Comments on "Exact Error Performance of Square Orthogonal Space-Time Block Coding with Channel Estimation"

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***Abstract*—In a recent paper [1], Garg *et al.* present an expression for the exact decoding error probability (DEP) of square orthogonal space-time block codes (OSTBCs) with imperfect channel estimation. We show that their DEP expression is only asymptotically correct and point out how to obtain the exact result for arbitrary signal-to-noise ratio.**

***Index Terms*—Space-time block coding, fading channels, channel estimation, error analysis.**

In [1], Garg *et al.* provide a general expression for the decoding error probability (DEP) of square linear orthogonal space-time block codes (OSTBCs) with $\mathcal{M}$-ary phase-shift keying ($\mathcal{M}$-PSK) signal constellations, on flat fading channels. Considering a wireless communication system with N transmit antennas and M receive antennas, the received $M \times N$ signal matrix $\mathbf{R}$ corresponding to the transmitted square OSTBC matrix $\mathbf{C}$ is given by

$$\mathbf{R} = \mathbf{HC} + \mathbf{N}, \tag{1}$$

where the $M \times N$ random channel matrix $\mathbf{H}$ and the $M \times N$ additive white Gaussian noise matrix $\mathbf{N}$ consist of i.i.d. zero-mean circularly symmetric complex Gaussian (ZMCSCG) random variables with variances Ω and $2N_0$, respectively. The entries of the $N \times N$ code matrix C depend linearly on K information symbols $s_1, \cdots, s_K$ and their complex conjugates, in such a way that

$$\mathbf{C}^H\mathbf{C} = \mathbf{C}\mathbf{C}^H = \|\mathbf{s}\|^2 \mathbf{I}_N, \tag{2}$$

where $\mathbf{I}_N$ is the $N \times N$ identity matrix and the symbol vector $\mathbf{s} = [s_1, s_2, \cdots, s_K]^T$ comprises the K information symbols.

Assuming least-squares (LS) or linear minimum mean-square error (MMSE) channel estimation from orthogonal pilot sequences ($\mathbf{C}_\mathrm{p}\mathbf{C}_\mathrm{p}^H = \beta\mathbf{I}_N$), it is shown in [1] that the channel estimate $\hat{\mathbf{H}}$ is given by

$$\hat{\mathbf{H}} = q\mathbf{H} + q\mathbf{N}_\mathrm{e}, \tag{3}$$

where the entries of $\mathbf{N}_\mathrm{e}$ are i.i.d. ZMCSCG random variables with variance $2N_0/\beta$, and q depends on the channel estimation strategy: $q = 1$ for LS estimation, and $q = (1 + 2N_0/(\Omega\beta))^{-1}$ for linear MMSE estimation.

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Maximum-likelihood (ML) detection boils down to the minimization of the objective function $\|\mathbf{R} - \hat{\mathbf{H}}\mathbf{C}(\mathbf{s})\|_F^2$ over all possible symbol vectors $\mathbf{s}$, with $\|\cdot\|_F$ denoting the Frobenius norm. Replacing in this objective function the channel estimate $\hat{\mathbf{H}}$ by (3) and the signal matrix $\mathbf{R}$ by $\mathbf{HC}(\tilde{\mathbf{s}}) + \mathbf{N}$ (with $\tilde{\mathbf{s}}$ denoting the symbol vector actually transmitted), gives rise to the decision variable $D_\mathbf{s}(\tilde{\mathbf{s}}, q)$ to be minimized over $\mathbf{s}$; $D_\mathbf{s}(\tilde{\mathbf{s}}, q)$ can be expressed as

$$D_\mathbf{s}(\tilde{\mathbf{s}}, q) = \|\tilde{\mathbf{s}} - q\mathbf{s} + \mathbf{w}(\mathbf{s}, q)\|^2 - \|\mathbf{w}(\mathbf{s}, q)\|^2 + \sum_{l=1}^{N} \|\mathbf{u}_l(\mathbf{s}, q)\|^2 . \tag{4}$$

Note that (4) is equivalent to [1, eq. (48)], with $\mathbf{z}(\mathbf{s}, q)\,\|\mathbf{H}\|_F^{-2}$ and $\mathbf{v}_l(\mathbf{s}, q)\,\|\mathbf{H}\|_F^{-1}$ replaced by $\mathbf{w}(\mathbf{s}, q)$ and $\mathbf{u}_l(\mathbf{s}, q)$, respectively, and the factor $\|\mathbf{H}\|_F^2$ (which does not affect the decision) removed. In (4), the entries of the vectors $\mathbf{u}_l(\mathbf{s}, q)$ and $\mathbf{w}(\mathbf{s}, q)$, conditioned on $\mathbf{H}$, are i.i.d. ZMCSCG random variables, with variances depending on the considered symbol vector $\mathbf{s}$ through $\|\mathbf{s}\|$ only. Hence, when considering $\mathcal{M}$-PSK constellations, these variances are independent of $\mathbf{s}$.

