mathrm{I}}^T, -\mathbf{h}_{2,\mathrm{R}}^T, \mathbf{h}_{1,\mathrm{I}}^T, -\mathbf{h}_{1,\mathrm{R}}^T\right]^T \end{cases}, \tag{23}$$

with $\hat{\mathbf{h}}_{i,\mathrm{R}} + j\hat{\mathbf{h}}_{i,\mathrm{I}}$ and $\mathbf{h}_{i,\mathrm{R}} + j\mathbf{h}_{i,\mathrm{I}}$, $i \in \{1, 2\}$, denoting the i-th column of $\hat{\mathbf{H}}$ and $\mathbf{H}$, respectively. It is easily verified that $\mathbf{h}_1'$, $\mathbf{h}_2'$, $\mathbf{h}_3'$ and $\mathbf{h}_4'$ are orthogonal. Let us consider an orthonormal coordinate system defined by the unit vectors $\{\mathbf{e}_i, i = 1, \cdots, 4N_\mathrm{r}\}$, with $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ and $\mathbf{e}_4$ directed along $\mathbf{h}_1'$, $\mathbf{h}_2'$, $\mathbf{h}_3'$ and $\mathbf{h}_4'$, respectively. The projections of $\hat{\mathbf{h}}'$ on $\mathbf{e}_i$ are denoted x_i, with $x_i = \hat{\mathbf{h}}'^T \mathbf{e_i}$, yielding $\|\hat{\mathbf{H}}\|^2 = |\hat{\mathbf{h}}'|^2 = x_1^2 + x_2^2 + x_3^2 + x_4^2 + z^2$, where

$$z^2 = \sum_{i=5}^{4N_\mathrm{r}} x_i^2. \tag{24}$$

Because of the specific choice of $\mathbf{e}_i$ for $1 \leq i \leq 4$, we have

$$x_1 = \frac{\hat{\mathbf{h}}'^T \mathbf{h}_1'}{|\mathbf{h}_1'|} = \frac{\Re\left\{\hat{\mathbf{h}}_1^H \mathbf{h}_1 + \hat{\mathbf{h}}_2^T \mathbf{h}_2^*\right\}}{\|\mathbf{H}\|}, \tag{25}$$

$$x_2 = \frac{\hat{\mathbf{h}}'^T \mathbf{h}_2'}{|\mathbf{h}_2'|} = \frac{\Im\left\{\hat{\mathbf{h}}_1^H \mathbf{h}_1 + \hat{\mathbf{h}}_2^T \mathbf{h}_2^*\right\}}{\|\mathbf{H}\|}, \tag{26}$$

$$x_3 = \frac{\hat{\mathbf{h}}'^T \mathbf{h}_3'}{|\mathbf{h}_3'|} = \frac{\Re\left\{\hat{\mathbf{h}}_1^H \mathbf{h}_2 - \hat{\mathbf{h}}_2^T \mathbf{h}_1^*\right\}}{\|\mathbf{H}\|}, \tag{27}$$

$$x_4 = \frac{\hat{\mathbf{h}}'^T \mathbf{h}_4'}{|\mathbf{h}_4'|} = \frac{\Im\left\{\hat{\mathbf{h}}_1^H \mathbf{h}_2 - \hat{\mathbf{h}}_2^T \mathbf{h}_1^*\right\}}{\|\mathbf{H}\|}. \tag{28}$$

Let us introduce the vector $\mathbf{x} = (x_1, x_2, x_3, x_4)$. From (18) and (20) it follows that $\mathrm{BER}_{1,\mathrm{R}}(\mathbf{H}, \hat{\mathbf{H}})$ is a function of only 6 random variables, x_1, x_2, x_3, x_4, z and $\|\mathbf{H}\|$, which we denote $\mathrm{BER}_{1,\mathrm{R}}(\mathbf{x}, z, \|\mathbf{H}\|)$. Taking (5) into account, it can be shown that x_1, x_2, x_3, x_4 and z (when conditioned on $\|\mathbf{H}\|$) are independent variables which satisfy the following properties:

- x_1 is a Gaussian variable with mean $\alpha\|\mathbf{H}\|$ and variance $\sigma_\mathbf{n}^2$.
- x_2, x_3 and x_4 are zero-mean Gaussian variables with variance $\sigma_\mathbf{n}^2$.
- $z/\sigma_\mathbf{n}$ is distributed according to the chi-distribution with $4N_\mathrm{r} - 4$ degrees of freedom [3].

Taking the above properties into account, $\mathrm{BER}_{1,\mathrm{R}}$ is obtained as

$$\mathrm{BER}_{1,\mathrm{R}} = \int \mathrm{BER}_{1,\mathrm{R}}(\mathbf{x}, z, u) \, p(\mathbf{x}, z \,|\, \|\mathbf{H}\| = u) p(u) \mathrm{d}\mathbf{x} \, \mathrm{d}z \, \mathrm{d}u. \tag{29}$$

where $p(u)$ is the pdf of the Frobenius norm $\|\mathbf{H}\|$ of the channel matrix $\mathbf{H}$. It is important to note that instead of the joint distribution $p(\mathbf{H})$ of the $2N_\mathbf{r}$ complex-valued fading gains, we need only the distribution of the Frobenius norm $\|\mathbf{H}\|$. Owing to the rotational symmetry of the M-QAM constellation and taking into account that the symbol vectors are equally likely, it can be verified that the BERs related to the in-phase and quadrature bits of s_1 and s_2 are equal. Hence, the BER resulting from Alamouti's code equals $\mathrm{BER}_{1,\mathrm{R}}$, and the BER evaluation involves an expectation over only 6 random variables. The expectation (29) can be evaluated numerically by approximating the 6-fold integral by a 6-fold sum, running over discretized versions of the continuous variables x_1, x_2, x_3, x_4, z and u.

