# Exact BER Analysis for QAM Transmission on Arbitrary Fading Channels with Maximal-Ratio Combining and Imperfect Channel Estimation

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*Abstract*— **In this contribution, we investigate the effect of imperfect channel estimation on the bit error rate (BER) performance of uncoded quadrature amplitude modulation (QAM) with maximal-ratio combining (MRC) multichannel reception. The propagation channels from the transmitter to each of the $N_\mathrm{r}$ receive antennas are assumed to be affected by (possibly correlated) flat block fading with an arbitrary fading distribution. The MRC receiver makes use of estimated channel coefficients, obtained from known pilot symbols sent among the data. The resulting average BER for QAM can easily be written as an expectation over $4N_\mathrm{r}$ random variables, but the computing time needed for its numerical evaluation increases exponentially with $N_\mathrm{r}$. We point out that the BER can be expressed in terms of the distribution of the norm of the channel vector, rather than the joint distribution of all channel coefficients. This allows to reduce the BER expression to an expectation over only $4$ random variables, irrespective of the number of receive antennas. Moreover, we show that for real-valued constellations and/or real-valued channels, the BER expression reduces to an expectation over less than $4$ variables. For practical BER levels, the numerical evaluation of the BER is much less time-consuming than a straightforward computer simulation. The presented BER expression is useful not only when the fading distribution is given in closed form, but also when only experimental data (e.g. a histogram) on the fading are available.**

## I. Introduction

To combat the detrimental effect of small-scale fading on the performance of wireless communication systems, several diversity techniques have been proposed to provide the receiver with independent replicas of the signal. Receive diversity is realized by using multiple antennas at the receiver side, which are sufficiently separated from each other to achieve uncorrelated fading. When the channel state information (CSI) is known at the receiver, Brennan showed that maximal-ratio combining (MRC) is the optimal way to combine the multiple received signals into a single signal with improved signal-to-noise ratio (SNR) [1]. A unified approach for the computation of the exact symbol error rate (SER) of linearly modulated signals over generalized fading channels with MRC is presented in [2]. The resulting expression involves a single finite-range integral, which can be easily computed numerically.

However, in practice the CSI is not a priori available, so the receiver has to estimate the diversity channels. The effect of estimation errors on the SNR of maximal-ratio combiners in Rayleigh fading channels was examined in [3] and [4]. The symbol error probability (SEP) of antenna subset diversity (SSD), including the case of MRC, was studied in [5] for $M$-ary quadrature amplitude modulation ($M$-QAM) and phase-shift keying ($M$-PSK) on Rayleigh fading channels with imperfect channel estimation (ICE). The exact bit error rate (BER) for square/rectangular QAM with MRC and ICE in non-identical Rayleigh fading channels was given in [6]. A similar analysis for Rician fading channels was provided in [7]. In [8], approximate BER expressions were given for $M$-QAM with both MRC and equal-gain combining (EGC) in Nakagami fading channels with ICE. In [9], the exact bit error probability (BEP) for MRC diversity systems utilizing binary phase-shift keying (BPSK) was derived for arbitrary fading channels with ICE. The resulting BEP expression requires the evaluation of a single finite-range integral provided that one can obtain the moment generating function (MGF) of the norm square of the channel vector. However, since the analysis in [9] is based on the result of [10, Appendix B], it cannot be extended to non-binary signaling constellations.

In this contribution, we provide an exact BER analysis for $M$-QAM signals sent over arbitrary (possibly correlated) flat-fading channels, with ICE and MRC at the receiver. In section II we describe the observation model which includes the $N_\mathrm{r}$ arbitrary fading channels. Section III presents a generic pilot-based channel estimation method (with least-squares estimation and linear minimum mean-square error (MMSE) estimation as special cases), and derives the statistical properties of the channel estimate. The mismatched receiver with MRC detection is briefly outlined in section IV. The main part of our contribution is in section V, where the exact BER expression is reduced to an expectation over (at most) $4$ variables, which allows numerical evaluation with a computing time that is independent of the number ($N_\mathrm{r}$) of receive antennas. In section VI the results obtained from the numerical evaluation of our BER expression are confirmed by computer simulations, assuming i.i.d. Nakagami-$m$ fading channels. Finally, conclusions are drawn in section VII. The main conclusion is that the presented BER expression can be used when the fading distribution is available in closed form

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or as an experimentally obtained histogram, and allows for a faster evaluation than by means of straightforward computer simulation.

