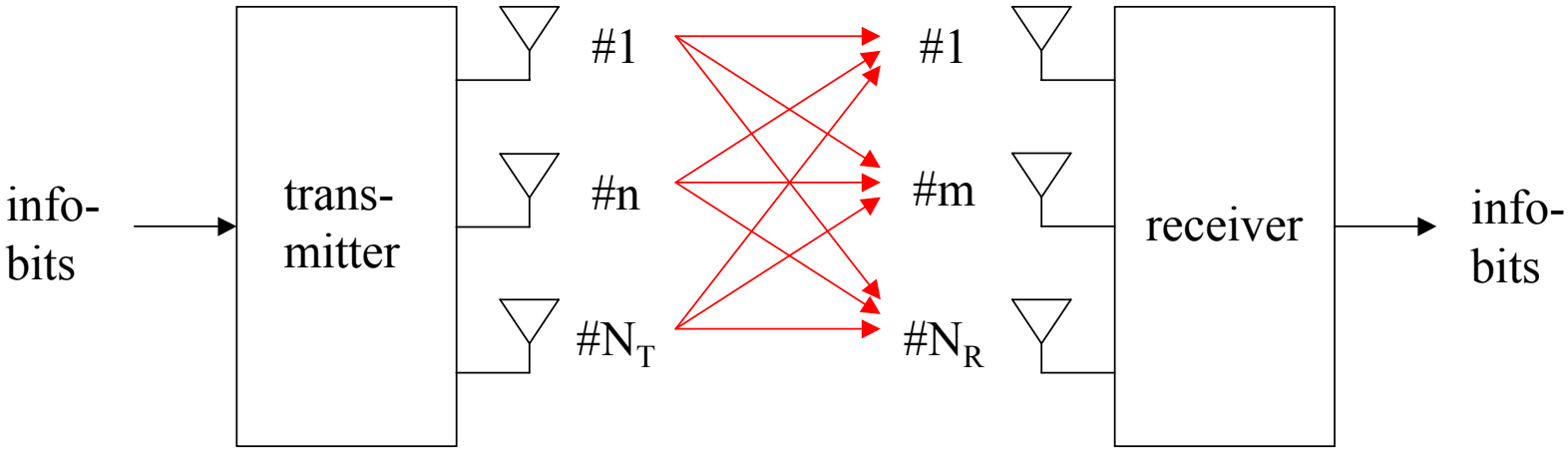
Multiple-input multiple-output (MIMO) communication systems

System model



N_T transmit antennas $\begin{pmatrix} a_1(k) \\ \vdots \\ a_{N_T}(k) \end{pmatrix}$ $\begin{pmatrix} r_1(k) \\ \vdots \\ r_{N_R}(k) \end{pmatrix}$ N_R receive antennas

k : time index (k = 1, ..., K)

(coded) data symbols $a_n(k)$, $E[|a_n(k)|^2] = E_s$

System model

$$r_m(k) = \sum_{n=1}^{N_T} h_{m,n} a_n(k) + w_m(k) \qquad (m = 1, ..., N_R;\ k = 1, ..., K)$$

$w_m(k)$: complex i.i.d. AWGN, $E[|w_m(k)|^2] = N_0$

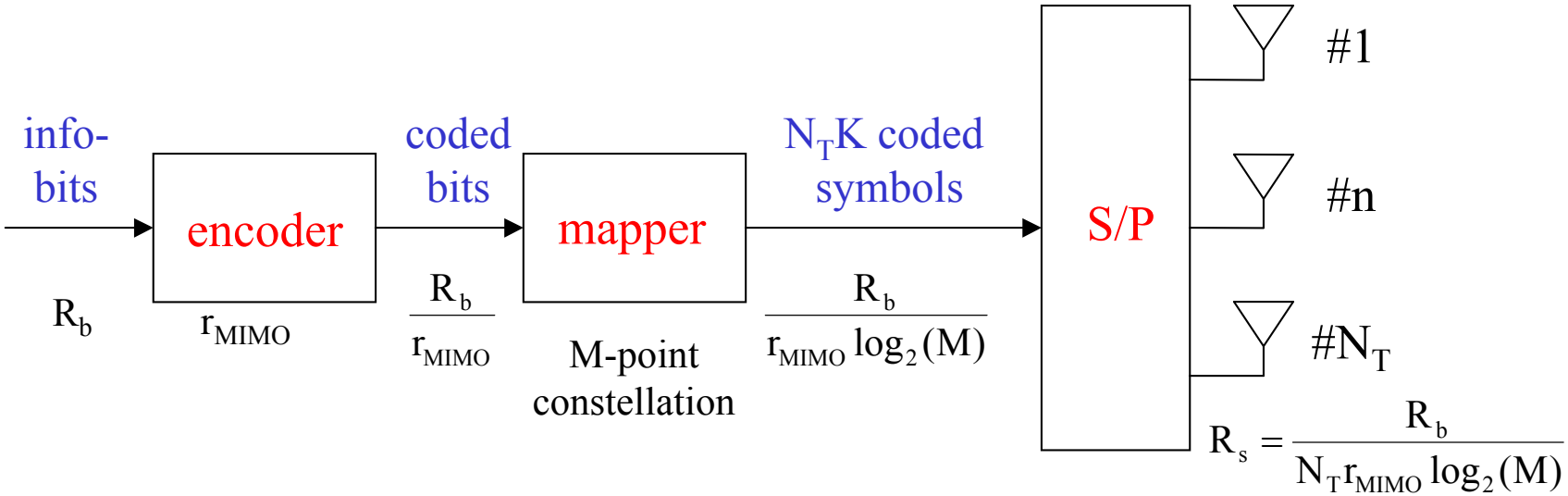
$h_{m,n}$: complex Gaussian zero-mean i.i.d. channel gains (*Rayleigh fading*), $E[|h_{m,n}|^2] = 1$

$$\underbrace{\begin{pmatrix} r_1(k) \\ \vdots \\ r_{N_R}(k) \end{pmatrix}}_{\mathbf{r}(k)} = \underbrace{\begin{pmatrix} h_{1,1} & \cdots & h_{1,N_T} \\ \vdots & \ddots & \vdots \\ h_{N_R,1} & \cdots & h_{N_R,N_T} \end{pmatrix}}_{\mathbf{H}} \cdot \underbrace{\begin{pmatrix} a_1(k) \\ \vdots \\ a_{N_T}(k) \end{pmatrix}}_{\mathbf{a}(k)} + \underbrace{\begin{pmatrix} w_1(k) \\ \vdots \\ w_{N_R}(k) \end{pmatrix}}_{\mathbf{w}(k)} \qquad \mathbf{r}(k) = \mathbf{H}\cdot\mathbf{a}(k) + \mathbf{w}(k)$$

$$\underbrace{(\mathbf{r}(1),\cdots,\mathbf{r}(k),\cdots,\mathbf{r}(K))}_{\mathbf{R}} = \mathbf{H}\cdot\underbrace{(\mathbf{a}(1),\cdots,\mathbf{a}(k),\cdots,\mathbf{a}(K))}_{\mathbf{A}} + \underbrace{(\mathbf{w}(1),\cdots,\mathbf{w}(k),\cdots,\mathbf{w}(K))}_{\mathbf{W}}$$

$$\mathbf{R} = \mathbf{H}\cdot\mathbf{A} + \mathbf{W}$$

System model



R_b : information bitrate
R_s : symbol rate *per transmit antenna*
E_b : energy per infobit
E_s : energy per symbol

