Exact BER Analysis for Orthogonal Space-Time Block Codes on Arbitrary Fading Channels with Imperfect Channel Estimation

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Abstract— **In this contribution, we provide an exact bit error rate (BER) analysis of orthogonal space-time block codes over arbitrarily distributed (possibly correlated) block fading channels with imperfect channel estimation. The transmitted symbols belong to a PAM or QAM signal constellation. With N_{t} and N_{r} denoting the number of transmit and receive antennas, respectively, the resulting average BER can easily be written as an expectation over $4N_{\mathrm{t}}N_{\mathrm{r}}$ real-valued random variables, but the computation time needed for its numerical evaluation increases exponentially with N_{t} and N_{r}. We provide a methodology that allows an efficient numerical evaluation of the exact BER with a computation time that is essentially independent of $N_{\mathrm{t}}N_{\mathrm{r}}$ and the signal-to-noise ratio.**

I. Introduction

The performance of wireless communication systems can be strongly enhanced by exploiting one or more diversity techniques which provide the receiver with independent replicas of the transmitted signal. A maximum diversity order of $N_{\mathrm{t}}N_{\mathrm{r}}$ (with N_{t} and N_{r} denoting the number of transmit and receive antennas, respectively) is achieved when applying the well-known transmit diversity technique of orthogonal space-time block codes (OSTBCs) [1]–[3] along with maximum-likelihood (ML) detection. Moreover, the ML detection algorithm reduces to symbol-by-symbol detection based only on linear processing at the receiver. Under the assumption of perfect channel knowlegde (PCK), numerous papers (e.g. [4]–[7]) deal with the bit error rate (BER) performance of OSTBCs.

Practical wireless systems, however, suffer from degraded BER performance as the receiver has to estimate the channel. Usually, systems with imperfect channel estimation (ICE) are investigated under the assumption of Rayleigh fading: an analytical expression for the BER of OSTBCs with minimum mean-square error (MMSE) channel estimation and pulse amplitude modulation (PAM) or quadrature amplitude modulation (QAM) constellations, was derived in [8]; for M-ary phase-shift keying (M-PSK) constellations, analytical BER expressions as well as the tight Chernoff bound were given in [9]; in [10], an eigenvalue approach was adopted in order to derive asymptotical pairwise error probability (PEP) expressions for STBCs with coherent and noncoherent receivers; by means of characteristic functions, an exact closed-form expression for the PEP of both orthogonal and non-orthogonal space-time codes was obtained in [11] in the case of least-squares (LS) channel estimation. In a fading environment, however, the PEP does not converge to the BER at high signal-to-noise ratio (SNR), such that only an upper bound on the BER can be derived. For PSK constellations, the exact decoding error probability (DEP) of OSTBCs under arbitrary fading conditions was presented in [12]. In spite of the extensive research on the effect of ICE on the performance of OSTBCs, a numerically efficient way to obtain the exact BER under generalized fading conditions and with PAM or QAM constellations, is still lacking.

In this contribution, we provide an exact BER analysis for OSTBCs on arbitrary flat-fading channels with ICE, for PAM and QAM constellations. For non-square OSTBCs, the resulting BER expression can be efficiently evaluated using a combination of numerical integration (over 3 variables) and Monte-Carlo integration (over $2N_{\mathrm{t}}N_{\mathrm{r}}$ variables) with importance sampling. Moreover, for square OSTBCs the evaluation of the BER requires numerical integration over 4 variables but no Monte-Carlo integration. Because of the high diversity order of OSTBCs, the numerical evaluation of the resulting BER expressions is much less time-consuming than a straightforward computer simulation.

Throughout this paper, the superscripts T and H represent the vector (matrix) transpose and conjugate transpose, respectively. $\mathbb{E}[x]$ denotes the expected value of x, while $\|\mathbf{X}\|$ and $\mathrm{tr}\,(\mathbf{X})$ refer to the Frobenius norm and the trace of $\mathbf{X}$, respectively.

II. Signal model

We consider a multiple-input multiple-output (MIMO) wireless communication system where the propagation channels between each of the N_{t} transmit and N_{r} receive antennas are affected by flat fading with an arbitrary distribution. The transmitted data frames consist of K_{p} known pilot symbols and K coded information symbols per transmit antenna; the pilot symbols enable channel estimation at the receiver. Within one frame of $N_{\mathrm{fr}} = K + K_{\mathrm{p}}$ symbols, the channel is assumed to remain constant (block fading), such that the $N_{\mathrm{r}} \times N_{\mathrm{fr}}$ received signal matrix $\mathbf{R}_{\mathrm{tot}}$ is given by