Our main comment pertains to an invalid simplification of the decision variable $D_\mathbf{s}(\tilde{\mathbf{s}}, q)$ which has been carried out in [1]. Considering that the variances of the ZMCSCG entries of $\mathbf{u}_l(\mathbf{s}, q)$ and $\mathbf{w}(\mathbf{s}, q)$ are independent of $\mathbf{s}$, Garg *et al.* neglect the dependency of the values of $\mathbf{u}_l(\mathbf{s}, q)$ and $\mathbf{w}(\mathbf{s}, q)$ on $\mathbf{s}$. Therefore, they make the following simplifications.

1) They drop the second and third terms in (4)
2) They replace $\mathbf{w}(\mathbf{s}, q)$ in the first term in (4) by a vector $\mathbf{w}(q)$, whose value does not depend on $\mathbf{s}$; the variances of the entries in $\mathbf{w}(\mathbf{s}, q)$ and $\mathbf{w}(q)$ are the same.

Based on the above approximations, the decision variable (4) is reduced to [1, eq. (51)]:

$$D_\mathbf{s}(\tilde{\mathbf{s}}, q) = \|\tilde{\mathbf{s}} - q\mathbf{s} + \mathbf{w}(q)\|^2 , \tag{5}$$

which allows to obtain the simple expression [1, eq. (52)] for the symbol error rate (SER), conditioned on $\mathbf{H}$. However, although their statistics are independent of $\mathbf{s}$, the respective values of $\mathbf{u}_l(\mathbf{s}, q)$ and $\mathbf{w}(\mathbf{s}, q)$ do depend on $\mathbf{s}$; therefore, their dependency on $\mathbf{s}$ should be preserved when minimizing $D_\mathbf{s}(\tilde{\mathbf{s}}, q)$ over $\mathbf{s}$. Hence, the approach from [1] does not yield the exact error performance.

In order to clearly illustrate the impact of these approximations we consider the simple example of uncoded single-input single-output (SISO) binary phase-shift keying (BPSK) ($\mathcal{M} = 2$) transmission with LS channel estimation ($q = 1$). In this case, the received signal (1) corresponding to the transmitted symbol $\tilde{s}$, and the LS channel estimate (3) reduce to $r = h\tilde{s} + n$ and $\hat{h} = h + n_\mathrm{e}$, respectively. For this example, it is easily verified that the correct version (4) of the decision variable reduces to

$$D_s(\tilde{s}) = |\tilde{s} - s + w(s)|^2, \tag{6}$$

with

$$w(s) = \frac{n - n_e s}{h}. \tag{7}$$

Clearly, the value of $w(s)$ from (6) depends on s, although its variance is independent of s. Taking the BPSK constellation into account, $D_s(\tilde{s})$ has to be minimized over $s \in \{\tilde{s}, -\tilde{s}\}$. Hence, a symbol error occurs if

$$D_{\tilde{s}}(\tilde{s}) > D_{-\tilde{s}}(\tilde{s}), \tag{8}$$

which reduces to the condition

$$\Re\{\tilde{s}w^*(-\tilde{s})\} < -|\tilde{s}|^2 + \frac{|w(\tilde{s})|^2 - |w(-\tilde{s})|^2}{4}. \tag{9}$$

When neglecting the s-dependency of $w(s)$, the decision variable (6) is approximated by (see [1, eq. (38)])

$$D_{\mathbf{s}}(\tilde{s}) = |\tilde{s} - s + w|^2, \tag{10}$$

and the condition for a symbol error to occur then becomes

$$\Re\{\tilde{s}w^*\} < -|\tilde{s}|^2, \tag{11}$$

where w has the same variance as $w(s)$ in (9). It is readily verified from (9) and (11) that neglecting the s-dependency of $w(s)$ corresponds to neglecting the noise×noise contribution (i.e., the terms involving $|w(\tilde{s})|^2$ and $|w(-\tilde{s})|^2$) from (9).