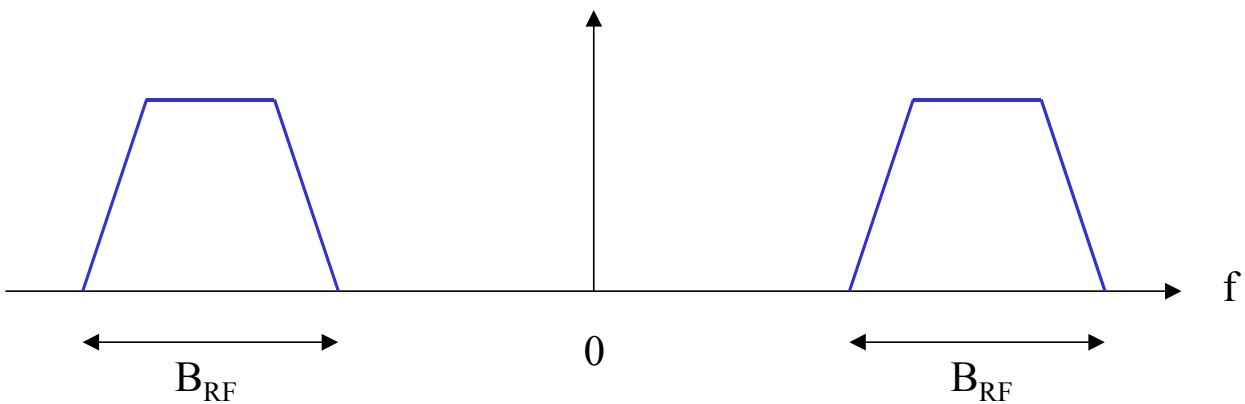
$$r_{MIMO} = \frac{\#\text{infobits}}{\#\text{coded bits}} \qquad \text{(code rate)}$$

$$0 < r_{MIMO} \leq 1$$

$$E_s = E_b r_{MIMO}\log_2(M) \qquad R_b = R_s N_T r_{MIMO}\log_2(M)$$

System model

Spectrum transmitted bandpass signal



$B_{RF} \geq R_s$ (necessary condition to avoid ISI between successive symbols)

All N_T transmitted signals are simultaneously in the same frequency interval

Bandwidth efficiency :

$$R_b/R_s = N_T \cdot r_{MIMO} \cdot \log_2(M) \qquad \text{(bit/s/Hz)}$$

ML detection

Log-likelihood function ln(p(**R** |**A**, **H**) :

$$-N_0 \ln p(\mathbf{R} \mid \mathbf{A}, \mathbf{H}) = \sum_{m=1}^{N_R} \sum_{k=1}^{K} \left| r_m(k) - \sum_{n=1}^{N_T} h_{m,n} a_n(k) \right|^2$$

$$= \| \mathbf{R} - \mathbf{H} \cdot \mathbf{A} \|^2 = \mathrm{Tr}[(\mathbf{R} - \mathbf{H} \cdot \mathbf{A})^H (\mathbf{R} - \mathbf{H} \cdot \mathbf{A})]$$

ML detection : $\hat{\mathbf{A}} = \arg\min_{\mathbf{A}} \| \mathbf{R} - \mathbf{H} \cdot \mathbf{A} \|^2$ (minimization over all symbol matrices that obey the encoding rule)

Frobenius norm of **X** : $\| \mathbf{X} \|^2 = \sum_{i,j} | x_{i,j} |^2 = \mathrm{Tr}[\mathbf{X}^H \mathbf{X}] = \mathrm{Tr}[\mathbf{X} \cdot \mathbf{X}^H]$

Trace of square matrix **M** : $\mathrm{Tr}[\mathbf{M}] = \sum_i m_{i,i}$

Properties : $\mathrm{Tr}[\mathbf{P} \cdot \mathbf{Q}] = \mathrm{Tr}[\mathbf{Q} \cdot \mathbf{P}]$, $\mathrm{Tr}[\mathbf{M}] = \mathrm{Tr}[\mathbf{M}^T]$

$\mathbf{X}^H$ denotes Hermitian transpose (= complex conjugate transpose $(\mathbf{X}^*)^T$)

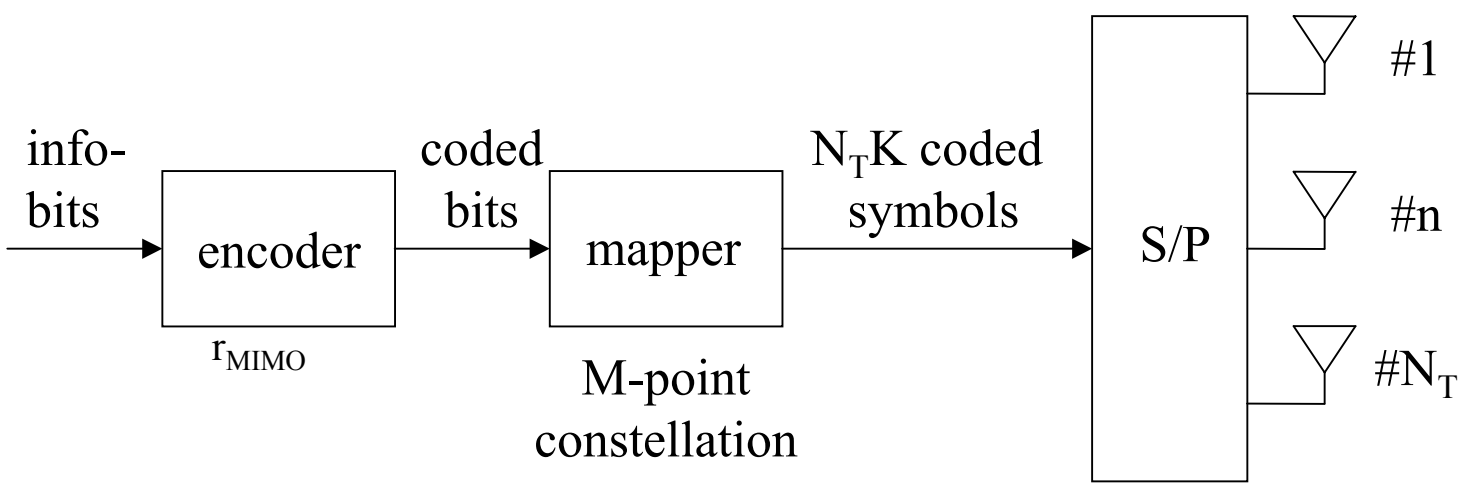
Bit error rate (BER)

$$\mathrm{BER} = \frac{E[\#\,\text{infobit errors}]}{\#\,\text{infobits transmitted}}$$

symbol matrix **A** represents $N_T K r_{MIMO} \log_2(M)$ infobits

$$\mathrm{BER} = \frac{\sum_{\mathbf{A}^{(0)},\mathbf{A}^{(i)}} N_{\text{info}}^{(i,0)} \Pr[\hat{\mathbf{A}} = \mathbf{A}^{(i)}, \mathbf{A} = \mathbf{A}^{(0)}]}{N_T K \log_2(M) r_{MIMO}}$$

$N_{\text{info}}^{(i,0)}$: # infobits in which $\mathbf{A}^{(i)}$ and $\mathbf{A}^{(i)}$ are different



