Exact BER Analysis for Square Orthogonal Space-Time Block Codes on Arbitrary Fading Channels with Imperfect Channel Estimation

Lennert Jacobs and Marc Moeneclaey
Ghent University, TELIN Department, DIGCOM Group
Sint-Pietersnieuwstraat 41, B-9000 Gent, Belgium
{Lennert.Jacobs,Marc.Moeneclaey}@telin.ugent.be

***Abstract*— In this contribution, we examine the effect of imperfect channel estimation on the bit error rate (BER) performance of square orthogonal space-time block codes (OSTBCs). The propagation channels from the N_t transmit antennas to each of the N_r receive antennas are assumed to be affected by (possibly correlated) flat block fading with an arbitrary distribution. The transmitted symbols belong to a PAM or QAM constellation.**

The resulting average BER can easily be written as an expectation over $4N_\mathrm{t}N_\mathrm{r}$ real-valued random variables, but the computing time needed for its numerical evaluation increases exponentially with N_t and N_r. However, we point out that expressing the BER in terms of the distribution of the Frobenius norm of the channel matrix, allows to reduce the BER expression for any number of receive antennas to an expectation over only $2N_\mathrm{s}+2$ random variables (with N_s denoting the number of information symbols in the OSTBC matrix). Due to the high diversity order of OSTBCs, the numerical evaluation of this expectation is much less time-consuming than a straightforward computer simulation. Moreover, the presented BER expression is useful not only when the fading distribution is given in closed form, but also when only experimental data (e.g. a histogram) on the fading are available.

I. Introduction

In order to improve the error performance of wireless communication systems, several diversity techniques have been proposed to provide the receiver with independent replicas of the transmitted signal. Spatial diversity can be exploited by using multiple transmit and/or receive antennas. Combined with proper space-time coding, a maximum diversity order of $N_\mathrm{t}N_\mathrm{r}$ (with N_t and N_r denoting the number of transmit and receive antennas, respectively) can be achieved. Orthogonal space-time block codes (OSTBCs) [1]–[3] achieve this full spatial diversity, and their maximum-likelihood (ML) detection algorithm reduces to symbol-by-symbol detection based only on linear processing at the receiver. Therefore, these codes are a very appealing transmit diversity technique. Under the assumption of perfect channel knowledge (PCK), the bit error rate (BER) performance of OSTBCs has been studied extensively in e.g. [4]–[7].

In realistic wireless systems, however, the channel state information (CSI) is not a priori available and the receiver has to estimate the channel response, which inevitably results in a performance degradation. Usually, the system performance is investigated under the assumption of Rayleigh fading: in case of minimum mean-square error (MMSE) channel estimation, an analytical expression for the BER of OSTBCs with pulse amplitude modulation (PAM) or quadrature amplitude modulation (QAM) constellations, was derived in [8]; for orthogonal space-time block coded systems employing M-ary phase-shift keying (M-PSK) modulation, analytical BER expressions as well as the tight Chernoff bound were given in [9]; By means of an eigenvalue approach, asymptotical pairwise error probability (PEP) expressions were derived in [10] for quite general STBCs with coherent and noncoherent receivers; an exact closed-form expression for the PEP of both orthogonal and non-orthogonal space-time codes in the case of least-squares channel estimation was obtained in [11] by means of characteristic functions. However, from the PEP only an upper bound on the BER can be derived, which in a fading environment does not converge to the true BER at high signal-to-noise ratio (SNR). Expressions for the exact decoding error probability (DEP) of OSTBCs were presented in [12], for the case of fading channels with an arbitrary distribution, but the analysis is only applicable to PSK constellations. Although the impact of imperfect channel estimation on the performance of OSTBCs has been examined extensively, exact BER expressions for OSTBCs with PAM and QAM constellations on generalized fading channels with imperfect channel estimation (ICE) are still lacking in the open literature.

In this contribution, we provide an exact BER analysis for square OSTBCs with PAM or QAM constellations on arbitrary flat-fading channels with ICE. We assume a linear pilot-based channel estimation method, with the well-known least-squares estimation and linear MMSE estimation as special cases. The resulting BER expression involves an expectation over at most $2N_\mathrm{s}+2$ variables (with N_s denoting the number of information symbols in the OSTBC matrix), that allows numerical evaluation with a computing time that is independent of the number of receive antennas. The presented BER expression can be used when the fading distribution of the Frobenius norm of the channel matrix is available in closed form or as an experimentally obtained histogram, and allows for a faster evaluation than by means of straightforward computer simulation, for practical values of the operating BER.

978-1-4244-2517-4/09/$20.00

Throughout this paper, the superscripts T and H represent the vector (matrix) transpose and conjugate transpose, respectively. $\Re\{x\}$, $\Im\{x\}$ and $\mathbb{E}[x]$ denote the real part, the imaginary part and the expected value of x, respectively, while $\|\mathbf{X}\|$ and $\mathrm{tr}(\mathbf{X})$ refer to the Frobenius norm and the trace of $\mathbf{X}$, respectively.