$$\mathbf{R}_{\mathrm{tot}} = [\mathbf{R}_{\mathrm{p}}, \mathbf{R}] = \mathbf{H}\,[\mathbf{A}_{\mathrm{p}}, \mathbf{A}] + [\mathbf{W}_{\mathrm{p}}, \mathbf{W}]\,, \tag{1}$$

where the $N_\mathrm{t} \times K_\mathrm{p}$ pilot matrix $\mathbf{A}_\mathrm{p}$ and the $N_\mathrm{t} \times K$ data matrix $\mathbf{A}$ consist of the pilot symbols and the coded data symbols, respectively, transmitted at each transmit antenna. The $N_\mathrm{r} \times N_\mathrm{t}$ channel matrix $\mathbf{H}$ consists of the $N_\mathrm{t} N_\mathrm{r}$ (possibly correlated) complex fading gains, distributed according to an arbitrary joint pdf $p(\mathbf{H})$. The entries of the $N_\mathrm{r} \times N_\mathrm{fr}$ noise matrix $[\mathbf{W}_\mathrm{p}, \mathbf{W}]$ represent independent and identically distributed (i.i.d.) zero-mean circularly symmetric complex Gaussian (ZMCSCG) random variables with variance N_0.

We consider OSTBCs from complex linear processing orthogonal designs [2], which are defined by a $N_\mathrm{t} \times K_\mathrm{c}$ coded symbol matrix $\mathbf{C}$, whose entries are linear combinations of N_s information symbols $s_1, s_2, \cdots, s_{N_\mathrm{s}}$ and their complex conjugates. Hence, $\mathbf{C}$ can be written as

$$\mathbf{C} = \sum_{i=1}^{N_\mathrm{s}} \left(\mathbf{C}_i s_i + \mathbf{C}_i^{'} s_i^* \right), \tag{2}$$

where the $N_\mathrm{t} \times K_\mathrm{c}$ matrices $\mathbf{C}_i$ and $\mathbf{C}_i^{'}$ comprise the coefficients of s_i and s_i^* in the matrix $\mathbf{C}$, respectively. Assuming that K is a multiple of K_c, the transmitted data symbol matrix $\mathbf{A} = \sqrt{E_\mathrm{s}}\,[\mathbf{C}(1), \cdots, \mathbf{C}(K/K_\mathrm{c})]$ then consists of K/K_c coded symbol matrices $\mathbf{C}(k)$, with k denoting the block index. Also, without loss of generality, we assume that the matrices $\mathbf{C}$ are scaled in such way that they satisfy the following orthogonality condition (we omit the block index k for notational convenience):

$$\mathbf{C}\mathbf{C}^H = \left(\lambda \sum_{i=1}^{N_\mathrm{s}} |s_i|^2 \right) \mathbf{I}_{N_\mathrm{t}}, \tag{3}$$

where $\lambda = K_\mathrm{c}/N_\mathrm{s}$ and $\mathbf{I}_{N_\mathrm{t}}$ is the $N_\mathrm{t} \times N_\mathrm{t}$ identity matrix. In this way, the average energy of the transmitted coded symbols is given by E_s when adopting a normalized symbol constellation ($\mathbb{E}[|s_i|^2] = 1$) for the information symbols. In the same way, E_p denotes the average energy of the pilot symbols.

III. Pilot-based channel estimation

We consider orthogonal pilot sequences, i.e. $\mathbf{A}_\mathrm{p}\mathbf{A}_\mathrm{p}^H = K_\mathrm{p} E_\mathrm{p} \mathbf{I}_{N_\mathrm{t}}$, and linear channel estimates of the form

$$\hat{\mathbf{H}} = \frac{\alpha}{K_\mathrm{p} E_\mathrm{p}} \mathbf{R}_\mathrm{p} \mathbf{A}_\mathrm{p}^H, \tag{4}$$

with $\alpha \in \mathbb{R}$. Substituting $\mathbf{R}_\mathrm{p}$ from (1) in (4) yields

$$\hat{\mathbf{H}} = \alpha \mathbf{H} + \mathbf{N}, \tag{5}$$

where the estimation noise term $\mathbf{N} = (\alpha/(K_\mathrm{p} E_\mathrm{p}))\, \mathbf{W}_\mathrm{p} \mathbf{A}_\mathrm{p}^H$ is independent of $\mathbf{H}$ and its entries represent ZMCSCG random variables, whose real and imaginary parts have a variance $\sigma_\mathbf{n}^2 = \alpha^2 N_0/(2K_\mathrm{p} E_\mathrm{p})$. Hence, when conditioned on the channel $\mathbf{H}$, the estimated channel coefficients $\hat{\mathbf{H}}_{i,j}$ ($1 \le i \le N_\mathrm{r}, 1 \le j \le N_\mathrm{t}$) are circularly symmetric complex Gaussian random variables with mean $\alpha \mathbf{H}_{i,j}$ and variance $2\sigma_\mathbf{n}^2$. Both LS and linear MMSE estimation satisfy (4) with $\alpha = 1$ and $\alpha = K_\mathrm{p} E_\mathrm{p}/(K_\mathrm{p} E_\mathrm{p} + N_0)$, respectively [13].