Bit error rate (BER)

$$\Pr[\hat{\mathbf{A}} = \mathbf{A}^{(i)}, \mathbf{A} = \mathbf{A}^{(0)}] = \Pr[\hat{\mathbf{A}} = \mathbf{A}^{(i)} \mid \mathbf{A} = \mathbf{A}^{(0)}]\Pr[\mathbf{A} = \mathbf{A}^{(0)}]$$

$$\Pr[\mathbf{A} = \mathbf{A}^{(0)}] = 2^{-N_T K \log_2(M) r_{MIMO}} = M^{-N_T K \cdot r_{MIMO}}$$

$$\Pr[\hat{\mathbf{A}} = \mathbf{A}^{(i)} \mid \mathbf{A} = \mathbf{A}^{(0)}] = E_{\mathbf{H}}\left[\Pr[\hat{\mathbf{A}} \neq \mathbf{A}^{(0)} \mid \mathbf{A} = \mathbf{A}^{(0)}, \mathbf{H}]\right]$$ (averaging over statistics of **H**)

$$\Pr[\hat{\mathbf{A}} = \mathbf{A}^{(i)} \mid \mathbf{A} = \mathbf{A}^{(0)}, \mathbf{H}] \leq PEP^{(i,0)}(\mathbf{H})$$

$$PEP^{(i,0)}(\mathbf{H}) = \Pr[||\mathbf{R} - \mathbf{H}\cdot\mathbf{A}^{(i)}||^2 < ||\mathbf{R} - \mathbf{H}\cdot\mathbf{A}^{(0)}||^2 | \ \mathbf{A} = \mathbf{A}^{(0)}, \mathbf{H}]$$

$PEP^{(i,0)}(\mathbf{H})$: pairwise error probability, conditioned on **H**

$PEP^{(i,0)} = E_{\mathbf{H}}[PEP^{(i,0)}(\mathbf{H})]$: pairwise error probability, averaged over **H**

Bit error rate (BER)

$$\mathrm{BER} \le \frac{\sum_{\mathbf{A}_0}\left(\Pr[\mathbf{A}=\mathbf{A}^{(0)}]\sum_{i\neq 0} N_{\mathrm{info}}^{(i,0)}\mathrm{PEP}^{(i,0)}\right)}{N_T K \log_2(M) r_{\mathrm{MIMO}}}$$

Define error matrix : $\mathbf{\Delta}^{(i,0)} = \mathbf{A}^{(0)} - \mathbf{A}^{(i)}$

$$\mathrm{PEP}^{(i,0)}(\mathbf{H}) = Q\left(\sqrt{\frac{\|\mathbf{H}\cdot\mathbf{\Delta}^{(i,0)}\|^2}{2N_0}}\right) \le \frac{1}{2}\exp\left(\frac{-\|\mathbf{H}\cdot\mathbf{\Delta}^{(i,0)}\|^2}{4N_0}\right)$$

Averaging over Rayleigh fading channel statistics :

$$\mathrm{PEP}^{(i,0)} = E_{\mathbf{H}}\left[\mathrm{PEP}^{(i,0)}(\mathbf{H})\right] \le \frac{1}{2}\left(\det\left[\mathbf{I}_{N_T} + \frac{\mathbf{\Delta}^{(i,0)}\cdot(\mathbf{\Delta}^{(i,0)})^H}{4N_0}\right]\right)^{-N_R} = \frac{1}{2}\left(\prod_{n=1}^{N_T}\left(1+\frac{\lambda_n^{(i,0)}}{4N_0}\right)\right)^{-N_R}$$

$\{\lambda_n^{(i,0)}, n = 0,\ldots,N_T\}$ set of eigenvalues of $N_T x N_T$ matrix $\mathbf{\Delta}^{(i,0)}\cdot(\mathbf{\Delta}^{(i,0)})^H$
(these eigenvalues are real-valued non-negative)

To be further considered

Single-input single-output (SISO) systems : $N_T = N_R = 1$

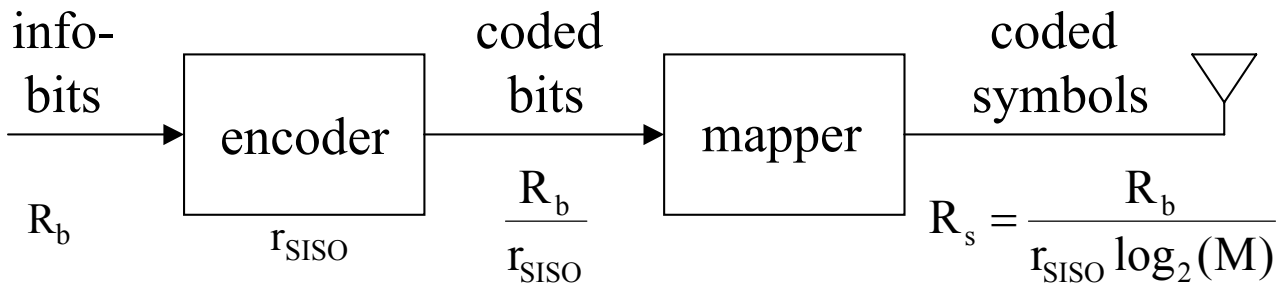
Single-input multiple-output (SIMO) systems : $N_T = 1$, $N_R \geq 1$

Multiple-input multiple-output (MIMO) systems : $N_T \geq 1$, $N_R \geq 1$

- Spatial multiplexing : ML detection, ZF detection
- Space-time coding : trellis coding, delay diversity, block coding

Single-input single-output (SISO) systems

SISO : observation model



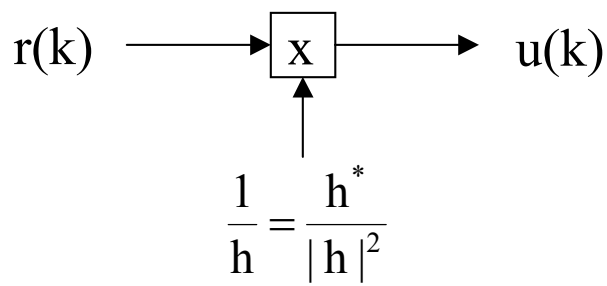
Observation model : $r(k) = h \cdot a(k) + w(k) \quad k = 1, ..., K$

Bandwidth efficiency : $R_b/R_s = r_{SISO} \cdot \log_2(M)$ (bit/s/Hz)

SISO : ML detection

$$\{\hat{a}(k)\} = \arg\min_{\{a(k)\}} \sum_k |r(k) - h \cdot a(k)|^2 = \arg\min_{\{a(k)\}} \sum_k |u(k) - a(k)|^2$$

$$u(k) = \frac{r(k)h^*}{|h|^2} = \frac{r(k)}{h}$$



ML detector looks for codeword that is closest (in Euclidean distance) to $\{u(k)\}$

$$\#\text{ codewords} = 2^{K \cdot \log_2(M) \cdot r_{SISO}} = M^{K \cdot r_{SISO}}$$

In general : decoding complexity exponential in K
(trellis codes, convolutional codes :
complexity *linear* in K by using Viterbi algorithm)