II. Signal model

Let us consider a multiple-input multiple-output (MIMO) wireless communication system with N_t transmit antennas and N_r receive antennas. The propagation channels between each transmit and receive antenna pair are affected by flat fading with an arbitrary distribution. The data transmission is organized in frames consisting of K_p known pilot symbols and K coded data symbols per transmit antenna; the pilot symbols enable channel estimation at the receiver. Since the channel is assumed to remain constant within one frame of $N_\mathrm{fr} = K + K_\mathrm{p}$ symbols (block fading), the $N_\mathrm{r} \times N_\mathrm{fr}$ received signal matrix $\mathbf{R}_\mathrm{tot}$ is given by

$$\mathbf{R}_\mathrm{tot} = [\mathbf{R}_\mathrm{p}, \mathbf{R}] = \mathbf{H}[\mathbf{A}_\mathrm{p}, \mathbf{A}] + \mathbf{W}, \tag{1}$$

where the $N_\mathrm{t} \times K_\mathrm{p}$ pilot matrix $\mathbf{A}_\mathrm{p}$ and the $N_\mathrm{t} \times K$ data matrix $\mathbf{A}$ consist of the pilot symbols and the coded data symbols, respectively, transmitted at each transmit antenna. The propagation channel is represented by the $N_\mathrm{r} \times N_\mathrm{t}$ complex random matrix $\mathbf{H}$. The $N_\mathrm{t}N_\mathrm{r}$ (possibly correlated) complex fading gains are distributed according to an arbitrary joint pdf $p(\mathbf{H})$. The $N_\mathrm{r} \times N_\mathrm{fr}$ noise matrix $\mathbf{W}$ consists of independent and identically distributed (i.i.d.) zero-mean circularly symmetric complex Gaussian (ZMCSCG) random variables with variance N_0.

Considering OSTBCs from complex linear processing orthogonal designs [2], an OSTBC is defined by a $N_\mathrm{t} \times K_\mathrm{c}$ coded symbol matrix $\mathbf{C}$, whose entries are linear combinations of N_s information symbols $s_1, s_2, \cdots, s_{N_\mathrm{s}}$ and their complex conjugate. Hence, $\mathbf{C}$ can be written as

$$\mathbf{C} = \sum_{i=1}^{N_\mathrm{s}} \left(\mathbf{C}_i s_i + \mathbf{C}_i^{'} s_i^*\right), \tag{2}$$

where the $N_\mathrm{t} \times K_\mathrm{c}$ matrices $\mathbf{C}_i$ and $\mathbf{C}_i^{'}$ comprise the coefficients of s_i and s_i^* in the matrix $\mathbf{C}$, respectively. Without loss of generality, we may assume that the matrices $\mathbf{C}$ are scaled in such way that they satisfy the following orthogonality condition:

$$\mathbf{C}\mathbf{C}^H = \left(\lambda \sum_{i=1}^{N_\mathrm{s}} |s_i|^2\right) \mathbf{I}_{N_\mathrm{t}}, \tag{3}$$

where $\lambda = K_\mathrm{c}/N_\mathrm{s}$ and $\mathbf{I}_{N_\mathrm{t}}$ is the $N_\mathrm{t} \times N_\mathrm{t}$ identity matrix. In this contribution, we consider square OSTBCs ($N_\mathrm{t} = K_\mathrm{c}$) with real-valued coefficient matrices $\mathbf{C}_i$ and $\mathbf{C}_i^{'}$, which satisfy

$$\mathbf{C}^H\mathbf{C} = \mathbf{C}\mathbf{C}^H = \left(\lambda \sum_{i=1}^{N_\mathrm{s}} |s_i|^2\right) \mathbf{I}_{N_\mathrm{t}}. \tag{4}$$

Examples of square OSTBCs satisfying (4) are the 2×2 Alamouti code [1] and the 4×4 codes given in [2, eq. 40], [3, eq. 62] and [13, eq. 41].

Assuming that K is a multiple of K_c, the transmitted data symbol matrix consists of K/K_c coded symbol matrices $\mathbf{C}(k)$, with k denoting the block index. In this way, we can write $\mathbf{A} = \sqrt{E_\mathrm{s}}\,[\mathbf{C}(1), \cdots, \mathbf{C}(K/K_\mathrm{c})]$. Considering a normalized information symbol constellation ($\mathbb{E}[|s_i|^2] = 1$), it follows from (3) that E_s denotes the average energy of the transmitted coded symbols:

$$\frac{1}{N_\mathrm{t}K}\mathbb{E}\left[\|\mathbf{A}\|^2\right] = E_\mathrm{s}. \tag{5}$$

Similarly, the average energy of the pilot symbols is E_p.