Increasing the total energy of the pilot symbols yields an accurate channel estimate, but reduces the symbol energy E_s available for data transmission. Denoting by E_b and $\gamma = E_\mathrm{p}/E_\mathrm{s}$ the energy per information bit and the ratio of the pilot energy E_p versus the symbol energy E_s, respectively, we have

$$E_\mathrm{s} = \frac{K}{K + \gamma K_\mathrm{p}} \rho \log_2(M) E_\mathrm{b}, \tag{6}$$

where $\rho = N_\mathrm{s}/(N_\mathrm{t} K_\mathrm{c})$ and M denote the code rate and the number of constellation points, respectively.

IV. ML Detection

We consider a mismatched ML receiver that uses $\hat{\mathbf{H}}$ in the same way an ML receiver would apply $\mathbf{H}$. Taking (3) into account, it can be shown that the detection algorithm for the information symbols s_i of a transmitted coded symbol matrix $\mathbf{C}$ reduces to symbol-by-symbol detection:

$$\hat{s}_i = \arg\min_{\tilde{s}} |u_i - \tilde{s}|\,,\ 1 \le i \le N_\mathrm{s}, \tag{7}$$

where the decision variables u_i are given by

$$u_i = \frac{\mathrm{tr}\left(\mathbf{C}_i^H \hat{\mathbf{H}}^H \mathbf{R}' + \mathbf{R}'^H \hat{\mathbf{H}} \mathbf{C}_i^{'} \right)}{\lambda \sqrt{E_\mathrm{s}} \left\| \hat{\mathbf{H}} \right\|^2}, \tag{8}$$

with $\mathbf{R}'$ denoting the received signal matrix corresponding to the transmitted coded symbol matrix $\mathbf{C}$.

V. Bit error rate analysis

In this section, we derive the BER for square M-QAM transmission with Gray mapping, which reduces to $\sqrt{M}$-PAM transmission for both the in-phase and quadrature information bits. The BER for M-PAM with Gray mapping can be obtained is a similar manner. For square M-QAM and equally likely symbol vectors $\mathbf{s} = (s_1, \cdots, s_{N_\mathrm{s}})$, the BERs related to the in-phase and quadrature bits of s_i are equal, and independent of the index i. Hence, the average BER for OSTBCs with M-QAM signaling equals the BER related to the in-phase information bits of the transmitted symbol s_i, irrespective of i. The latter BER is obtained by averaging the conditional BER related to the in-phase information bits of s_i (conditioned on the channel $\mathbf{H}$ and the channel estimate $\hat{\mathbf{H}}$):

$$\mathrm{BER} = \mathbb{E}_{\mathbf{H},\hat{\mathbf{H}}} \left[\mathrm{BER}_{i,\mathrm{R}}(\mathbf{H}, \hat{\mathbf{H}}) \right]. \tag{9}$$

Denoting by $\mathrm{BER}_{i,\mathrm{R}}(\mathbf{s}, \mathbf{H}, \hat{\mathbf{H}})$ the conditional BER related to the in-phase information bits of s_i (conditioned on $\mathbf{H}$, $\hat{\mathbf{H}}$ and the transmitted symbol vector $\mathbf{s}$), $\mathrm{BER}_{i,\mathrm{R}}(\mathbf{H}, \hat{\mathbf{H}})$ is given by

$$\mathrm{BER}_{i,\mathrm{R}}(\mathbf{H}, \hat{\mathbf{H}}) = \frac{1}{M^{N_\mathrm{s}}} \sum_{\mathbf{s} \in \Psi^{N_\mathrm{s}}} \mathrm{BER}_{i,\mathrm{R}}(\mathbf{s}, \mathbf{H}, \hat{\mathbf{H}}), \tag{10}$$

where Ψ denotes the normalized M-QAM constellation. By referring to the projection (on the real axis) of the decision area of a QAM symbol $b = b_\mathrm{R} + jb_\mathrm{I}$ as the decision region of b_R, we can write $\mathrm{BER}_{i,\mathrm{R}}(\mathbf{s}, \mathbf{H}, \hat{\mathbf{H}})$ as

$$\mathrm{BER}_{i,\mathrm{R}}(\mathbf{s}, \mathbf{H}, \hat{\mathbf{H}}) = \sum_{b_\mathrm{R} \in \Psi_\mathrm{R}} \frac{N(s_{i,\mathrm{R}}, b_\mathrm{R})}{\frac{1}{2}\log_2 M} P_{i,\mathrm{R}}(\mathbf{s}, b_\mathrm{R}, \mathbf{H}, \hat{\mathbf{H}}), \tag{11}$$