Fig. 1 shows the SER for SISO transmission with BPSK signaling in case of LS channel estimation with $\beta = 1$ versus the *average SNR per diversity branch*, which is defined as (see [1, eq. (60)])

$$\Gamma \triangleq \frac{\|\mathbf{s}\|^2}{K}\frac{\Omega}{2N_0}, \tag{12}$$

and reduces to $\Gamma = |s|^2\Omega/(2N_0)$ for uncoded SISO transmission. The following results are displayed: the correct SER according to (9), the approximate SER according to (11), and the SER that corresponds to perfect channel estimation (PCE); for each case, both analytical results and simulations are shown. From the figure we observe that the exact and the approximate SER curves for LS channel estimation differ mainly for low and moderate Γ, whereas they coincide for large Γ. This behavior at large Γ is consistent with our finding that the difference between (9) and (11) is in the noise×noise contribution, which becomes negligible at large Γ.

Now we point out how the correct SER can be obtained without any approximations in the case of Rayleigh fading. Let us decompose the channel $\mathbf{H}$ as

$$\mathbf{H} = \mu\hat{\mathbf{H}} + \mathbf{N}'_{\mathrm{e}}, \tag{13}$$

such that the additive white Gaussian noise term $\mathbf{N}'_{\mathrm{e}}$ and the channel estimate $\hat{\mathbf{H}}$ are statistically independent; it is readily verified that $\mu = (1 + 2N_0/(\Omega\beta))^{-1}$ for LS estimation, and $\mu = 1$ for linear MMSE estimation. The entries of $\mathbf{N}'_{\mathrm{e}}$ are i.i.d. ZMCSCG random variables with variance $2N_0\Omega/(\Omega\beta + 2N_0)$, irrespective of the channel estimation strategy. In this way, the received signal (1) becomes [2]

$$\mathbf{R} = \mu\hat{\mathbf{H}}\mathbf{C} + \mathbf{N}'_{\mathrm{e}}\mathbf{C} + \mathbf{N}, \tag{14}$$

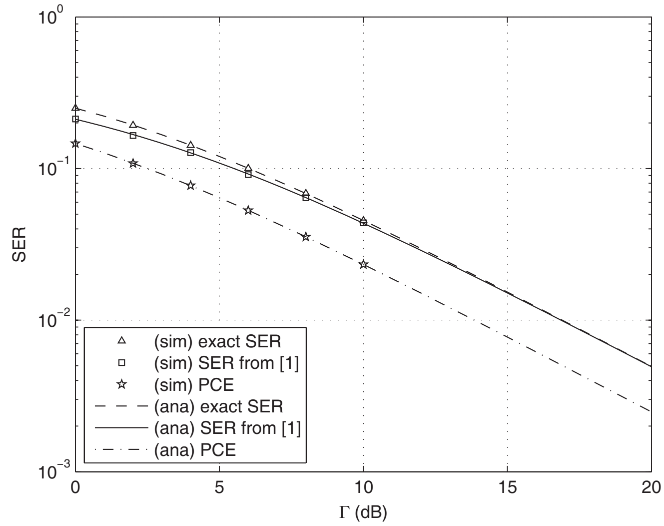

Fig. 1. SER for uncoded BPSK transmission on SISO Rayleigh fading channel with LS channel estimation ($\beta = 1$). The following results are displayed: the correct SER according to (9), the approximate SER according to (11), and the SER that corresponds to PCE; for each case, both analytical results (ana) and simulations (sim) are shown.

where $\mathbf{N}'_{\mathrm{e}}\mathbf{C}$ represents extra noise caused by imperfect channel estimation. Because of (2), the entries of $\mathbf{N}'_{\mathrm{e}}\mathbf{C}$ are i.i.d. ZMCSCG random variables with variance $2N_0\Omega\|\mathbf{s}\|^2/(\Omega\beta + 2N_0)$; for $\mathcal{M}$-PSK constellations, this variance is independent of the transmitted symbol vector $\mathbf{s}$. Expanding the objective function $\|\mathbf{R} - \hat{\mathbf{H}}\mathbf{C}(\mathbf{s})\|_F^2$ taking (14) into account, and keeping only terms that depend on $\mathbf{s}$, yields

$$\|\mathbf{R} - \hat{\mathbf{H}}\mathbf{C}(\mathbf{s})\|_F^2 \propto - 2\Re\{\mathrm{tr}(\mathbf{C}^H(\mathbf{s})\mathbf{H}_{\mathrm{eq}}^H(\mathbf{H}_{\mathrm{eq}}\mathbf{C}(\tilde{\mathbf{s}}) + \mathbf{N}_{\mathrm{eq}}))\}, \tag{15}$$