III. Pilot-based channel estimation

Using the known pilot matrix $\mathbf{A}_\mathrm{p}$ and the corresponding received signal matrix $\mathbf{R}_\mathrm{p}$, the receiver can estimate the channel matrix $\mathbf{H}$. We assume orthogonal pilot sequences, i.e. $\mathbf{A}_\mathrm{p}\mathbf{A}_\mathrm{p}^H = K_\mathrm{p}E_\mathrm{p}\mathbf{I}_{N_\mathrm{t}}$, and consider linear channel estimates of the form

$$\hat{\mathbf{H}} = \frac{\alpha}{K_\mathrm{p}E_\mathrm{p}}\mathbf{R}_\mathrm{p}\mathbf{A}_\mathrm{p}^H, \tag{6}$$

with $\alpha \in \mathbb{R}$, such that $\hat{\mathbf{H}}$ can be decomposed into the sum of two statistically independent contributions:

$$\hat{\mathbf{H}} = \alpha\mathbf{H} + \mathbf{N}. \tag{7}$$

It can be shown that the entries of $\mathbf{N} = (\alpha/(K_\mathrm{p}E_\mathrm{p}))\,\mathbf{W}_\mathrm{p}\mathbf{A}_\mathrm{p}^H$ are ZMCSCG random variables, whose real and imaginary parts have a variance $\sigma_\mathbf{n}^2 = \alpha^2 N_0/(2K_\mathrm{p}E_\mathrm{p})$. Hence, when conditioned on the channel $\mathbf{H}$, the channel estimate $\hat{\mathbf{H}}$ is a complex Gaussian random matrix with mean $\alpha\mathbf{H}$ and diagonal covariance matrix with diagonal elements $2\sigma_\mathbf{n}^2$. Both least-squares and linear MMSE estimation satisfy (6) with $\alpha = 1$ and $\alpha = K_\mathrm{p}E_\mathrm{p}/(K_\mathrm{p}E_\mathrm{p} + N_0)$, respectively [14].

An accurate channel estimate can be obtained by allocating a large total energy to pilot symbols. Increasing the total energy of the pilot symbols, however, reduces the symbol energy E_s available for data transmission. Denoting by E_b and $\gamma = E_\mathrm{p}/E_\mathrm{s}$ the energy per information bit and the ratio of the pilot energy E_p versus the symbol energy E_s, we have

$$E_\mathrm{s} = \frac{K}{K + \gamma K_\mathrm{p}}\rho\log_2(M)E_\mathrm{b}, \tag{8}$$

where $\rho = N_\mathrm{s}/(N_\mathrm{t}K_\mathrm{c})$ and M denote the code rate and the number of constellation points, respectively.

IV. ML Detection

When the receiver knows the channel matrix $\mathbf{H}$, ML detection is known to be the optimal detection algorithm for the data matrices $\mathbf{C}(k)$

$$\hat{\mathbf{C}}(k) = \arg\min_{\widetilde{\mathbf{C}}}\left\|\mathbf{R}'(k) - \sqrt{E_\mathrm{s}}\mathbf{H}\widetilde{\mathbf{C}}\right\|^2, \tag{9}$$

where $\mathbf{R}'(k)$ is the received signal matrix corresponding to the transmitted symbol matrix $\mathbf{C}(k)$, and the minimization is over the valid code matrices satisfying (2). However, in realistic applications the channel state information is not known, and the receiver has to estimate the channel. We consider a mismatched receiver that uses $\hat{\mathbf{H}}$ in the same way an ML receiver would apply $\mathbf{H}$ (we omit the index k for notational convenience):

$$\hat{\mathbf{C}} = \arg\min_{\widetilde{\mathbf{C}}} \left\| \mathbf{R}' - \sqrt{E_\mathrm{s}} \hat{\mathbf{H}} \widetilde{\mathbf{C}} \right\|^2 . \tag{10}$$

Taking (3) into account, the detection algorithm (10) for the information symbols s_i reduces to symbol-by-symbol detection:

$$\hat{s}_i = \arg\min_{s_i} |u_i - s_i| \,,\, 1 \leq i \leq N_\mathrm{s}, \tag{11}$$

where the decision variables u_i are given by

$$u_i = \frac{\mathrm{tr}\left(\mathbf{C}_i^H \hat{\mathbf{H}}^H \mathbf{R}' + \mathbf{R}'^H \hat{\mathbf{H}} \mathbf{C}_i'\right)}{\lambda \sqrt{E_\mathrm{s}} \left\|\hat{\mathbf{H}}\right\|^2}. \tag{12}$$

V. Bit error rate analysis

In this contribution, we consider square M-QAM transmission with Gray mapping, which reduces to $\sqrt{M}$-PAM transmission for both the in-phase and quadrature information bits. Because of the rotational symmetry of the M-QAM constellation and assuming that the symbol vectors $\mathbf{s} = (s_1, \cdots, s_{N_\mathrm{s}})$ are equally likely, it can be verified that the BERs related to the in-phase and quadrature bits of s_i are equal, and independent of the index i. Hence, the average BER of the OSTBC equals the BER of the in-phase information bits related to the transmitted symbol s_i, irrespective of i. The latter BER can be obtained by averaging the conditional BER $\mathrm{BER}_{i,\mathrm{R}}(\mathbf{H}, \hat{\mathbf{H}})$ (which is related to the in-phase information bits of s_i and conditioned on the channel $\mathbf{H}$ and the channel estimate $\hat{\mathbf{H}}$). In this way, the BER of the OSTBC is given by

$$\mathrm{BER} = \mathbb{E}_{\mathbf{H},\hat{\mathbf{H}}}\left[\mathrm{BER}_{i,\mathrm{R}}(\mathbf{H}, \hat{\mathbf{H}})\right]. \tag{13}$$