Throughout this paper, the superscripts $T$ and $H$ represent the vector (matrix) transpose and conjugate transpose, respectively. $\Re\{x\}$, $\Im\{x\}$, $sgn(x)$ and $\mathbb{E}[x]$ denote the real part, the imaginary part, the sign and the expected value of $x$, respectively, while $\|\mathbf{X}\|$ refers to the Frobenius norm of $\mathbf{X}$.

## II. Signal model

Let us consider a wireless single-input multiple-output (SIMO) communication system with 1 transmit antenna and $N_\mathrm{r}$ receive antennas. We assume the propagation channels between the transmit antenna and each of the receive antennas to be affected by flat fading with an arbitrary distribution. Transmission is organized in frames consisting of $K_\mathrm{p}$ known pilot symbols and $K$ uncoded data symbols. The pilot symbols are used by the receiver to estimate the channel, which is assumed to be constant within one frame of $N_\mathrm{fr} = K + K_\mathrm{p}$ symbols and changes independently from one frame to another (block fading). The $N_\mathrm{r} \times N_\mathrm{fr}$ received signal matrix $\mathbf{R}_\mathrm{tot}$ is given by

$$\mathbf{R}_\mathrm{tot} = [\mathbf{R}_\mathrm{p}\,\mathbf{R}] = \mathbf{h}\left[\sqrt{E_\mathrm{p}}\mathbf{a}_\mathrm{p}\ \sqrt{E_\mathrm{s}}\mathbf{a}\right] + [\mathbf{W}_\mathrm{p}\,\mathbf{W}], \tag{1}$$

where the $1 \times K_\mathrm{p}$ pilot vector $\mathbf{a}_\mathrm{p}$ and the $1 \times K$ data vector $\mathbf{a}$ consist of the pilot symbols and the information symbols, respectively. We assume a normalized $M$-QAM constellation for the information symbols ($\mathbb{E}\left[|\mathbf{a}(i)|^2\right] = 1$, $1 \le i \le K$), such that $E_\mathrm{s}$ denotes the average transmitted energy for the information symbols. Similarly, $E_\mathrm{p}$ denotes the energy of the transmitted pilot symbols. The propagation channel is represented by the $N_\mathrm{r} \times 1$ complex random vector $\mathbf{h}$. We consider an arbitrary joint pdf $p(\mathbf{h})$ of the (possibly correlated) $N_\mathrm{r}$ complex fading gains. The $N_\mathrm{r} \times N_\mathrm{fr}$ matrix $[\mathbf{W}_\mathrm{p}\,\mathbf{W}]$ describes additive spatially and temporally white noise and consists of i.i.d. zero-mean circularly symmetric complex Gaussian (ZMCSCG) random variables with variance $N_0$.

## III. Pilot-based channel estimation

The receiver estimates the channel vector $\mathbf{h}$ from the known pilot vector $\mathbf{a}_\mathrm{p}$ and the corresponding received signal matrix $\mathbf{R}_\mathrm{p}$. We assume linear channel estimates of the form

$$\hat{\mathbf{h}} = \frac{\alpha}{K_\mathrm{p}\sqrt{E_\mathrm{p}}}\mathbf{R}_\mathrm{p}\mathbf{a}_\mathrm{p}^H, \tag{2}$$

with $\alpha \in \mathbb{R}$, such that $\hat{\mathbf{h}}$ can be decomposed into the sum of two statistically independent contributions:

$$\hat{\mathbf{h}} = \alpha\mathbf{h} + \mathbf{n}, \tag{3}$$

where the entries of $\mathbf{n} = \left(\alpha/(K_\mathrm{p}\sqrt{E_\mathrm{p}})\right)\mathbf{W}_\mathrm{p}\mathbf{a}_\mathrm{p}^H$ are ZMCSCG random variables; the real and imaginary parts of the entries of $\mathbf{n}$ have a variance $\sigma_\mathbf{n}^2 = \alpha^2 N_0/(2K_\mathrm{p}E_\mathrm{p})$. Hence, when conditioned on $\mathbf{h}$, the channel estimate $\hat{\mathbf{h}}$ is a complex Gaussian random variable with mean $\alpha\mathbf{h}$ and diagonal covariance matrix with diagonal elements $2\sigma_\mathbf{n}^2$. Both least-squares and linear MMSE estimation satisfy (2) with $\alpha = 1$ and $\alpha = K_\mathrm{p}E_\mathrm{p}/(K_\mathrm{p}E_\mathrm{p} + N_0)$, respectively [11].