where Ψ_{R} contains the real parts of the M-QAM constellation points and $N(s_{i,\mathrm{R}}, b_{\mathrm{R}})$ denotes the Hamming distance between the in-phase bits of the transmitted symbol s_i and the in-phase bits of the detected symbol b. $P_{i,\mathrm{R}}(\mathbf{s}, b_{\mathrm{R}}, \mathbf{H}, \hat{\mathbf{H}})$ is the probability that the real part of the decision variable (8) is located inside the decision area of b_{R}, when the transmitted symbol vector $\mathbf{s}$, the channel $\mathbf{H}$ and the channel estimate $\hat{\mathbf{H}}$ are known:

$$P_{i,\mathrm{R}}(\mathbf{s}, b_{\mathrm{R}}, \mathbf{H}, \hat{\mathbf{H}}) = Pr\left[\hat{s}_{i,\mathrm{R}} = b_{\mathrm{R}} \middle| \mathbf{s}, \mathbf{H}, \hat{\mathbf{H}}\right]. \quad (12)$$

From (1) and (2), it follows that expanding the real part $u_{i,\mathrm{R}}$ of the decision variable (8) yields:

$$u_{i,\mathrm{R}} = u'_{i,\mathrm{R}} + n_{i,\mathrm{R}}, \quad (13)$$

where $u'_{i,\mathrm{R}}$ is a function of the transmitted symbol vector $\mathbf{s}$, the channel matrix $\mathbf{H}$ and the estimated channel matrix $\hat{\mathbf{H}}$ and $n_{i,\mathrm{R}}$ represents zero-mean Gaussian noise with variance $N_0/(2\lambda E_{\mathrm{s}}\|\hat{\mathbf{H}}\|^2)$. Denoting by $d_1(b_{\mathrm{R}})$ and $d_2(b_{\mathrm{R}})$ the boundaries of the decision area of b_{R}, with $d_1(b_{\mathrm{R}}) < d_2(b_{\mathrm{R}})$ (we set $d_1(b_{\mathrm{R}}) = -\infty$ ($d_2(b_{\mathrm{R}}) = \infty$) if b is a left (right) outer constellation point), the probability (12) reduces to

$$P_{i,\mathrm{R}}(\mathbf{s}, b_{\mathrm{R}}, \mathbf{H}, \hat{\mathbf{H}}) = Q_1 - Q_2, \quad (14)$$

where the quantities Q_k, $k \in \{1, 2\}$, are given by

$$Q_k = Q\left(\sqrt{2\lambda\frac{E_{\mathrm{s}}}{N_0}}\|\hat{\mathbf{H}}\|\left(d_k(b_{\mathrm{R}}) - u'_{\mathrm{i,R}}\right)\right), \quad (15)$$

and $Q(.)$ is defined as the complementary cumulative distribution function of a zero-mean Gaussian random variable with unit variance. Hence, the BER (9) for OSTBCs with M-QAM signaling can be expressed as

$$\mathrm{BER} = \int \mathrm{BER}_{i,\mathrm{R}}(\mathbf{H}, \hat{\mathbf{H}})p(\hat{\mathbf{H}}|\mathbf{H})p(\mathbf{H})\mathrm{d}\hat{\mathbf{H}}\mathrm{d}\mathbf{H}, \quad (16)$$

where $p(\hat{\mathbf{H}}|\mathbf{H})$ is the joint Gaussian pdf of $N_{\mathrm{t}}N_{\mathrm{r}}$ complex-valued random variables, and $p(\mathbf{H})$ represents the arbitrary joint distribution of $N_{\mathrm{t}}N_{\mathrm{r}}$ complex-valued fading gains. As the numerical evaluation of the general expression (16) requires taking the expectation over $4N_{\mathrm{t}}N_{\mathrm{r}}$ real-valued variables, the associated computation time increases exponentially with the number of transmit and receive antennas; this makes (16) unsuitable for numerical evaluation.

However, the computational complexity related to the evaluation of (16) can be decreased significantly by reducing the number of random variables involved in the expectation (16). To this end, we introduce the $(2N_{\mathrm{t}}N_{\mathrm{r}} \times 1)$ real-valued column vectors $\hat{\mathbf{h}}$ and $\mathbf{h}$, which contain all elements of $\hat{\mathbf{H}}$ and $\mathbf{H}$, respectively:

$$\begin{cases} \hat{\mathbf{h}} = \left[\hat{\mathbf{h}}_{1,\mathrm{R}}^T, \hat{\mathbf{h}}_{1,\mathrm{I}}^T, \cdots, \hat{\mathbf{h}}_{N_{\mathrm{s}},\mathrm{R}}^T, \hat{\mathbf{h}}_{N_{\mathrm{s}},\mathrm{I}}^T\right]^T \\ \mathbf{h} = \left[\mathbf{h}_{1,\mathrm{R}}^T, \mathbf{h}_{1,\mathrm{I}}^T, \cdots, \mathbf{h}_{N_{\mathrm{s}},\mathrm{R}}^T, \mathbf{h}_{N_{\mathrm{s}},\mathrm{I}}^T\right]^T \end{cases}, \quad (17)$$

with $\hat{\mathbf{h}}_{i,\mathrm{R}} + j\hat{\mathbf{h}}_{i,\mathrm{I}}$ and $\mathbf{h}_{i,\mathrm{R}} + j\mathbf{h}_{i,\mathrm{I}}$ denoting the i-th column of $\hat{\mathbf{H}}$ and $\mathbf{H}$, respectively. It is easily seen that $|\hat{\mathbf{h}}| = \|\hat{\mathbf{H}}\|$ and $|\mathbf{h}| = \|\mathbf{H}\|$. Under the assumption of real-valued coefficient matrices $\mathbf{C}_i$, $\mathbf{C}_j$, $\mathbf{C}_i'$ and $\mathbf{C}_j'$, it can be shown that $u'_{i,\mathrm{R}}$ in (15) can be written as

$$u'_{i,\mathrm{R}} = \frac{1}{\lambda|\hat{\mathbf{h}}|^2}\sum_{j=1}^{N_{\mathrm{s}}}\left[s_{j,\mathrm{R}}\left(\hat{\mathbf{h}}^T\mathbf{g}_{i,\mathrm{R}}(j)\right) - s_{j,\mathrm{I}}\left(\hat{\mathbf{h}}^T\mathbf{g}_{i,\mathrm{I}}(j)\right)\right], \quad (18)$$

where the $(2N_{\mathrm{t}}N_{\mathrm{r}} \times 1)$ column vectors $\mathbf{g}_{i,\mathrm{R}}(j)$ and $\mathbf{g}_{i,\mathrm{I}}(j)$, $1 \le j \le N_{\mathrm{s}}$, are given by

$$\begin{cases} \mathbf{g}_{i,\mathrm{R}}(j) = \mathbf{M}_{i,\mathrm{R}}(j)\mathbf{h} \\ \mathbf{g}_{i,\mathrm{I}}(j) = \mathbf{M}_{i,\mathrm{I}}(j)\mathbf{h} \end{cases}, \quad (19)$$

and the $(2N_{\mathrm{t}}N_{\mathrm{r}} \times 2N_{\mathrm{t}}N_{\mathrm{r}})$ matrices $\mathbf{M}_{i,\mathrm{R}}(j)$ and $\mathbf{M}_{i,\mathrm{I}}(j)$ depend on the coefficient matrices $\mathbf{C}_i$, $\mathbf{C}_j$, $\mathbf{C}_i'$ and $\mathbf{C}_j'$. For any given i, it can be shown that the vectors $\mathbf{g}_{i,\mathrm{R}}(j)$, with $j \neq i$, and $\mathbf{g}_{i,\mathrm{I}}(j)$ are orthogonal to $\mathbf{g}_{i,\mathrm{R}}(i) = \lambda\mathbf{h}$:

$$\begin{cases} \mathbf{g}_{i,\mathrm{R}}(i)^T\mathbf{g}_{i,\mathrm{R}}(j) = \lambda^2|\mathbf{h}|^2\delta_{i-j} \\ \mathbf{g}_{i,\mathrm{R}}(i)^T\mathbf{g}_{i,\mathrm{I}}(j) = 0 \end{cases}, \quad (20)$$

where δ_{i-j} denotes the Kronecker delta. The term in $s_{i,\mathrm{R}}$ in (18) is the useful term, whereas the term in $s_{i,\mathrm{I}}$ and the terms in $s_{j,\mathrm{R}}$ and $s_{j,\mathrm{I}}$, with $j \neq i$, represent interference from the quadrature component of s_i, and the in-phase and quadrature components of the transmitted symbols s_j, respectively. In case of PCK ($\hat{\mathbf{h}} = \mathbf{h}$), it is readily verified that $u'_{i,\mathrm{R}} = s_{i,\mathrm{R}}$. Let us now introduce the $(2N_{\mathrm{t}}N_{\mathrm{r}} \times 2N_{\mathrm{t}}N_{\mathrm{r}})$ matrix $\mathbf{M}$

$$\mathbf{M} = \sum_{\substack{j=1 \\ j\neq i}}^{N_s}\left(s_{j,\mathrm{R}}\mathbf{M}_{i,\mathrm{R}}(j)\right) - \sum_{j=1}^{N_s}\left(s_{j,\mathrm{I}}\mathbf{M}_{i,\mathrm{I}}(j)\right), \quad (21)$$