where $\tilde{\mathbf{s}}$ is the symbol vector actually transmitted, and $\Re\{\cdot\}$ and $\mathrm{tr}(\cdot)$ denote the real part and the trace, respectively. The entries of $\mathbf{H}_{\mathrm{eq}} = \mu\hat{\mathbf{H}}$ and $\mathbf{N}_{\mathrm{eq}} = \mathbf{N}'_{\mathrm{e}}\mathbf{C}(\tilde{\mathbf{s}}) + \mathbf{N}$ are i.i.d. ZMCSCG random variables not depending on $\mathbf{s}$, with respective variances $\sigma^2_{\mathbf{H}_{\mathrm{eq}}}$ and $\sigma^2_{\mathbf{N}_{\mathrm{eq}}}$ given by

$$\sigma^2_{\mathbf{H}_{\mathrm{eq}}} = \frac{\Omega^2\beta}{\Omega\beta + 2N_0}, \tag{16}$$

and

$$\sigma^2_{\mathbf{N}_{\mathrm{eq}}} = \frac{2N_0\Omega\|\tilde{\mathbf{s}}\|^2}{\Omega\beta + 2N_0} + 2N_0. \tag{17}$$

In the case of PCE, the objective function still satisfies (15), but with $\mathbf{H}_{\mathrm{eq}}$ and $\mathbf{N}_{\mathrm{eq}}$ replaced by $\mathbf{H}$ and $\mathbf{N}$ from (1). Hence, ML detection of the symbol vector $\tilde{\mathbf{s}}$ reduces to

$$\hat{\mathbf{s}}_{\mathrm{PCE}} = \arg\max_{\mathbf{s}}\Big(\Re\left\{\mathrm{tr}\left(\mathbf{C}^H(\mathbf{s})\mathbf{H}^H\left(\mathbf{H}\mathbf{C}(\tilde{\mathbf{s}}) + \mathbf{N}\right)\right)\right\}\Big). \tag{18}$$

For imperfect channel estimation (ICE), the ML detection algorithm can be written as

$$\hat{\mathbf{s}}_{\mathrm{ICE}} = \arg\max_{\mathbf{s}}\Big(\Re\left\{\mathrm{tr}\left(\mathbf{C}^H(\mathbf{s})\bar{\mathbf{H}}^H\left(\bar{\mathbf{H}}\mathbf{C}(\tilde{\mathbf{s}}) + \sqrt{\gamma}\bar{\mathbf{N}}\right)\right)\right\}\Big), \tag{19}$$

where we have introduced the matrices $\bar{\mathbf{H}} = \sqrt{\xi}\mathbf{H}_{\mathrm{eq}}$ and $\bar{\mathbf{N}} = \mathbf{N}_{\mathrm{eq}}/\sqrt{\psi}$, with $\gamma = \xi\psi$. Moreover, as the entries in each of the matrices $\mathbf{H}_{\mathrm{eq}}$, $\mathbf{H}$, $\mathbf{N}_{\mathrm{eq}}$ and $\mathbf{N}$ are i.i.d. ZMCSCG random

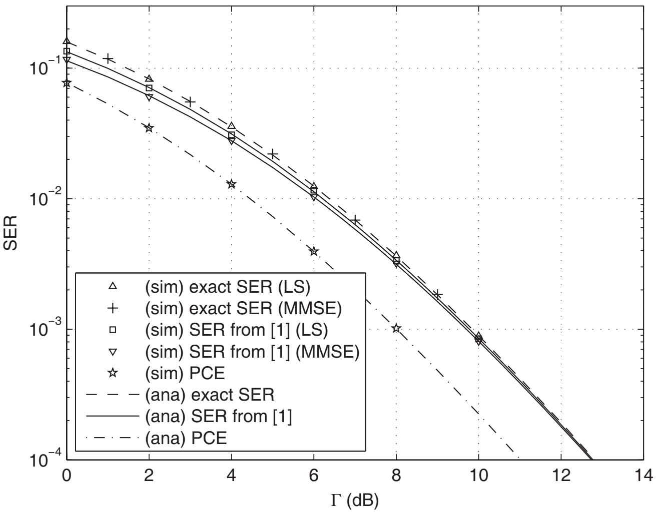

Fig. 2. SER for Alamouti's code on MIMO Rayleigh fading channel, with $N = 2$, $K = 2$, $M = 2$, QPSK signaling and imperfect channel estimation ($\beta = 4$). Both LS and linear MMSE channel estimation are considered. Also shown is the SER in the case of PCE. For each case, both analytical results (ana) and simulations (sim) are shown.

variables with known variances, the scaling factors ξ and ψ can be chosen in such a way that the statistics of $\bar{\mathbf{H}}$ and $\bar{\mathbf{N}}$ are identical to the statistics of $\mathbf{H}$ and $\mathbf{N}$, respectively. For this selection of ξ and ψ, γ can easily be shown to reduce to

$$\gamma = 1 + \frac{\|\tilde{\mathbf{s}}\|^2}{\beta} + \frac{2N_0}{\Omega\beta}, \tag{20}$$

which does not depend on $\tilde{\mathbf{s}}$ for $\mathcal{M}$-PSK constellations.