SISO : PEP computation

Conditional PEP : $$\mathrm{PEP}^{(i,0)}(h) \le \frac{1}{2}\exp\left(\frac{-|h|^2\, d^2_{E,(i,0)}}{4N_0}\right)$$

$$d^2_{E,(i,0)} = \sum_{k=1}^{K} |a^{(0)}(k) - a^{(i)}(k)|^2$$

squared Euclidean distance between codewords $\{a^{(0)}(k)\}$ and $\{a^{(i)}(k)\}$

$$d^2_{E,(i,0)} \propto E_s = r_{SISO}\log_2(M)E_b$$

Average PEP, AWGN ($|h| = 1$) : $$\mathrm{PEP}^{(i,0)} \le \frac{1}{2}\exp\left(\frac{-d^2_{E,(i,0)}}{4N_0}\right)$$

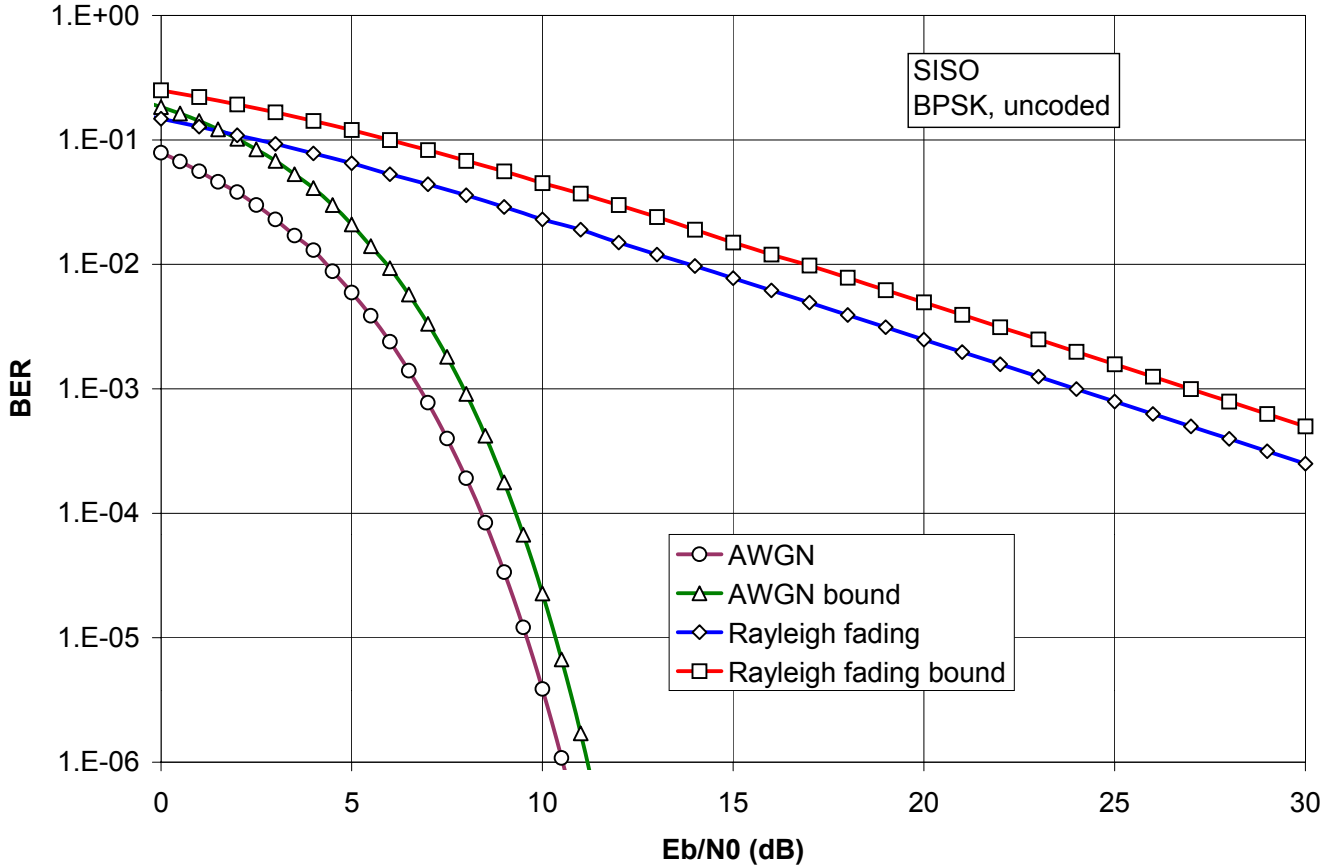
decreases exponentially with E_b/N_0

Average PEP, Rayleigh fading : $$\mathrm{PEP}^{(i,0)} \le \frac{1}{2}\left(1 + \frac{d^2_{E,(i,0)}}{4N_0}\right)^{-1}$$

inversely proportional to E_b/N_0

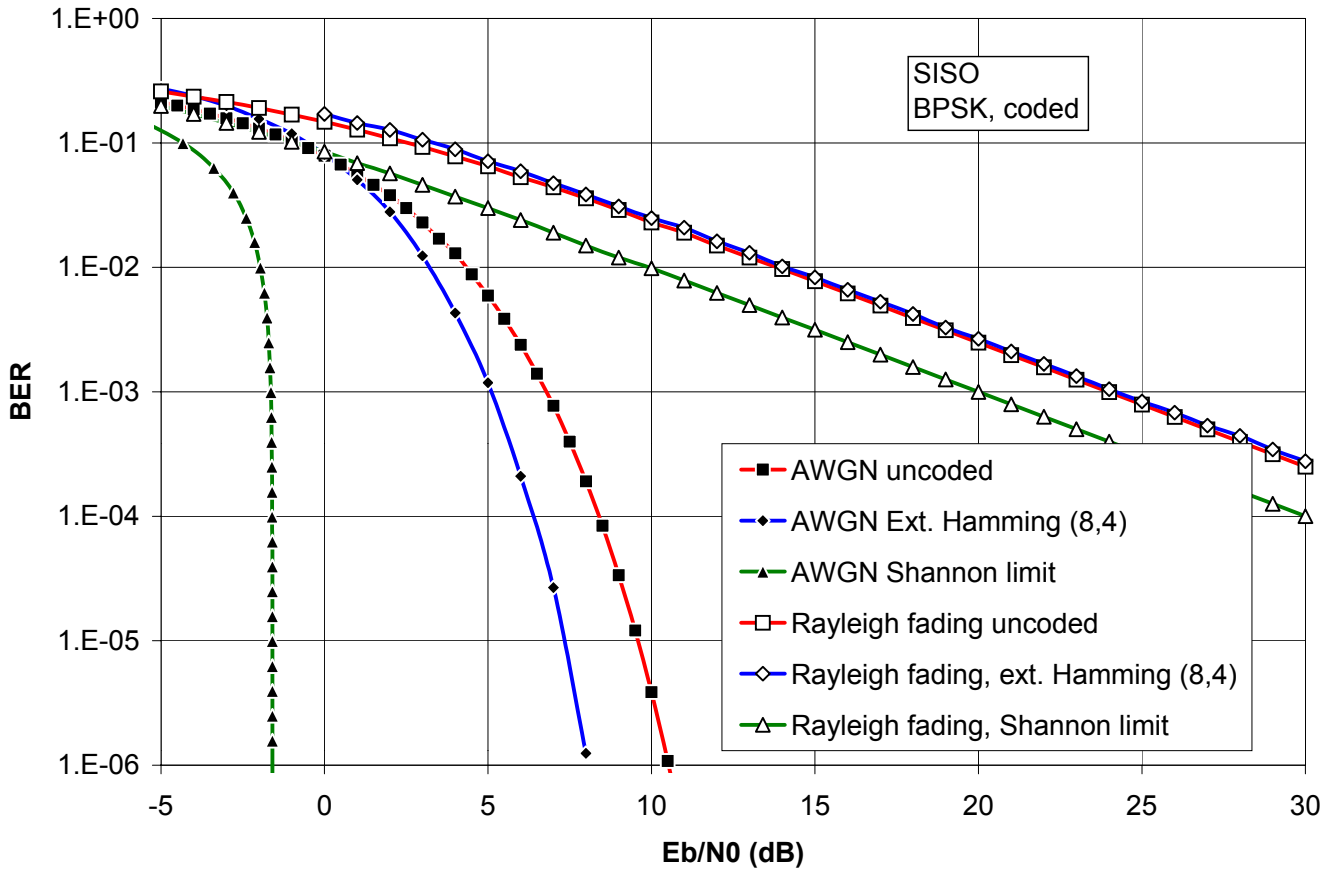
(occasionally, $|h|$ becomes very small $\Rightarrow$ large PEP)