Denoting by $\mathrm{BER}_{i,\mathrm{R}}(\mathbf{s}, \mathbf{H}, \hat{\mathbf{H}})$ the conditional BER related to the in-phase information bits of s_i (conditioned on $\mathbf{H}$, $\hat{\mathbf{H}}$ and the symbol vector $\mathbf{s}$), $\mathrm{BER}_{i,\mathrm{R}}(\mathbf{H}, \hat{\mathbf{H}})$ is given by

$$\mathrm{BER}_{i,\mathrm{R}}(\mathbf{H}, \hat{\mathbf{H}}) = \frac{1}{M^{N_\mathrm{s}}} \sum_{\mathbf{s} \in \Psi^{N_\mathrm{s}}} \mathrm{BER}_{i,\mathrm{R}}(\mathbf{s}, \mathbf{H}, \hat{\mathbf{H}}), \tag{14}$$

where Ψ denotes the normalized M-QAM constellation. Referring to the projection (on the real axis) of the decision area of a QAM symbol b, for which the in-phase component b_R is different from the in-phase component $s_{i,\mathrm{R}}$ of the transmitted symbol s_i, as the decision region of b_R, we can write $\mathrm{BER}_{i,\mathrm{R}}(\mathbf{s}, \mathbf{H}, \hat{\mathbf{H}})$ as

$$\mathrm{BER}_{i,\mathrm{R}}(\mathbf{s}, \mathbf{H}, \hat{\mathbf{H}}) = \sum_{b_\mathrm{R} \in \Psi_\mathrm{R}} \frac{N(s_{\mathrm{i,R}}, b_\mathrm{R})}{\frac{1}{2}\log_2 M} P_{i,\mathrm{R}}(\mathbf{s}, b_\mathrm{R}, \mathbf{H}, \hat{\mathbf{H}}), \tag{15}$$

where Ψ_R is the set consisting of the real parts of the constellation points, $N(s_{\mathrm{i,R}}, b_\mathrm{R})$ represents the Hamming distance between the in-phase bits of the transmitted symbol s_i and the in-phase bits of the detected symbol b, and $P_{i,\mathrm{R}}(\mathbf{s}, b_\mathrm{R}, \mathbf{H}, \hat{\mathbf{H}})$ is the probability that the real part of the decision variable u_i is located inside the decision area of b_R (when the transmitted symbol vector $\mathbf{s}$, the channel $\mathbf{H}$ and the channel estimate $\hat{\mathbf{H}}$ are known):

$$P_{i,\mathrm{R}}(\mathbf{s}, b_\mathrm{R}, \mathbf{H}, \hat{\mathbf{H}}) = Pr\left[\hat{s}_{\mathrm{i,R}} = b_\mathrm{R} \middle| \mathbf{s}, \mathbf{H}, \hat{\mathbf{H}}\right]. \tag{16}$$

Taking (1) and (2) into account, expanding the real part of the decision variable (12) yields:

$$u_{\mathrm{i,R}} = u'_{i,\mathrm{R}} + \frac{\Re\left\{\mathrm{tr}\left(\mathbf{C}_i^H \hat{\mathbf{H}}^H \mathbf{W}' + \mathbf{W}'^H \hat{\mathbf{H}} \mathbf{C}_i'\right)\right\}}{\lambda \sqrt{E_\mathrm{s}} \left\|\hat{\mathbf{H}}\right\|^2}, \tag{17}$$

where $u'_{i,\mathrm{R}}$ is a function of the transmitted symbol vector $\mathbf{s}$, the channel matrix $\mathbf{H}$ and the estimated channel matrix $\hat{\mathbf{H}}$. The second term in (17) represents zero-mean Gaussian noise with variance $N_0/(2\lambda E_\mathrm{s}\|\hat{\mathbf{H}}\|^2)$. Let $d_1(b_\mathrm{R})$ and $d_2(b_\mathrm{R})$ denote the boundaries of the projection of the decision area of b on the real axis, with $d_1(b_\mathrm{R}) < d_2(b_\mathrm{R})$; we set $d_1(b_\mathrm{R}) = -\infty$ ($d_2(b_\mathrm{R}) = \infty$) if b is a left (right) outer constellation point. In this way, (16) reduces to

$$P_{i,\mathrm{R}}(\mathbf{s}, b_\mathrm{R}, \mathbf{H}, \hat{\mathbf{H}}) = Q_1 - Q_2, \tag{18}$$

where the quantities Q_k, $k \in \{1, 2\}$, are given by

$$Q_k = Q\left(\sqrt{2\lambda \frac{E_\mathrm{s}}{N_0}} \|\hat{\mathbf{H}}\| \left(d_k(b_\mathrm{R}) - u'_{\mathrm{i,R}}\right)\right). \tag{19}$$

Here, $Q(.)$ denotes the Gaussian Q-function, defined as

$$Q(x) \triangleq \frac{1}{\sqrt{2\pi}} \int_x^\infty \exp\left(-\frac{x^2}{2}\right) \mathrm{d}x. \tag{20}$$