which is a function of the transmitted symbol vector $\mathbf{s}$ and the coefficient matrices $\mathbf{C}_i$, $\mathbf{C}_j$, $\mathbf{C}_i'$ and $\mathbf{C}_j'$ through the matrices $\mathbf{M}_{i,\mathrm{R}}(j)$ and $\mathbf{M}_{i,\mathrm{I}}(j)$. In this way, (18) reduces to

$$u'_{i,\mathrm{R}} = \frac{1}{\lambda|\hat{\mathbf{h}}|^2}\left(\lambda s_{i,\mathrm{R}}\hat{\mathbf{h}}^T\mathbf{h} + \hat{\mathbf{h}}^T\mathbf{M}\mathbf{h}\right). \quad (22)$$

Because it follows from (19) and (20) that $\mathbf{M}\mathbf{h}$ is orthogonal to $\mathbf{h}$, we can define an orthonormal coordinate system with $2N_{\mathrm{t}}N_{\mathrm{r}}$ unit vectors $\{\mathbf{e}_n, n = 1, \cdots, 2N_{\mathrm{t}}N_{\mathrm{r}}\}$, with $\mathbf{e}_1$ and $\mathbf{e}_2$ directed along $\mathbf{h}$ and $\mathbf{M}\mathbf{h}$, respectively. The projections of $\hat{\mathbf{h}}$ on $\mathbf{e}_n$ are denoted x_n, which yields

$$|\hat{\mathbf{h}}|^2 = x_1^2 + x_2^2 + z^2, \quad (23)$$

with

$$z^2 = \sum_{n=3}^{2N_{\mathrm{t}}N_{\mathrm{r}}} x_n^2. \quad (24)$$

Because of the specific choice of $\mathbf{e}_1$ and $\mathbf{e}_2$, (22) reduces to

$$u'_{i,\mathrm{R}} = \frac{1}{\lambda(x_1^2 + x_2^2 + z^2)}\left(\lambda x_1 s_{i,\mathrm{R}}|\mathbf{h}| + x_2\|\mathbf{M}\mathbf{h}\|\right), \quad (25)$$

such that $\mathrm{BER}_{i,\mathrm{R}}(\mathbf{H}, \hat{\mathbf{H}})$ depends on the actual channel $\mathbf{H}$ through the random vector $\mathbf{h}$, and on the estimated channel $\hat{\mathbf{H}}$ through only 3 random variables: x_1, x_2 and z. Hence, we denote $\mathrm{BER}_{i,\mathrm{R}}(\mathbf{H}, \hat{\mathbf{H}})$ by $\mathrm{BER}_{i,\mathrm{R}}(x_1, x_2, z, \mathbf{h})$. Taking

(5) into account, it can be shown that x_1, x_2 and z, when conditioned on $\mathbf{h}$, are independent variables which satisfy the following properties:

- x_1 is a Gaussian variable with mean $\alpha|\mathbf{h}|$ and variance $\sigma_{\mathbf{n}}^2$.
- x_2 is a zero-mean Gaussian variable with variance $\sigma_{\mathbf{n}}^2$.
- $z/\sigma_{\mathbf{n}}$ is distributed according to the chi-distribution with $2N_{\mathrm{t}}N_{\mathrm{r}} - 2$ degrees of freedom [14].

Taking the above properties into account, the BER (16) reduces to an expectation over $2N_{\mathrm{t}}N_{\mathrm{r}} + 3$ random variables (i.e., x_1, x_2, z and the $2N_{\mathrm{t}}N_{\mathrm{r}}$ components of $\mathbf{h}$). Hence, the computation time needed for its numerical evaluation increases exponentially with $N_{\mathrm{t}}N_{\mathrm{r}}$. However, the expectation over $\mathbf{h}$ in (16) can be approximated by Monte-Carlo integration with importance sampling (IS) [15], yielding:

$$\mathrm{BER} = \frac{1}{L}\sum_{l=1}^{L}\int \mathrm{BER}_{i,\mathrm{R}}\left(x_1, x_2, z, \mathbf{h}_l\right) p\left(x_1, x_2, z\middle|\mathbf{h}_l\right)\frac{p(\mathbf{h}_l)}{q(\mathbf{h}_l)}\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}z, \quad (26)$$

where $p(\mathbf{h})$ denotes the joint distribution of the channel vector $\mathbf{h}$, $p\left(x_1, x_2, z\middle|\mathbf{h}\right)$ denotes the joint distribution of x_1, x_2 and z, conditioned on $\mathbf{h}$, and the L samples $\mathbf{h}_l$ are generated according to a joint sampling distribution $q(\mathbf{h})$. For each realization of $\mathbf{h}_l$, the expectation over x_1, x_2 and z can be evaluated numerically by approximating the 3-fold integral by a 3-fold sum, running over discretized versions of the continuous variables. In the case of traditional Monte-Carlo integration, with $q(\mathbf{h}) \triangleq p(\mathbf{h})$, the variance of the terms in (26) may become very large as compared to their squared mean, requiring a large number of samples L. However, the variance reduction technique of IS allows to minimize L, by selecting an optimal sampling pdf $q(\mathbf{h})$ that reduces the variance of the terms in (26). In this way, the number of samples L required to obtain an accurate BER result (26) is much less than for traditional Monte-Carlo integration.