SISO : numerical results



BER fading channel >> BER AWGN channel

SISO : numerical results

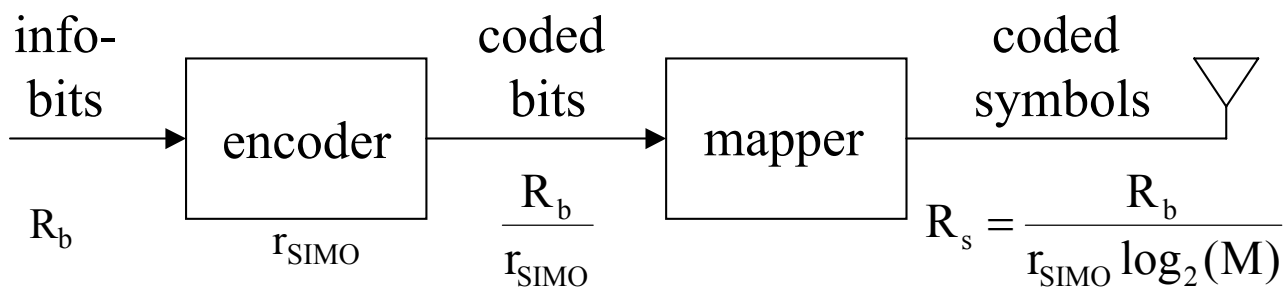


coding gain on fading channel << coding gain on AWGN channel

Single-input multiple-output (SIMO) systems

SIMO : observation model

Same transmitter as for SISO; N_R antennas at receiver



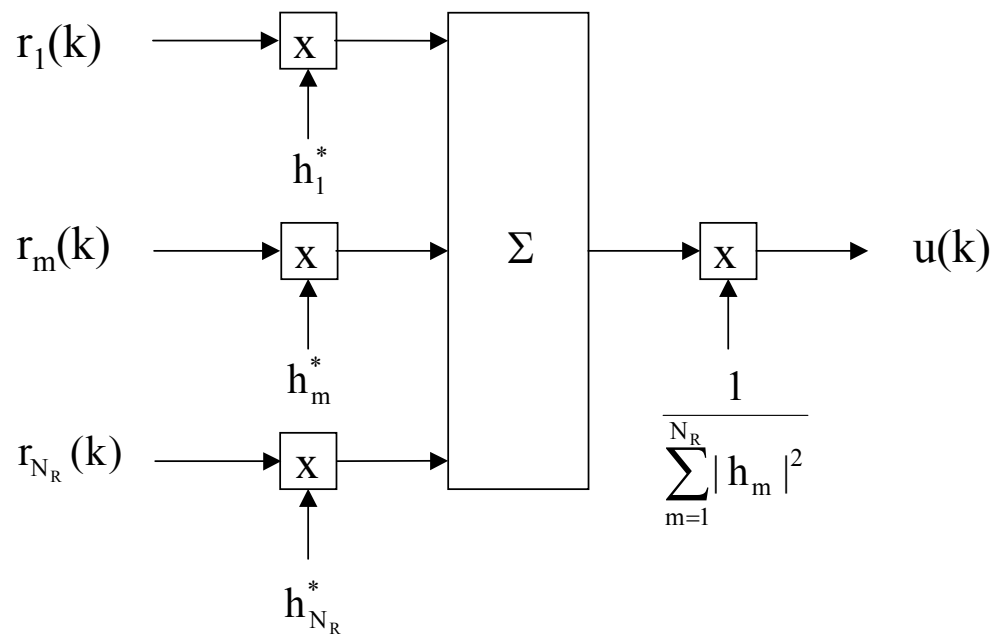
Observation model : $r_m(k) = h_m a(k) + w_m(k)$ $m = 1, ..., N_R$; $k = 1, ..., K$

Bandwidth efficiency : $R_b/R_s = r_{SIMO} \cdot \log_2(M)$ (bit/s/Hz)

SIMO : ML detection

$$\{\hat{a}(k)\} = \arg\min_{\{a(k)\}} \sum_k \sum_{m=1}^{N_R} |r_m(k) - h_m a(k)|^2 = \arg\min_{\{a(k)\}} \sum_k |u(k) - a(k)|^2$$

$$u(k) = \frac{\sum_{m=1}^{N_R} h_m^* r_m(k)}{\sum_{m=1}^{N_R} |h_m|^2}$$



ML detector looks for codeword that is closest to $\{u(k)\}$

Same decoding complexity as for SISO

SIMO : PEP computation

Conditional PEP : $$\mathrm{PEP}^{(i,0)}(\mathbf{h}) \le \frac{1}{2}\exp\left(\frac{-N_R d^2_{E,(i,0)}(|h|^2)_{avg}}{4N_0}\right) \qquad (|h|^2)_{avg} = \frac{1}{N_R}\sum_{m=1}^{N_R}|h_m|^2$$

PEP, AWGN ($|h_m| = 1$) : $$\mathrm{PEP}^{(0,i)} \le \frac{1}{2}\exp\left(\frac{-N_R d^2_{E,(i,0)}}{4N_0}\right)$$

decreases exponentially with E_b/N_0

total received energy per symbol increases by factor N_R as compared to SISO
$\Rightarrow$ same PEP as with SISO, but with E_b replaced by $N_R E_b$
("array gain" of $10\log(N_R)$ dB)