Hence, the BER (13) of the OSTBC can be expressed as

$$\mathrm{BER} = \int \mathrm{BER}_{i,\mathrm{R}}(\mathbf{H}, \hat{\mathbf{H}}) p(\hat{\mathbf{H}}|\mathbf{H}) p(\mathbf{H}) \mathrm{d}\hat{\mathbf{H}} \mathrm{d}\mathbf{H}, \tag{21}$$

where $p(\hat{\mathbf{H}}|\mathbf{H})$ is the joint Gausian pdf of $N_\mathrm{t}N_\mathrm{r}$ complex-valued random variables, with mean $\alpha\mathbf{H}$ and diagonal covariance matrix with diagonal elements $2\sigma_\mathbf{n}^2$, and $p(\mathbf{H})$ represents the arbitrary joint distribution of $N_\mathrm{t}N_\mathrm{r}$ complex-valued fading gains. Although the general expression (21), which allows correlation between the components of $\mathbf{H}$, is conceptually simple, it is not well suited for numerical evaluation as its evaluation requires taking the expectation over $4N_\mathrm{t}N_\mathrm{r}$ real-valued variables; hence, the associated computing time increases exponentially with the number of transmit and receive antennas.

In order to avoid the computational complexity corresponding to the evaluation of (21), we will manipulate (21) into an expectation over only $2N_\mathrm{s} + 2$ variables. Therefore, we introduce the $(2N_\mathrm{t}N_\mathrm{r} \times 1)$ real-valued column vectors $\hat{\mathbf{h}}$ and $\mathbf{h}$, which consist of all elements of $\hat{\mathbf{H}}$ and $\mathbf{H}$, respectively:

$$\begin{cases} \hat{\mathbf{h}} = \left[\hat{\mathbf{h}}_{1,\mathrm{R}}^T, \hat{\mathbf{h}}_{1,\mathrm{I}}^T, \cdots, \hat{\mathbf{h}}_{N_\mathrm{s},\mathrm{R}}^T, \hat{\mathbf{h}}_{N_\mathrm{s},\mathrm{I}}^T\right]^T \\ \mathbf{h} = \left[\mathbf{h}_{1,\mathrm{R}}^T, \mathbf{h}_{1,\mathrm{I}}^T, \cdots, \mathbf{h}_{N_\mathrm{s},\mathrm{R}}^T, \mathbf{h}_{N_\mathrm{s},\mathrm{I}}^T\right]^T \end{cases}, \tag{22}$$

with $\hat{\mathbf{h}}_{i,\mathrm{R}} + j\hat{\mathbf{h}}_{i,\mathrm{I}}$ and $\mathbf{h}_{i,\mathrm{R}} + j\mathbf{h}_{i,\mathrm{I}}$ denoting the i-th column of $\hat{\mathbf{H}}$ and $\mathbf{H}$, respectively. In this way, it can be shown that $u'_{i,\mathrm{R}}$ reduces to

$$u'_{i,\mathrm{R}} = \frac{1}{\lambda\|\hat{\mathbf{H}}\|^2} \sum_{j=1}^{N_\mathrm{s}} \Big[s_{j,\mathrm{R}} \left(\hat{\mathbf{h}}^T \mathbf{g}_{i,\mathrm{R}}(j)\right) - s_{j,\mathrm{I}} \left(\hat{\mathbf{h}}^T \mathbf{g}_{i,\mathrm{I}}(j)\right)\Big], \quad (23)$$

where the $(2N_\mathrm{t}N_\mathrm{r} \times 1)$ column vectors $\mathbf{g}_{i,\mathrm{R}}(j)$ and $\mathbf{g}_{i,\mathrm{I}}(j)$ are given by

$$\begin{cases} \mathbf{g}_{i,\mathrm{R}}(j) = \mathbf{M}_{i,\mathrm{R}}(j)\mathbf{h} \\ \mathbf{g}_{i,\mathrm{I}}(j) = \mathbf{M}_{i,\mathrm{I}}(j)\mathbf{h} \end{cases}, \quad (24)$$

and the $(2N_\mathrm{t}N_\mathrm{r} \times 2N_\mathrm{t}N_\mathrm{r})$ matrices $\mathbf{M}_{i,\mathrm{R}}(j)$ and $\mathbf{M}_{i,\mathrm{I}}(j)$ depend on the coefficient matrices $\mathbf{C}_i$, $\mathbf{C}_j$, $\mathbf{C}'_i$ and $\mathbf{C}'_j$. The term in $s_{i,\mathrm{R}}$ in (23) is the useful term, whereas the term in $s_{i,\mathrm{I}}$ represents interference from the quadrature component of s_i. The terms in $s_{j,\mathrm{R}}$ and $s_{j,\mathrm{I}}$ ($j \neq i$) represent interference from the in-phase and quadrature components of the transmitted symbols s_j, respectively. For square OSTBCs satisfying (4), it can be shown that, for any given i, the following orthogonality conditions hold:

$$\begin{cases} \mathbf{g}_{i,\mathrm{R}}(j)^T \mathbf{g}_{i,\mathrm{I}}(j') = 0 \\ \mathbf{g}_{i,\mathrm{R}}(j)^T \mathbf{g}_{i,\mathrm{R}}(j') = \lambda^2 \|\mathbf{H}\|^2 \delta_{j-j'} \\ \mathbf{g}_{i,\mathrm{I}}(j)^T \mathbf{g}_{i,\mathrm{I}}(j') = \lambda^2 \|\mathbf{H}\|^2 \delta_{j-j'} \end{cases}, \quad (25)$$

where $\delta_{j-j'}$ denotes the Kronecker delta. Moreover, since $\mathbf{M}_{i,\mathrm{R}}(i) = \lambda \mathbf{I}_{2N_\mathrm{t}N_\mathrm{r}}$, it is readily verified that $u'_{\mathrm{i,R}} = s_{\mathrm{i,R}}$ in case of PCK ($\hat{\mathbf{h}} = \mathbf{h}$).