Allocating a large total energy $K_\mathrm{p}E_\mathrm{p}$ to pilot symbols yields an accurate channel estimate, but on the other hand gives rise to a reduction of the symbol energy $E_\mathrm{s}$. When $E_\mathrm{b}$ denotes the energy per information bit and $\gamma \triangleq E_\mathrm{p}/E_\mathrm{s}$ is the ratio of the pilot energy to the data energy, we have

$$E_\mathrm{s} = \frac{K}{K + \gamma K_\mathrm{p}}\log_2(M)E_\mathrm{b}, \tag{4}$$

where $M$ denotes the number of constellation points. Hence, $E_\mathrm{s}$ decreases when the number of pilot symbols $K_\mathrm{p}$ is increased.

## IV. Maximal-ratio combining

When the CSI is known by the receiver, ML detection of the transmitted data yields optimum performance:

$$\hat{\mathbf{a}} = \arg\min_{\tilde{\mathbf{a}}}\left\|\mathbf{R} - \sqrt{E_\mathrm{s}}\mathbf{h}\tilde{\mathbf{a}}\right\|^2. \tag{5}$$

Here, $\tilde{\mathbf{a}}$ ranges over all possible symbol vectors of length $K$. Writing $\mathbf{R} = [\mathbf{r}_1 \cdots \mathbf{r}_K]$, it can be easily verified that the ML algorithm (5) reduces to symbol-by-symbol detection for the elements of the data vector $\mathbf{a}$:

$$\hat{a}_k = \arg\min_{\tilde{a}}|u_k - \tilde{a}|,\ 1 \le k \le K, \tag{6}$$

where the decision variable $u_k$ is given by

$$u_k = \frac{\mathbf{h}^H\mathbf{r}_k}{\sqrt{E_\mathrm{s}}\,|\mathbf{h}|^2}. \tag{7}$$

Hence, the ML detection algorithm reduces to multichannel reception with MRC [2]. Since the CSI is not known by the receiver, we assume a mismatched receiver which uses the estimated channel vector $\hat{\mathbf{h}}$ in the same way an MRC receiver with perfect channel knowledge (PCK) would use the actual channel vector $\mathbf{h}$. In this way, the decision variable becomes

$$u_k = \frac{\hat{\mathbf{h}}^H\mathbf{r}_k}{\sqrt{E_\mathrm{s}}|\hat{\mathbf{h}}|^2}. \tag{8}$$

In this paper, we consider square $M$-QAM transmission with Gray mapping, which is equivalent to $\sqrt{M}$-PAM transmission for both the in-phase and quadrature information bits. The in-phase (quadrature) bits corresponding to the transmitted information symbol $a$ (we omit the time index $k$ for notational convenience), are detected correctly when the real (imaginary) part of the decision variable $u$ is located inside the projection of the decision area of $a$ on the real (imaginary) axis. Hence, we can compute the BER as the average of the BERs for the in-phase and the quadrature bits.

## V. Bit error rate analysis

Let us first concentrate on the BER related to the in-phase information bits, which is obtained by averaging the conditional BER (conditioned on the channel $\mathbf{h}$ and the channel estimate $\hat{\mathbf{h}}$):

$$\mathrm{BER}_\mathrm{R} = \mathbb{E}_{\mathbf{h},\hat{\mathbf{h}}}\left[\mathrm{BER}_\mathrm{R}(\mathbf{h},\hat{\mathbf{h}})\right]. \tag{9}$$

$\mathrm{BER_R}(\mathbf{h},\hat{\mathbf{h}})$ from (9) can be expressed in terms of $\mathrm{BER_R}(a,\mathbf{h},\hat{\mathbf{h}})$, the BER conditioned on the transmitted symbol ($a$), the true channel vector ($\mathbf{h}$), and the estimated channel vector ($\hat{\mathbf{h}}$):

$$\mathrm{BER_R}(\mathbf{h},\hat{\mathbf{h}}) = \frac{1}{M}\sum_{a\in\Psi}\mathrm{BER_R}(a,\mathbf{h},\hat{\mathbf{h}}), \tag{10}$$