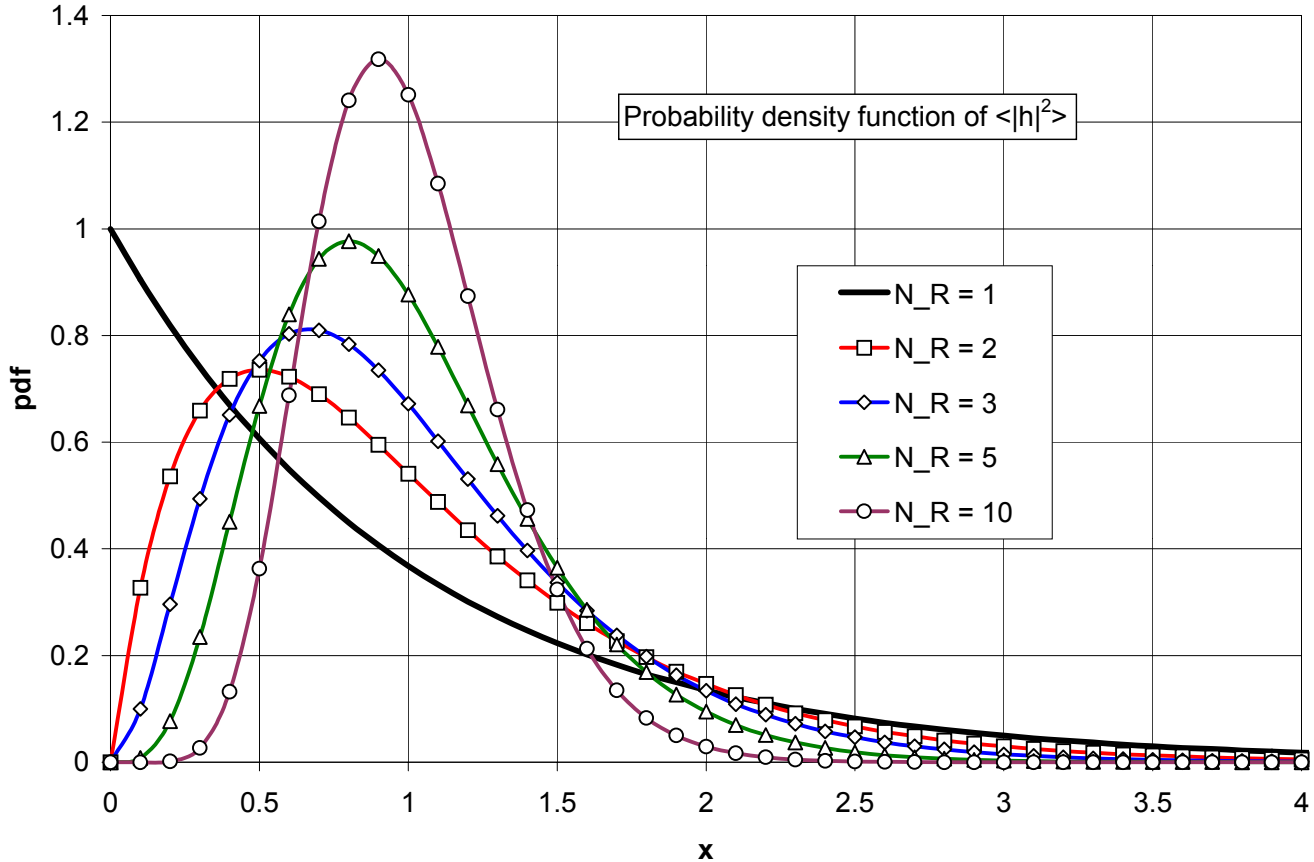
PEP, Rayleigh fading : $$\mathrm{PEP}^{(i,0)} \le \frac{1}{2}\left(1+\frac{d^2_{E,(i,0)}}{4N_0}\right)^{-N_R}$$

inversely proportional to (E_b/N_0)^N_R

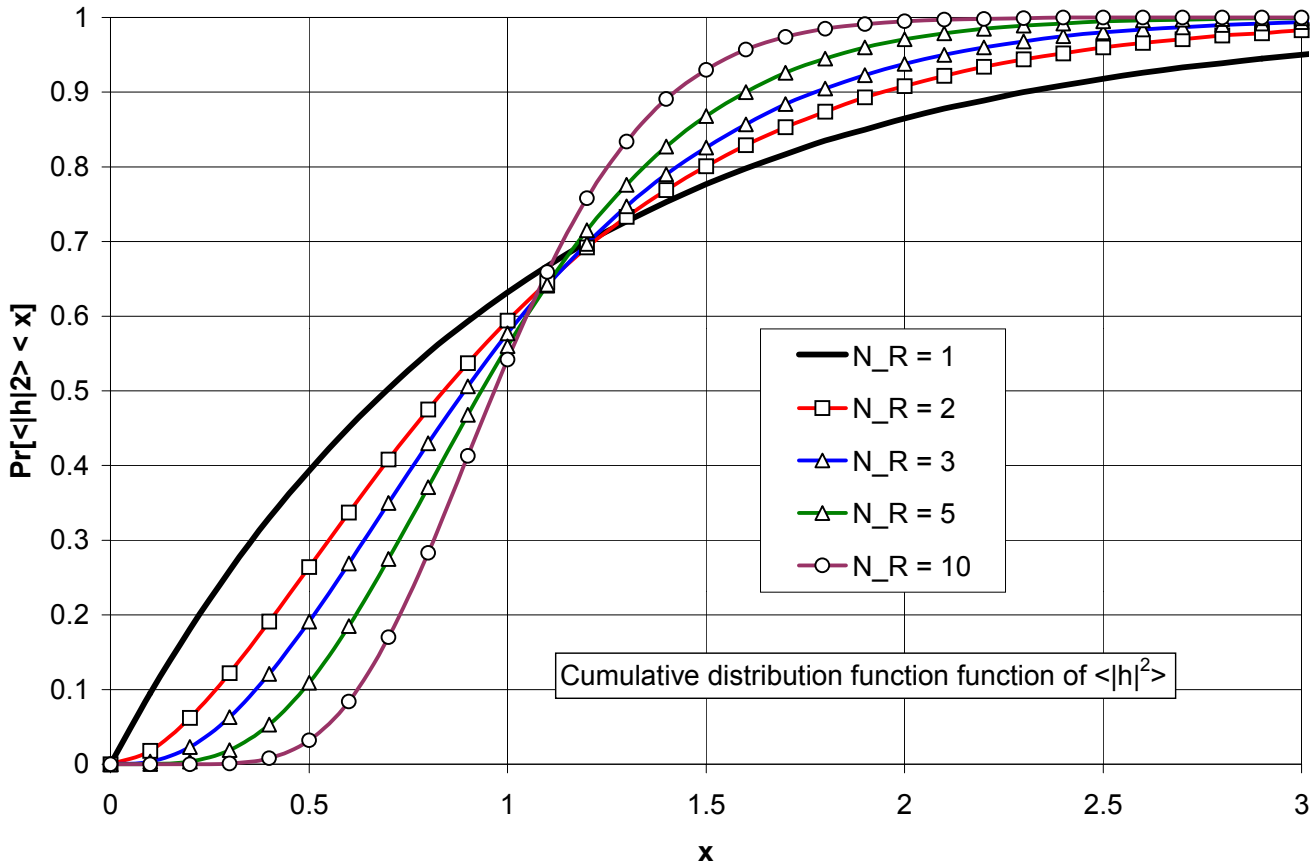
PEP is smaller than in SISO setup with E_b replaced by $N_R E_b$

- "array gain" of $10\log(N_R)$ dB
- additional "diversity gain" (diversity order N_R) :
  probability that $(|h|^2)_{avg}$ is small, decreases with N_R