Let us now define an orthonormal coordinate system with $2N_\mathrm{t}N_\mathrm{r}$ unit vectors $\{\mathbf{e}_{i,\mathrm{R}}(j), \mathbf{e}_{i,\mathrm{I}}(j), j = 1, \cdots, N_\mathrm{t}N_\mathrm{r}\}$, with $\mathbf{e}_{i,\mathrm{R}}(j)$ and $\mathbf{e}_{i,\mathrm{I}}(j)$, $1 \leq j \leq N_\mathrm{s}$, directed along $\mathbf{g}_{i,\mathrm{R}}(j)$ and $\mathbf{g}_{i,\mathrm{I}}(j)$, respectively. The projections of $\hat{\mathbf{h}}$ on $\mathbf{e}_{i,\mathrm{R}}(j)$ and $\mathbf{e}_{i,\mathrm{I}}(j)$ are denoted $x_{i,\mathrm{R}}(j)$ and $x_{i,\mathrm{I}}(j)$, respectively, which yields

$$\|\hat{\mathbf{H}}\|^2 = |\hat{\mathbf{h}}|^2 = \sum_{j=1}^{N_\mathrm{s}} \left(x_{i,\mathrm{R}}^2(j) + x_{i,\mathrm{I}}^2(j)\right) + z^2, \quad (26)$$

with

$$z^2 = \sum_{i=N_\mathrm{s}+1}^{N_\mathrm{t}N_\mathrm{r}} \left(x_{i,\mathrm{R}}^2(j) + x_{i,\mathrm{I}}^2(j)\right). \quad (27)$$

Because of the specific choice of $\mathbf{e}_{i,\mathrm{R}}(j)$ and $\mathbf{e}_{i,\mathrm{I}}(j)$ for $1 \leq i \leq N_\mathrm{s}$ and taking (25) into account, we have

$$x_{i,\mathrm{R}}(j) = \hat{\mathbf{h}}^T \mathbf{e}_{i,\mathrm{R}}(j) = \frac{\hat{\mathbf{h}}^T \mathbf{g}_{i,\mathrm{R}}(j)}{\lambda\|\mathbf{H}\|}, \quad (28)$$

$$x_{i,\mathrm{I}}(j) = \hat{\mathbf{h}}^T \mathbf{e}_{i,\mathrm{I}}(j) = \frac{\hat{\mathbf{h}}^T \mathbf{g}_{i,\mathrm{I}}(j)}{\lambda\|\mathbf{H}\|}. \quad (29)$$

From (23) and (19) it follows that $\mathrm{BER}_{i,\mathrm{R}}(\mathbf{H}, \hat{\mathbf{H}})$ is a function of only $2N_\mathrm{s}+2$ random variables: $x_{i,\mathrm{R}}(j)$ and $x_{i,\mathrm{I}}(j)$ ($1 \leq j \leq N_\mathrm{s}$), z and $\|\mathbf{H}\|$. Hence, we denote $\mathrm{BER}_{i,\mathrm{R}}(\mathbf{H}, \hat{\mathbf{H}})$ by $\mathrm{BER}_{i,\mathrm{R}}(\mathbf{x}, z, \|\mathbf{H}\|)$, where the vector $\mathbf{x}$ contains the $2N_\mathrm{s}$ components $x_{i,\mathrm{R}}(j)$ and $x_{i,\mathrm{I}}(j)$, $1 \leq j \leq N_\mathrm{s}$. Taking (7) into account, it can be shown that z and the entries of $\mathbf{x}$, when conditioned on $\|\mathbf{H}\|$, are independent variables which satisfy the following properties:

- $x_{i,\mathrm{R}}(i)$ is a Gaussian variable with mean $\alpha\|\mathbf{H}\|$ and variance $\sigma_\mathbf{n}^2$.
- $x_{i,\mathrm{R}}(j)$ ($j \neq i$) and $x_{i,\mathrm{I}}(j)$ are zero-mean Gaussian variables with variance $\sigma_\mathbf{n}^2$.
- $z/\sigma_\mathbf{n}$ is distributed according to the chi-distribution with $2N_\mathrm{t}N_\mathrm{r} - 2N_\mathrm{s}$ degrees of freedom [15].