Although the BER expression (26) allows for efficient numerical evaluation, the BER computation can be strongly simplified for *square OSTBCs* ($N_{\mathrm{t}} = K_{\mathrm{c}}$), which satisfy

$$\mathbf{C}^H\mathbf{C} = \mathbf{C}\mathbf{C}^H = \left(\lambda\sum_{i=1}^{N_{\mathrm{s}}}|s_i|^2\right)\mathbf{I}_{N_{\mathrm{t}}}. \quad (27)$$

For these OSTBCs, the vectors $\mathbf{g}_{i,\mathrm{R}}(j)$ and $\mathbf{g}_{i,\mathrm{I}}(j)$ in (18) are mutually orthogonal, such that the orthogonality conditions (20) reduce to

$$\begin{cases}\mathbf{g}_{i,\mathrm{R}}(j)^T\mathbf{g}_{i,\mathrm{I}}(j') = 0\\ \mathbf{g}_{i,\mathrm{R}}(j)^T\mathbf{g}_{i,\mathrm{R}}(j') = \lambda^2|\mathbf{h}|^2\delta_{j-j'}\\ \mathbf{g}_{i,\mathrm{I}}(j)^T\mathbf{g}_{i,\mathrm{I}}(j') = \lambda^2|\mathbf{h}|^2\delta_{j-j'}\end{cases}. \quad (28)$$

Taking the above properties into account, the Frobenius norm of $\mathbf{Mh}$ in (25) is shown to be $\|\mathbf{Mh}\| = \lambda|\mathbf{h}|\sqrt{|\mathbf{s}|^2 - s_{i,\mathrm{R}}^2}$. Hence, $\mathrm{BER}_{i,\mathrm{R}}(\mathbf{H}, \hat{\mathbf{H}})$ is a function of only 4 random variables: x_1, x_2, z and the norm $|\mathbf{h}|$ of the channel vector $\mathbf{h}$. Denoting the latter conditional BER by $\mathrm{BER}_{i,\mathrm{R}}(x_1, x_2, z, |\mathbf{h}|)$, the BER (16) for square OSTBCs reduces to

$$\mathrm{BER} = \int \mathrm{BER}_{i,\mathrm{R}}\left(x_1, x_2, z, u\right) p\left(x_1, x_2, z\middle||\mathbf{h}| = u\right)p(u)\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}z\mathrm{d}u, \quad (29)$$

where $p(u)$ is the pdf of $|\mathbf{h}|$. It is important to note that instead of the joint distribution $p(\mathbf{h})$ of the $2N_{\mathbf{t}}N_{\mathbf{r}}$ real-valued fading gains, we need only the distribution of the norm $|\mathbf{h}|$. The BER (29) can be evaluated numerically by approximating the 4-fold integral by a 4-fold sum, running over discretized versions of the continuous variables x_1, x_2, z and u; the associated computation time is independent of $N_{\mathrm{t}}N_{\mathrm{r}}$ and the SNR. Because the variable u in (29) is to be discretized, the pdf $p(u)$ of $|\mathbf{h}|$ can be available either in analytical form or as a (discrete) histogram obtained from experiments. Examples of square OSTBCs satisfying (27) are the 2×2 space-time block code proposed by Alamouti [1] and the 4×4 codes given in [2, eq. 40], [3, eq. 62] and [16, eq. 41].

VI. Numerical results

To obtain our numerical results, we consider a MIMO propagation channel consisting of $N_{\mathrm{t}}N_{\mathrm{r}}$ i.i.d. complex-valued channel coefficients with uniformly distributed phases and fading evelopes that are distributed according to the Nakagami-m distribution, with parameters m and Ω [17]. The Nakagami-m distribution is known to accurately model a broad variety of fading environments and includes both the Rayleigh distribution ($m = 1$) and the one-sided Gaussian distribution ($m = 0.5$).