where $\Psi$ denotes the normalized $M$-QAM constellation. An in-phase decision error occurs when the real part $u_\mathrm{R}$ of the decision variable $u$ is inside the projection (on the real axis) of the decision area of a QAM symbol $b$ for which the in-phase component $b_\mathrm{R}$ is different from the in-phase component $a_\mathrm{R}$ of the transmitted symbol $a$; this projection will be referred to as the decision region of $b_\mathrm{R}$. In this way, $\mathrm{BER_R}(a,\mathbf{h},\hat{\mathbf{h}})$ is given by

$$\mathrm{BER_R}(a,\mathbf{h},\hat{\mathbf{h}}) = \sum_{b_\mathrm{R}\in\Psi_\mathrm{R}}\frac{N(a_\mathrm{R},b_\mathrm{R})}{\frac{1}{2}\log_2 M}P(a,b_\mathrm{R},\mathbf{h},\hat{\mathbf{h}}), \tag{11}$$

where $N(a_\mathrm{R},b_\mathrm{R})$ represents the Hamming distance between the in-phase bits of the transmitted symbol $a$ and the in-phase bits of the detected symbol $b$. $P(a,b_\mathrm{R},\mathbf{h},\hat{\mathbf{h}})$ is the probability that the real part of the decision variable $u$ is located inside the decision area of $b_\mathrm{R}$, when the symbol $a$ has been transmitted, and the channel $\mathbf{h}$ and the channel estimate $\hat{\mathbf{h}}$ are known:

$$P(a,b_\mathrm{R},\mathbf{h},\hat{\mathbf{h}}) = Pr\left[\hat{a}_\mathrm{R}=b_\mathrm{R}\middle|a,\mathbf{h},\hat{\mathbf{h}}\right]. \tag{12}$$

Expanding the real part of the decision variable (8) yields:

$$u_\mathrm{R} = u'_\mathrm{R} + \frac{\Re\left\{\hat{\mathbf{h}}^H\mathbf{w}\right\}}{\sqrt{E_\mathrm{s}}\left|\hat{\mathbf{h}}\right|^2}, \tag{13}$$

with

$$u'_\mathrm{R} = a_\mathrm{R}\frac{\Re\left\{\hat{\mathbf{h}}^H\mathbf{h}\right\}}{|\hat{\mathbf{h}}|^2} - a_\mathrm{I}\frac{\Im\left\{\hat{\mathbf{h}}^H\mathbf{h}\right\}}{|\hat{\mathbf{h}}|^2}. \tag{14}$$

The first term in (14) is the useful term, while the second term represents interference from the quadrature component $a_\mathrm{I}$. Note that in case of PCK ($\hat{\mathbf{h}}=\mathbf{h}$), the interference term is zero and $u'_\mathrm{R}=a_\mathrm{R}$. When the boundaries of the decision region of $b_\mathrm{R}$ are denoted $g_1(b_\mathrm{R})$ and $g_2(b_\mathrm{R})$, with $g_1(b_\mathrm{R})<g_2(b_\mathrm{R})$ (we set $g_1(b_\mathrm{R})=-\infty$ ($g_2(b_\mathrm{R})=\infty$) if $b$ is a left (right) outer constellation point), the probability (12) reduces to

$$P(a,b_\mathrm{R},\mathbf{h},\hat{\mathbf{h}}) = Q_1 - Q_2, \tag{15}$$

where the functions $Q_i$, $i\in\{1,2\}$, are given by

$$Q_i = Q\left(\sqrt{2\frac{E_\mathrm{s}}{N_0}}|\hat{\mathbf{h}}|\left(g_i(b_\mathrm{R})-u'_\mathrm{R}\right)\right). \tag{16}$$

In (16), $Q(.)$ is the Gaussian Q-function, defined as

$$Q(x) \triangleq \frac{1}{\sqrt{2\pi}}\int_x^\infty \exp\left(-\frac{x^2}{2}\right)\mathrm{d}x. \tag{17}$$