Taking the above properties into account, (21) reduces to

$$\mathrm{BER} = \int \mathrm{BER}_{i,\mathrm{R}}(\mathbf{x}, z, u)\, p\left(\mathbf{x}, z \middle| \|\mathbf{H}\| = u\right) p(u) \mathrm{d}\mathbf{x}\mathrm{d}z\mathrm{d}u. \quad (30)$$

where $p(u)$ is the pdf of the Frobenius norm $\|\mathbf{H}\|$ of the channel matrix $\mathbf{H}$. It is important to note that instead of the joint distribution $p(\mathbf{H})$ of the $N_\mathbf{t}N_\mathbf{r}$ complex-valued fading gains, we need only the distribution of the Frobenius norm $\|\mathbf{H}\|$. The evaluation of the BER involves an expectation over only $2N_\mathrm{s}+2$ random variables, which can be evaluated numerically by approximating the $(2N_\mathrm{s}+2)$-fold integral (30) by a $(2N_\mathrm{s}+2)$-fold sum, running over discretized versions of the continuous variables.

The number of random variables to be considered in the expectation (30) is reduced in the following cases:

1) When the product of the number of transmit and receive antennas equals the number of information symbols in the square OSTBC matrix ($N_\mathrm{t}N_\mathrm{r} = N_\mathrm{s}$), the dimension of $\hat{\mathbf{h}}'$ is $2N_\mathrm{s}$, such that the only projections are $x_{i,\mathrm{R}}(j)$ and $x_{i,\mathrm{I}}(j)$ ($1 \leq j \leq N_\mathrm{s}$). Hence, the BER computation involves averaging over only $2N_\mathrm{s}+1$ variables: $x_{i,\mathrm{R}}(j)$ and $x_{i,\mathrm{I}}(j)$ ($1 \leq j \leq N_\mathrm{s}$), and $\|\mathbf{H}\|$.
2) For PAM constellations, we have $s_{j,\mathrm{I}} = 0$, so that $u'_{i,\mathrm{R}}$ from (23) does not depend on $x_{i,\mathrm{I}}(j)$. Hence, the BER is obtained by averaging over only $N_\mathrm{s}+2$ variables, $x_{i,\mathrm{R}}(j)$ ($1 \leq j \leq N_\mathrm{s}$), z' and $\|\mathbf{H}\|$, where
$$\|\hat{\mathbf{H}}\|^2 = \sum_{i=1}^{N_\mathrm{s}} \left(x_{i,\mathrm{R}}^2(j)\right) + z'^2 \quad (31)$$
and $z'/\sigma_\mathbf{n}$ has a chi-distribution with $2N_\mathrm{t}N_\mathrm{r} - N_\mathrm{s}$ degrees of freedom.

The above analysis is not only applicable to complex-valued channel gains. Also real-valued propagation channels (along with PAM constellations) can be considered. The resulting BER is nearly the same as for PAM transmission over complex-valued channels (averaging over $N_\mathrm{s}+2$ variables, $x_{i,\mathrm{R}}(j)$ ($1 \leq j \leq N_\mathrm{s}$), z' and $\|\mathbf{H}\|$), except for the variable $z'/\sigma_\mathbf{n}$ being distributed according to a chi-distribution with $N_\mathrm{t}N_\mathrm{r} - N_\mathrm{s}$ (instead of $2N_\mathrm{t}N_\mathrm{r} - N_\mathrm{s}$) degrees of freedom. For specific cases, the number of random variables involved in the expectation can be further reduced. According to [12], only a twofold integral must be computed in the case of M-PSK constellations and arbitrary fading.

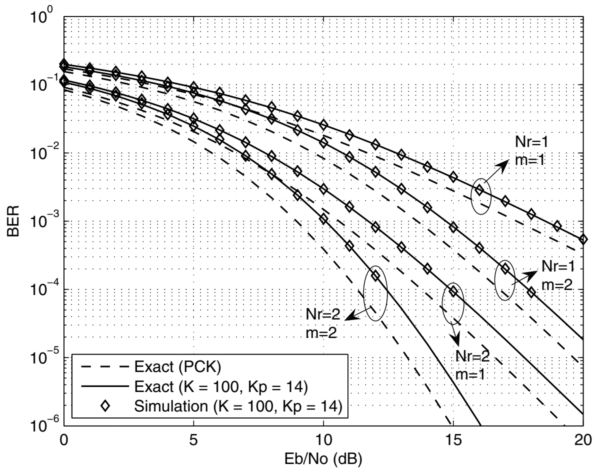

Fig. 1. BER for Alamouti's code on complex-valued Nakagami-m fading channel, 16-QAM

VI. Numerical results

To obtain our numerical results, we use Alamouti's code [1] ($N_\mathrm{t} = K_\mathrm{c} = N_\mathrm{s} = 2$) and linear MMSE channel estimation with $E_\mathrm{p} = E_\mathrm{s}$. The complex-valued MIMO propagation channel consists of $2N_\mathrm{r}$ i.i.d. Nakagami-m fading channels, with parameters m and Ω [16], where we assume $\Omega = 1$.

For both the mismatched receiver and the PCK receiver, fig. 1 displays the exact BER curves in case of 16-QAM transmission, for several values of m and N_r. It is easily derived that the Frobenius norm $\|\mathbf{H}\|$ of the MIMO channel is distributed according to a Nakagami-m distribution with parameters $2N_\mathrm{r}m$ and $2N_\mathrm{r}\Omega$. The BER curves for the mismatched receiver are obtained by numerically computing the expectation (30) over $2N_\mathrm{s} + 2 = 6$ variables, whereas the BER of the PCK receiver involves the expectation over $\|\mathbf{H}\|$ only. Also shown in the figure are straightforward computer simulation results for the mismatched receiver that confirm the result obtained from numerical averaging.