Fig. 1 displays the exact BER curves for a 2×2 MIMO system employing Alamouti's code [1] ($N_{\mathrm{t}} = K_{\mathrm{c}} = N_{\mathrm{s}} = 2$), along with 16-QAM signaling. The results are shown for both a PCK receiver and a mismatched receiver employing LS channel estimation with blocks of $K = 200$ information symbols and $K_{\mathrm{p}} = 20$ pilot symbols, and for several values of the fading parameter m. The BER curves for the mismatched receiver are obtained by numerically computing the expectation (29), whereas the BER of the PCK receiver involves the expectation over $|\mathbf{h}|$ only. It is readily verified that the norm $|\mathbf{h}|$ of the 2×2 MIMO channel is distributed according to a Nakagami-m distribution with parameters $4m$ and 4Ω. Also shown in the figure are straightforward computer simulation results for the mismatched receiver that confirm the analytical results obtained from numerical evaluation.

Fig. 2 displays the exact BER curves for the non-square OSTBC given by [2, eq. 39] ($N_{\mathrm{t}} = N_{\mathrm{s}} = 3$, $K_{\mathrm{c}} = 4$), in case of 4-QAM signaling and $m = 2$. Both a PCK receiver and a mismatched receiver employing linear MMSE channel estimation with blocks of $K = 100$ information symbols and $K_{\mathrm{p}} = 16$ pilot symbols are considered, for several values of N_{r}. The BER curves for the mismatched receiver are obtained by numerically evaluating the expectation (26), where $q(\mathbf{h})$ corresponds to i.i.d. Nakagami-m random variables with parameters $m' = m$ and $\Omega' = \left(\frac{1}{\Omega} + \frac{\lambda d^2}{m}\frac{E_s}{N_0}\right)^{-1}$, with $d = \sqrt{3/(2(M-1))}$ denoting half the distance between adjacent

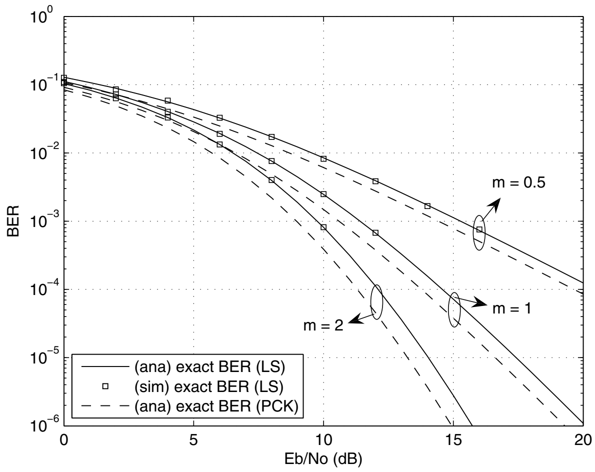

Fig. 1. BER for Alamouti's code on Nakagami-m fading channels with LS channel estimation (16-QAM, $N_{\mathrm{t}} = N_{\mathrm{r}} = 2$, $m \in \{0.5, 1, 2\}$). Both analytical results (ana) from numerical evaluation and simulations (sim) are shown.

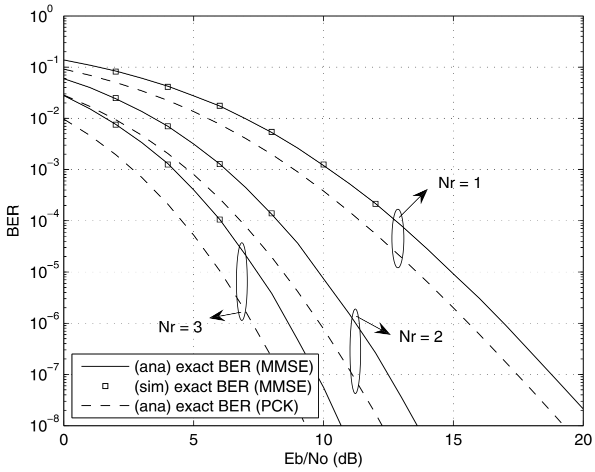

Fig. 2. BER for the non-square OSTBC given by [2, eq. 39] on Nakagami-m fading channels with linear MMSE channel estimation (4-QAM, $m = 2$, $N_{\mathrm{t}} = 3$, $N_{\mathrm{r}} \in \{1, 2, 3\}$). Both analytical results (ana) from numerical evaluation and simulations (sim) are shown.

M-QAM constellation points. In this way, a very accurate BER result can be obtained with $L = 100$ channel realizations, irrespective of N_{r} and the SNR. Again, computer simulation results for the mismatched receiver match the numerically obtained BER curves.

VII. Conclusions and Remarks

In this paper, we investigated the exact BER performance of OSTBCs on arbitrarily distributed (possibly correlated) fading channels with ICE. For PAM and QAM signal constellations, we proposed a methodology that allows an efficient numerical evaluation of the BER with a computation time that is essentially independent of $N_{\mathrm{t}} N_{\mathrm{r}}$ and the SNR.

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