Hence, (9) can be expressed as

$$\mathrm{BER_R} = \int \mathrm{BER_R}(\mathbf{h},\hat{\mathbf{h}})p(\hat{\mathbf{h}}|\mathbf{h})p(\mathbf{h})\mathrm{d}\hat{\mathbf{h}}\mathrm{d}\mathbf{h}, \tag{18}$$

where $p(\hat{\mathbf{h}}|\mathbf{h})$ is the joint Gausian pdf of $N_\mathrm{r}$ complex-valued random variables, with mean $\alpha\mathbf{h}$ and diagonal covariance matrix with diagonal elements $2\sigma_\mathbf{n}^2$, and $p(\mathbf{h})$ represents the arbitrary fading distribution. Note that this general description also allows to assess correlated channels. Taking into account that the components of $\mathbf{h}$ and $\hat{\mathbf{h}}$ are complex-valued, the evaluation of (18) requires taking the expectation over $4N_\mathrm{r}$ real-valued variables. Hence, despite its conceptual simplicity, (18) is not well suited for numerical evaluation, since the associated computing time increases exponentially with the number $N_\mathrm{r}$ of receive antennas.

In order to avoid the computational complexity of evaluating (18), we will manipulate (18) into an expectation over only 4 variables. Therefore, we introduce the following real-valued vectors:

$$\begin{cases}\hat{\mathbf{h}}' = \left[\hat{\mathbf{h}}_\mathrm{R}^T,\hat{\mathbf{h}}_\mathrm{I}^T\right]^T\\ \mathbf{h}_1' = \left[\mathbf{h}_\mathrm{R}^T,\mathbf{h}_\mathrm{I}^T\right]^T\\ \mathbf{h}_2' = \left[\mathbf{h}_\mathrm{I}^T,-\mathbf{h}_\mathrm{R}^T\right]^T\end{cases}, \tag{19}$$

where $\hat{\mathbf{h}}=\hat{\mathbf{h}}_\mathrm{R}+j\hat{\mathbf{h}}_\mathrm{I}$ and $\mathbf{h}=\mathbf{h}_\mathrm{R}+j\mathbf{h}_\mathrm{I}$. It is readily verified that $\mathbf{h}_1'$ and $\mathbf{h}_2'$ are orthogonal. Also, we introduce an orthonormal coordinate system defined by the unit vectors $\{\mathbf{e}_i, i=1,\cdots,2N_\mathrm{r}\}$, with $\mathbf{e}_1$ and $\mathbf{e}_2$ directed along $\mathbf{h}_1'$ and $\mathbf{h}_2'$, respectively. The projection of $\hat{\mathbf{h}}'$ on $\mathbf{e}_i$ is denoted $x_i$, with $x_i=\hat{\mathbf{h}}'^T\mathbf{e_i}$, yielding $|\hat{\mathbf{h}}|^2=|\hat{\mathbf{h}}'|^2=x_1^2+x_2^2+z^2$, where

$$z^2 = \sum_{i=3}^{2N_\mathrm{r}} x_i^2. \tag{20}$$

Because of the specific directions of $\mathbf{e}_1$ and $\mathbf{e}_2$, it can be shown that $x_1$ and $x_2$ are given by

$$x_1 = \frac{\hat{\mathbf{h}}'^T\mathbf{h}_1'}{|\mathbf{h}_1'|} = \frac{\Re\left\{\hat{\mathbf{h}}^H\mathbf{h}\right\}}{|\mathbf{h}|}, \tag{21}$$

$$x_2 = \frac{\hat{\mathbf{h}}'^T\mathbf{h}_2'}{|\mathbf{h}_2'|} = \frac{\Im\left\{\hat{\mathbf{h}}^H\mathbf{h}\right\}}{|\mathbf{h}|}. \tag{22}$$

Taking (14) and (16) into account, it follows that the conditional BER (10) is a function of only 4 random variables, $x_1$, $x_2$, $z$ and $|\mathbf{h}|$, which we denote $\mathrm{BER_R}(x_1,x_2,z,|\mathbf{h}|)$. Taking (3) into account, it can be shown that $x_1$, $x_2$ and $z$ (when conditioned on $|\mathbf{h}|$) are independent variables which satisfy the following properties:

- $x_1$ is a Gaussian random variable with mean $\alpha|\mathbf{h}|$ and variance $\sigma_\mathbf{n}^2$.
- $x_2$ is a Gaussian random variable with zero-mean and variance $\sigma_\mathbf{n}^2$.
- $z/\sigma_\mathbf{n}$ is distributed according to a chi distribution with $2N_\mathrm{r}-2$ degrees of freedom [12].