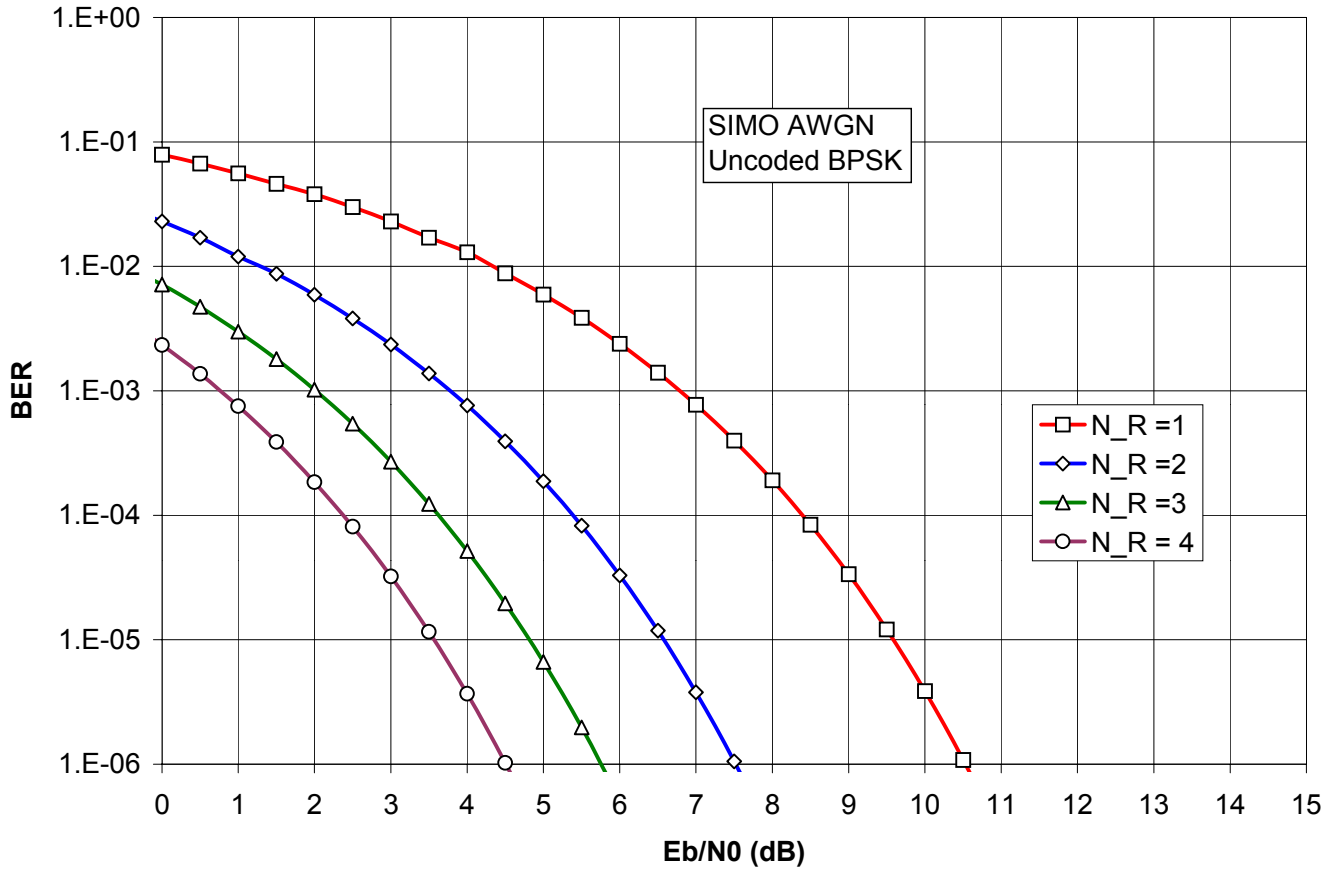
SISO : statistical properties of $<|h|^2>$