The number of random variables to be considered in the expectation (29) is reduced in the following cases:

1) In the case of single-input single-output (SISO) transmission ($N_\mathrm{r} = 1$), the dimension of $\hat{\mathbf{h}}'$ is 4, so the only projections are x_1, x_2, x_3 and x_4. Hence the BER computation involves averaging over only 5 variables, x_1, x_2, x_3, x_4 and $\|\mathbf{H}\|$.
2) For PAM constellations, we have $s_{1,\mathrm{I}} = s_{2,\mathrm{I}} = 0$, so that $u_{1,\mathrm{R}}'$ from (18) does not depend on x_2 and x_4. Hence, the BER (which equals the in-phase BER) is obtained by averaging over only 4 variables, x_1, x_3, z' and $\|\mathbf{H}\|$, where $\|\hat{\mathbf{H}}\|^2 = x_1^2 + x_3^2 + z'^2$ and $z'/\sigma_\mathbf{n}$ has a chi-distribution with $4N_\mathrm{r} - 2$ degrees of freedom.

Till now we have considered complex-valued channel gains. In some applications (e.g., ultra wideband communication), channel models with real-valued gains are more appropriate (and the transmitted symbols are real-valued too). The BER result in this case is nearly the same as for PAM transmission over complex-valued channels (averaging over 4 variables, x_1, x_3, z' and $\|\mathbf{H}\|$), the only difference being that on real-valued channels the variable $z'/\sigma_\mathbf{n}$ has $2N_\mathrm{r} - 2$ (instead of $4N_\mathrm{r} - 2$) degrees of freedom. For specific cases, a further reduction of the number of random variables can be obtained. According to [6], only a twofold integral must be computed in the case of M-PSK constellations and arbitrary fading.

VI. Numerical results

To obtain our numerical results, we assume MMSE channel estimation with $E_\mathrm{p} = E_\mathrm{s}$ and $2N_\mathrm{r}$ independent and identically distributed (i.i.d.) Nakagami-m fading channels, with parameters m and Ω [14].

Fig. 1 shows the exact BER curves for Alamouti's code and 16-QAM transmission over a complex-valued MIMO channel, for both the mismatched receiver and the PCK receiver, and for several values of m and N_r. It is easily derived that the Frobenius norm $\|\mathbf{H}\|$ of the MIMO channel follows a Nakagami-m distribution with parameters $2N_\mathrm{r}m$ and $2N_\mathrm{r}\Omega$. The curves corresponding to the mismatched receiver represent the numerically computed expectation over 6 variables resulting from (29), whereas the BER of the PCK receiver involves the expectation over $\|\mathbf{H}\|$ only. Also shown in the figure are straightforward computer simulation results for the mismatched receiver that confirm the result obtained from (but require considerably more computing time then) numerical averaging.

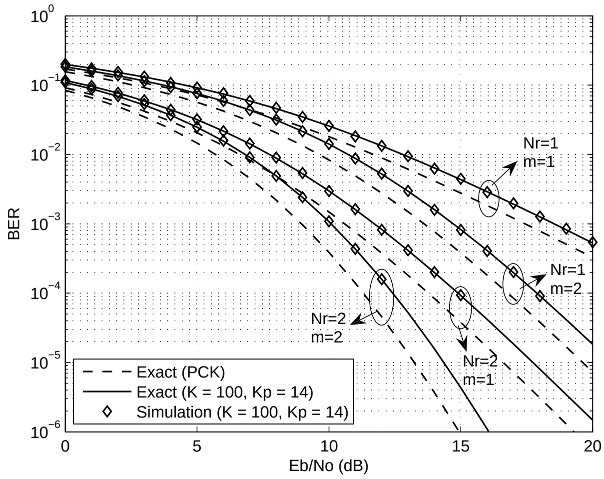

Fig. 1. Complex-valued Nakagami-m fading channel, 16-QAM

VII. Conclusions and remarks

In this contribution, we investigated the effect of imperfect channel estimation on the BER performance of Alamouti's code. The transmitted symbols belong to a PAM or QAM constellation. We assumed pilot symbol assisted channel estimation and $2N_\mathrm{r}$ propagation channels affected by flat block fading with arbitrary pdf. The main conclusions are the following:

- A conceptually simple BER expression can be obtained in the form of an expectation over $8N_\mathrm{r}$ random variables. However, its numerical evaluation requires a computing time that increases dramatically with N_r.
- We have reduced the BER expression to a form that contains an expectation over only 6 variables, irrespective of the value of N_r. Rather than the joint distribution of all $2N_\mathrm{r}$ complex-valued channel gains, this expression depends on the distribution of only the Frobenius norm of the channel matrix. We have pointed out that for some cases (SISO, real-valued channel, PAM constellation, and combinations thereof) the number of variables involved in the expectation can be further reduced.
- Evaluation of the reduced BER expression generally requires a (6-fold or less) numerical integration. This BER expression can be used when the pdf of the Frobenius norm of the channel matrix is available in closed form or has been determined experimentally (histogram). Comparing the computing times resulting from the numerical averaging and from straightforward simulation, it turns out that the numerical averaging is to be preferred for BER values of practical interest.

The reviewers brought to our attention that OSTBCs can be represented by an equivalent SISO channel, which allows to exploit SISO channel results for studying OSTBCs [13], [12], [7]. This avenue will be explored in further research.

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