According to the above properties, the in-phase BER (9) can be obtained as

$$\mathrm{BER_R} = \int \mathrm{BER_R}(x_1,x_2,z,u) \\ p(x_1,x_2,z\big||\mathbf{h}|=u)p(u)\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}z\mathrm{d}u, \tag{23}$$

where $p(u)$ is the pdf of the norm $|\mathbf{h}|$ of the channel vector $\mathbf{h}$. It is important to note that instead of the joint distribution $p(\mathbf{h})$ of the $N_\mathbf{r}$ complex-valued fading gains, we need only the distribution of the norm $|\mathbf{h}|$. Deriving in the same way as above the BER related to the quadrature bits, the resulting BER expression for QAM transmission is obtained as the arithmetical average of the in-phase and quadrature BERs. Hence, the BER for QAM involves an expectation over only 4 random variables.

The expectation (23) is evaluated numerically by approximating the 4-fold integral by a 4-fold sum, running over discretized versions of the continuous variables $x_1$, $x_2$, $z$ and $u$. Considering (10)-(12), (14)-(16) and the pdfs of the independent variables $x_1$, $x_2$ and $z$ (conditioned on $|\mathbf{h}|$), the integrand in (23) is, apart from $p(u)$, the product of known analytical functions. The pdf $p(u)$ of $|\mathbf{h}|$ can be available either in analytical form or obtained from experiments.

The number of random variables to be considered in the expectation (23) is further reduced in the following cases:

1) In the case of single-input single-output (SISO) transmission ($N_\mathrm{r} = 1$), the dimension of $\hat{\mathbf{h}}'$ is two, so the only projections are $x_1$ and $x_2$. Hence the BER computation involves averaging over only 3 variables, $x_1$, $x_2$ and $|h|$.
2) For PAM constellations, we have $a_I = 0$, so that $u'_\mathrm{R}$ from (14) does not depend on $x_2$. Hence, the BER (which equals the in-phase BER) is obtained by averaging over only 3 variables, $x_1$, $z'$ and $|\mathbf{h}|$, where $z'$ is obtained by changing in (20) the lower summation index into 2. The variable $z'/\sigma_\mathbf{n}$ follows a chi distribution with $2N_\mathrm{r} - 1$ degrees of freedom.

Till now we have considered complex-valued channel gains. In some applications (e.g., ultra wideband communication) channel models with real-valued gains are more appropriate. The BER result for PAM transmission over real-valued channels is nearly the same as for PAM over complex-valued channels (averaging over 3 variables, $x_1$, $z'$ and $|\mathbf{h}|$). The variable $z'/\sigma_\mathbf{n}$, however, is distributed according to a chi distribution with $N_\mathrm{r} - 1$ (instead of $2N_\mathrm{r} - 1$) degrees of freedom. For specific cases, a further reduction of the number of random variables can be obtained. In the case of binary phase-shift keying (BPSK) transmission over a real-valued SISO propagation channel, it can be shown that the BER is given by

$$\mathrm{BER} = \mathbb{E}_{h,\hat{h}}\left[Q\left(\sqrt{2\frac{E_\mathrm{s}}{N_0}}sgn(\hat{h})h\right)\right]. \quad (24)$$

Writing $A_1 = \sqrt{2E_\mathrm{s}/N_0}$ and $A_2 = \sqrt{2K_\mathrm{p}E_\mathrm{p}/N_0}$, it is easily derived that (24) simplifies to:

$$\mathrm{BER} = \mathbb{E}_{|h|}[Q(A_1|h|) + Q(A_2|h|) - 2Q(A_1|h|)Q(A_2|h|)]. \quad (25)$$

In this case, we need to take the expectation w.r.t. a single variable only. Moreover, we obtain an exact closed-form expression for the BER for BPSK transmission over a real-valued SISO propagation channel affected by Nakagami-$m$ fading with integer $m$ [13], using the result from [14, eq. (6)].