VII. Conclusions and remarks

In this contribution, we investigated the effect of ICE on the BER performance of square OSTBCs on arbitrary distributed (possibly correlated) flat-fading propagation channels. We assumed a generic pilot-based linear channel estimation method, which includes the well-known least-squares estimation and linear MMSE estimation as special cases. The resulting exact BER expression involves an expectation over only $2N_\mathrm{s} + 2$ variables, irrespective of the number of receive antennas. Moreover, we have shown that for some cases ($N_\mathrm{t}N_\mathrm{r} = N_\mathrm{s}$, real-valued channel, PAM constellation, and combinations thereof) the number of variables involved in the expectation can be further reduced. Generally, the evaluation of the BER expression requires a numerical integration over at most $2N_\mathrm{s} + 2$ variables. Due to the high diversity order of OSTBCs, straightforward simulations are extremely time-consuming, such that the numerical averaging is to be preferred for BER values of practical interest. Moreover, the BER expression is useful not only when the pdf of the Frobenius norm of the channel matrix is available in closed form but also when only experimental data (e.g. a histogram) on the fading are available.

Acknowledgments

The authors wish to acknowledge the activity of the Network of Excellence in Wireless COMmunications NEWCOM++ of the European Commission (contract n. 216715) that motivated this work. The first author also gratefully acknowledges the support from the Fund for Scientific Research in Flanders (FWO-Vlaanderen).

References

[1] S. Alamouti, "A simple transmit diversity technique for wireless communications," *IEEE J. Select. Areas Commun.*, vol. 16, pp. 1459–1478, Oct. 1998.

[2] V. Tarokh, H. Jafarkhani, and A. Calderbank, "Space-time block codes from orthogonal designs," *IEEE Trans. Inform. Theory*, vol. 45, no. 5, pp. 1456–1467, Jul. 1999.

[3] X.-B. Liang, "Orthogonal designs with maximal rates," *IEEE Trans. Inform. Theory*, vol. 49, no. 10, pp. 2468–2503, Oct. 2003.

[4] I.-M. Kim and V. Tarokh, "Variable-rate space-time block codes in M-ary PSK systems," *IEEE J. Select. Areas Commun.*, vol. 21, no. 3, pp. 362–373, Apr. 2003.

[5] G. Femenias, "BER performance of linear STBC from orthogonal designs over MIMO correlated Nakagami-m fading channels," *IEEE Trans. Veh. Technol.*, vol. 53, no. 2, pp. 307–317, Mar. 2004.

[6] S.-H. Kim, I.-S. Kang, and J.-S. No, "Exact bit error probability of orthogonal space-time block codes with QAM," in *Proc. IEEE Int. Symp. on Information Theory (ISIT)*, Yokohama, Japan, 29 Jun.-4 Jul. 2003, p. 63.

[7] I.-M. Kim, "Exact BER analysis of OSTBCs in spatially correlated MIMO channels," *IEEE Trans. Commun.*, vol. 54, no. 8, pp. 1365–1373, Aug. 2006.

[8] L. Jacobs and M. Moeneclaey, "Effect of MMSE channel estimation on BER performance of orthogonal space-time block codes in Rayleigh fading channels," *Accepted for publication in IEEE Trans. Commun.*

[9] C. Shan, P. Kam, and A. Nallanathan, "Theoretical performance of space-time block coded system with channel estimation," in *Proc. IEEE Global Telecommunications Conf. (Globecom)*, Dallas, USA, 29 Nov.-3 Dec. 2004, pp. 3666–3670 Vol.6.

[10] M. Brehler and M. Varanasi, "Asymptotic error probability analysis of quadratic receivers in Rayleigh-fading channels with applications to a unified analysis of coherent and noncoherent space-time receivers," *IEEE Trans. Inform. Theory*, vol. 47, no. 6, pp. 2383–2399, Sep. 2001.

[11] P. Garg, R. Mallik, and H. Gupta, "Performance analysis of space-time coding with imperfect channel estimation," *IEEE Trans. Wireless Commun.*, vol. 4, no. 1, pp. 257–265, Jan. 2005.

[12] ——, "Exact error performance of square orthogonal space-time block coding with channel estimation," *IEEE Trans. Commun.*, vol. 54, no. 3, pp. 430–437, Mar. 2006.

[13] G. Ganesan and P. Stoica, "Space-time block codes: A maximum SNR approach," *IEEE Trans. Inform. Theory*, vol. 47, no. 4, pp. 1650–1656, May 2001.

[14] G. Taricco and E. Biglieri, "Space-time decoding with imperfect channel estimation," *IEEE Trans. Wireless Commun.*, vol. 4, no. 4, pp. 1874–1888, Jul. 2005.

[15] M. Evans, N. Hastings, and B. Peacock, *Statistical Distributions 3rd ed.* New York: Wiley, 2000.

[16] M. Nakagami, "The m-distribution—A general formula for intensity distribution of rapid fading," in *Statistical Methods in Radio Wave Propagation*. New York: Pergamon, 1960, pp. 3–36.