Let us denote by $\mathrm{SER}_{\mathrm{PCE}}(\Gamma)$ the SER resulting from (18) as a function of Γ. An analytical SER expression for square OSTBCs with $\mathcal{M}$-PSK signaling and perfect channel estimation is easily obtained (e.g., following a similar analysis as in [1, Sect. IV]):

$$\mathrm{SER}_{\mathrm{PCE}}(\Gamma) = \frac{1}{\pi}\int_0^{\frac{(\mathcal{M}-1)\pi}{\mathcal{M}}} \left(1 + \Gamma\frac{\sin^2\left(\frac{\pi}{\mathcal{M}}\right)}{\sin^2\phi}\right)^{-MN} d\phi. \tag{21}$$

Since $\bar{\mathbf{H}}$ and $\bar{\mathbf{N}}$ in (19) are identically distributed as $\mathbf{H}$ and $\mathbf{N}$ in (18), respectively, it is easily seen that the SER in the case of imperfect channel estimation is given by

$$\mathrm{SER}_{\mathrm{ICE}}(\Gamma) = \mathrm{SER}_{\mathrm{PCE}}(\Gamma/\gamma), \tag{22}$$

with γ according to (20). As γ does not depend on q, both channel estimation strategies (LS, MMSE) yield the same SER (this is because a real-valued scaling of the channel estimate does not affect the decision in case of $\mathcal{M}$-PSK constellations). For increasing Γ, γ converges to $1 + (\|\tilde{\mathbf{s}}\|^2/\beta)$.

Now we consider the approach from [1]. Expansion of the objective function $\|\mathbf{R} - \hat{\mathbf{H}}\mathbf{C}(\mathbf{s})\|_F^2$, using (3) and neglecting the dependency of $\mathbf{N}_{\mathrm{e}}\mathbf{C}(\mathbf{s})$ on $\mathbf{s}$ yields (15), with $\mathbf{H}_{\mathrm{eq}} = \mathbf{H}$ and $\mathbf{N}_{\mathrm{eq}}$ denoting a Gaussian matrix whose entries have the same variance as those from $\mathbf{N} - q\mathbf{N}_{\mathrm{e}}\mathbf{C}(\mathbf{s})$; this variance equals $2N_0(1 + q^2\|\mathbf{s}\|^2/\beta)$. Hence, the approach from [1] gives rise to $\mathrm{SER}_{\mathrm{ICE}}(\Gamma) = \mathrm{SER}_{\mathrm{PCE}}(\Gamma/\gamma')$, with γ' given by

$$\gamma' = 1 + \frac{q^2\|\mathbf{s}\|^2}{\beta}. \tag{23}$$

Note that γ' depends on q and is different from γ in (20) for both LS and linear MMSE channel estimation. Nevertheless, taking into account that $q = 1$ in the case of LS estimation, and $q \to 1$ when $\Gamma \to \infty$ in the case of linear MMSE estimation, it is readily verified that γ' converges to γ for increasing Γ, for both estimation strategies. This indicates that the error performance results from [1] are asymptotically correct for both LS and linear MMSE channel estimation.

Fig. 2 shows the SER curves for Alamouti's code [3] ($N = K = 2$) with quaternary phase-shift keying (QPSK) ($\mathcal{M} = 4$) signaling, operating over a 2×2 MIMO Rayleigh fading channel ($M = 2$). For both LS and linear MMSE channel estimation ($\beta = 4$), the correct SER according to (22) and the approximate SER from [1] are displayed. Also shown in the figure is the SER that corresponds to perfect channel estimation (PCE). For each case, both analytical results and simulations are shown. From the figure we observe that the exact SER curves for LS and linear MMSE channel estimation coincide, as expected for $\mathcal{M}$-PSK constellations. The SER curves resulting from [1], however, are different for LS and MMSE channel estimation and clearly differ from the exact SER for low and moderate Γ. Nevertheless, for large Γ the approximate and the exact curves converge, which is consistent with (20) and (23).

Since the DEP follows directly from the SER [1, eq. (54)], the above comments and conclusions also pertain to the DEP expressions resulting from [1]. The exact DEP for square OSTBCs with $\mathcal{M}$-PSK constellations and imperfect channel estimation is easily obtained from (22).

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