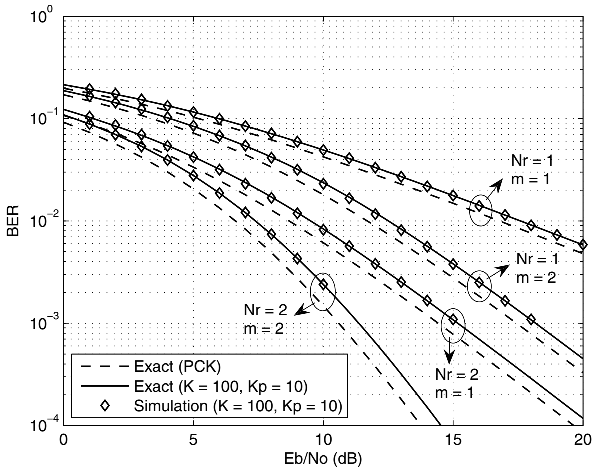


Fig. 1. Complex-valued Nakagami-$m$ fading channel, 16-QAM

## VI. Numerical results

In this section, we illustrate the exact BER analysis by comparing several analytical BER curves resulting from numerical evaluation with straightforward simulation results. We assume linear MMSE channel estimation ($E_\mathrm{p} = E_\mathrm{s}$) at the receiver and $N_\mathrm{r}$ independent and identically distributed (i.i.d.) Nakagami-$m$ fading channels, with parameters $m$ and $\Omega$ [13].

Fig. 1 shows the exact BER curves for 16-QAM transmission over a complex-valued SIMO channel, for both the mismatched receiver and the PCK receiver, and for several values of $m$ and $N_\mathrm{r}$. It is easily derived that the norm $|\mathbf{h}|$ of the SIMO channel is distributed according to a Nakagami-$m$ distribution with parameters $N_\mathrm{r}m$ and $N_\mathrm{r}\Omega$. The curves corresponding to the mismatched receiver represent the numerically computed expectation over 4 variables resulting from (23), whereas the BER of the PCK receiver involves the expectation over $|\mathbf{h}|$ only. Also shown in the figure are straightforward computer simulation results for the mismatched receiver that confirm the result obtained from numerical averaging.

Fig. 2 illustrates the exact BER curves for BPSK transmission over a real-valued SISO channel, for both the mismatched receiver and the PCK receiver, and for several values of $m$. The curves corresponding to the mismatched receiver result from the closed-form expression derived in Section V. The BER of the PCK receiver can be computed in a similar manner. Computer simulation results for the mismatched receiver again confirm the analytical result.

## VII. Conclusions and remarks

In this contribution, we investigated the exact BER performance of $M$-QAM with MRC multichannel reception under the assumption of pilot symbol assisted channel estimation at the receiver. The $N_\mathrm{r}$ propagation channels between the

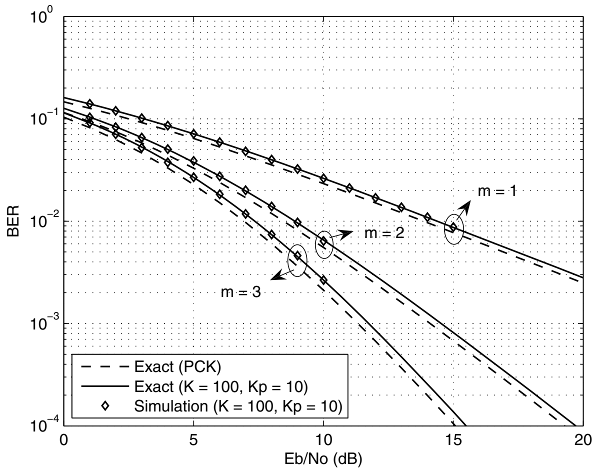


Fig. 2. Real-valued Nakagami-$m$ fading channel ($N_{\mathrm{r}} = 1$), BPSK

transmitter and each of the receive antennas were assumed to be affected by (possibly correlated) flat block fading with arbitrary pdf. The main conclusions are the following:

- A conceptually simple BER expression can be obtained in the form of an expectation over $4N_{\mathrm{r}}$ real-valued random variables. However, the computing time needed for its numerical evaluation increases exponentially with $N_{\mathrm{r}}$.
- We have reduced the BER expression to an expectation over only $4$ variables, irrespective of the value of $N_{\mathrm{r}}$. This expression depends on the distribution of only the norm of the channel vector, instead of the joint distribution of all complex-valued channel gains. Moreover, for some cases (SISO, real-valued channel, PAM constellation, BPSK constellation, and combinations thereof) the number of variables involved in the expectation can be further reduced.
- Evaluation of the reduced BER expression generally requires a (4-fold or less) numerical integration. This integration can be carried out when the pdf of the norm of the channel vector is available in closed form or has been obtained from experiments (histogram). Comparing the computing times resulting from the numerical averaging and from straightforward simulation, it turns out that the numerical averaging is to be preferred for operating BER values of practical interest.

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