SISO : statistical properties of <|h|^2>

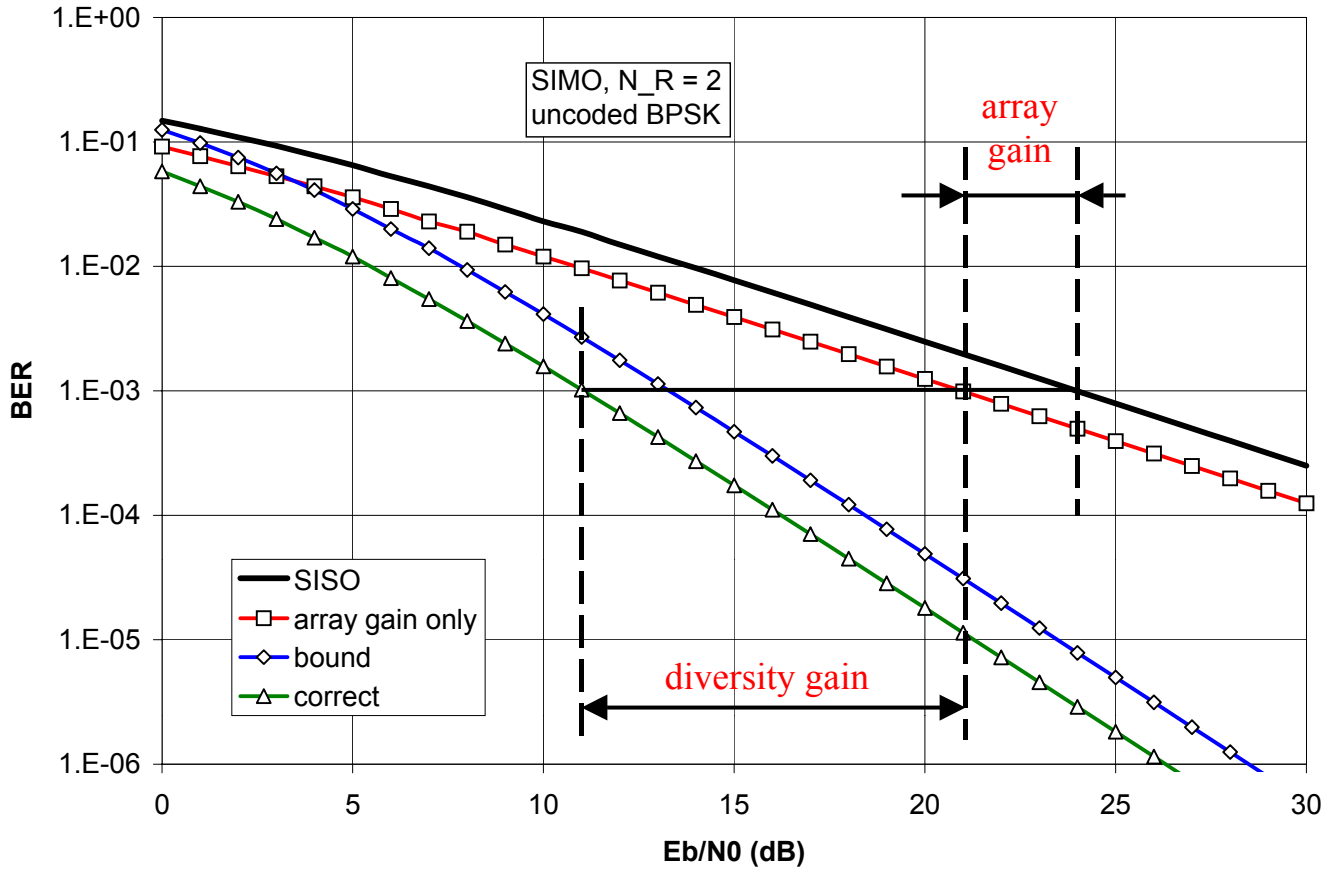


SIMO : numerical results



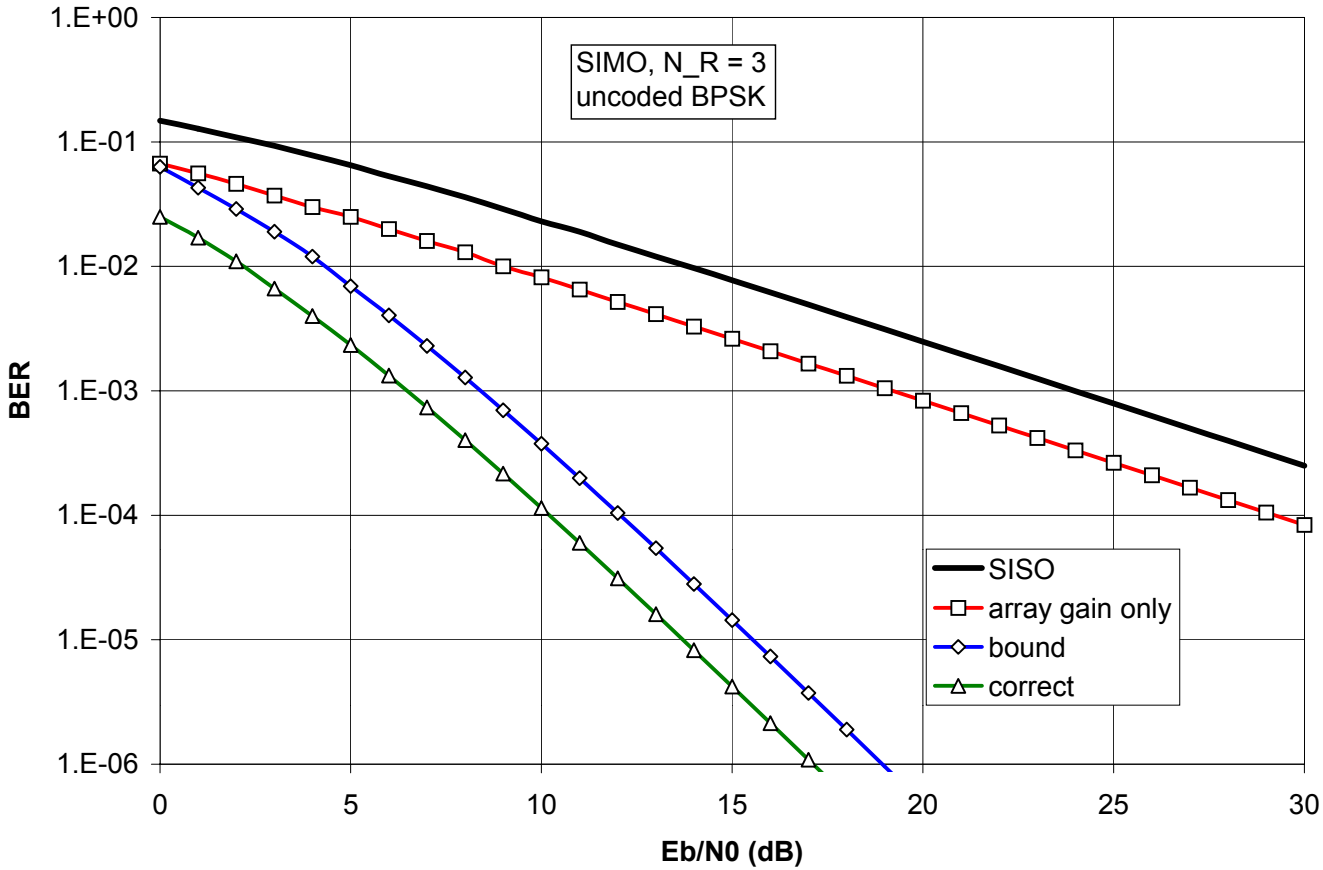
AWGN channel : array gain of $10 \cdot \log(N_R)$ dB

SIMO : numerical results



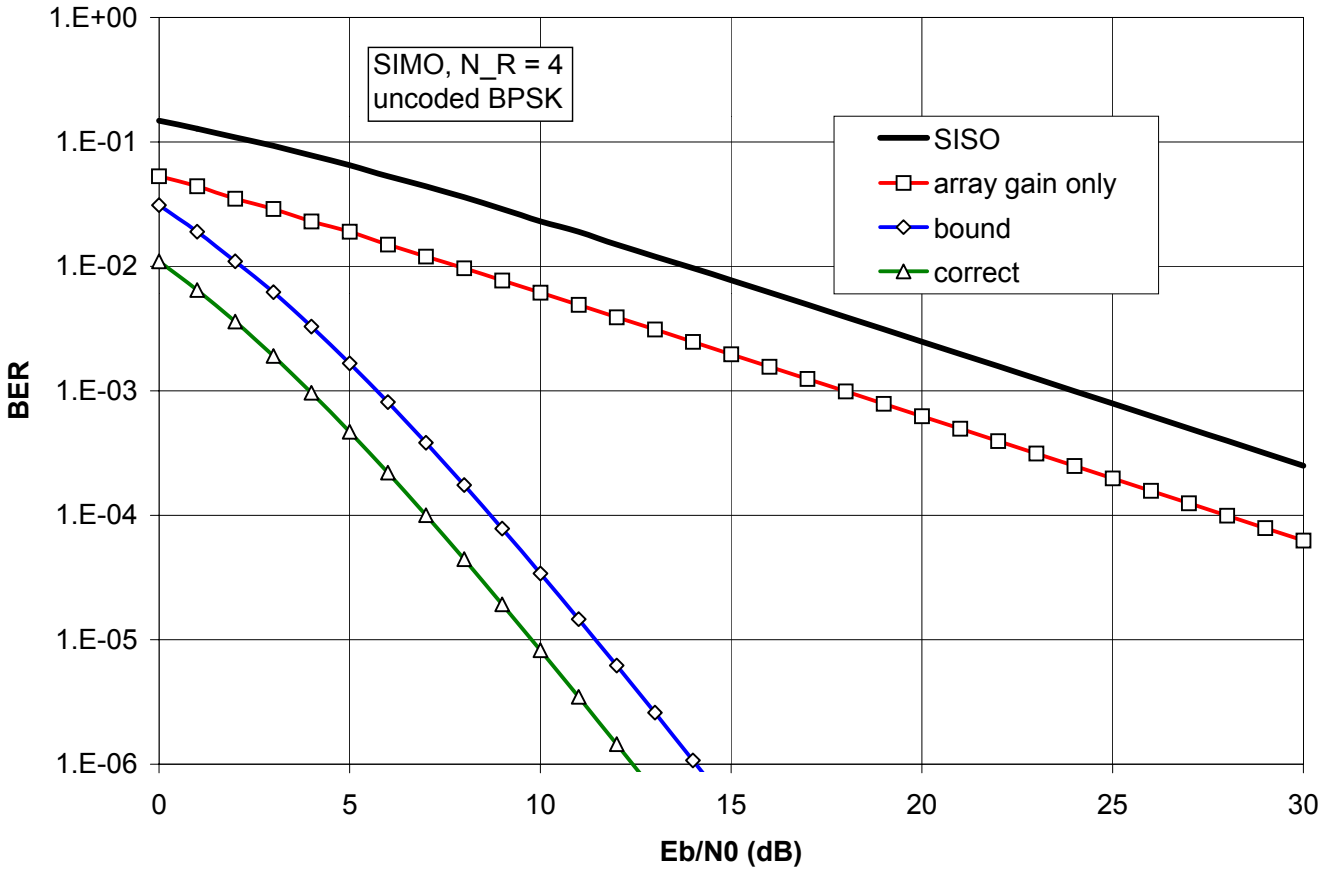
fading channel : array gain + diversity gain

SIMO : numerical results



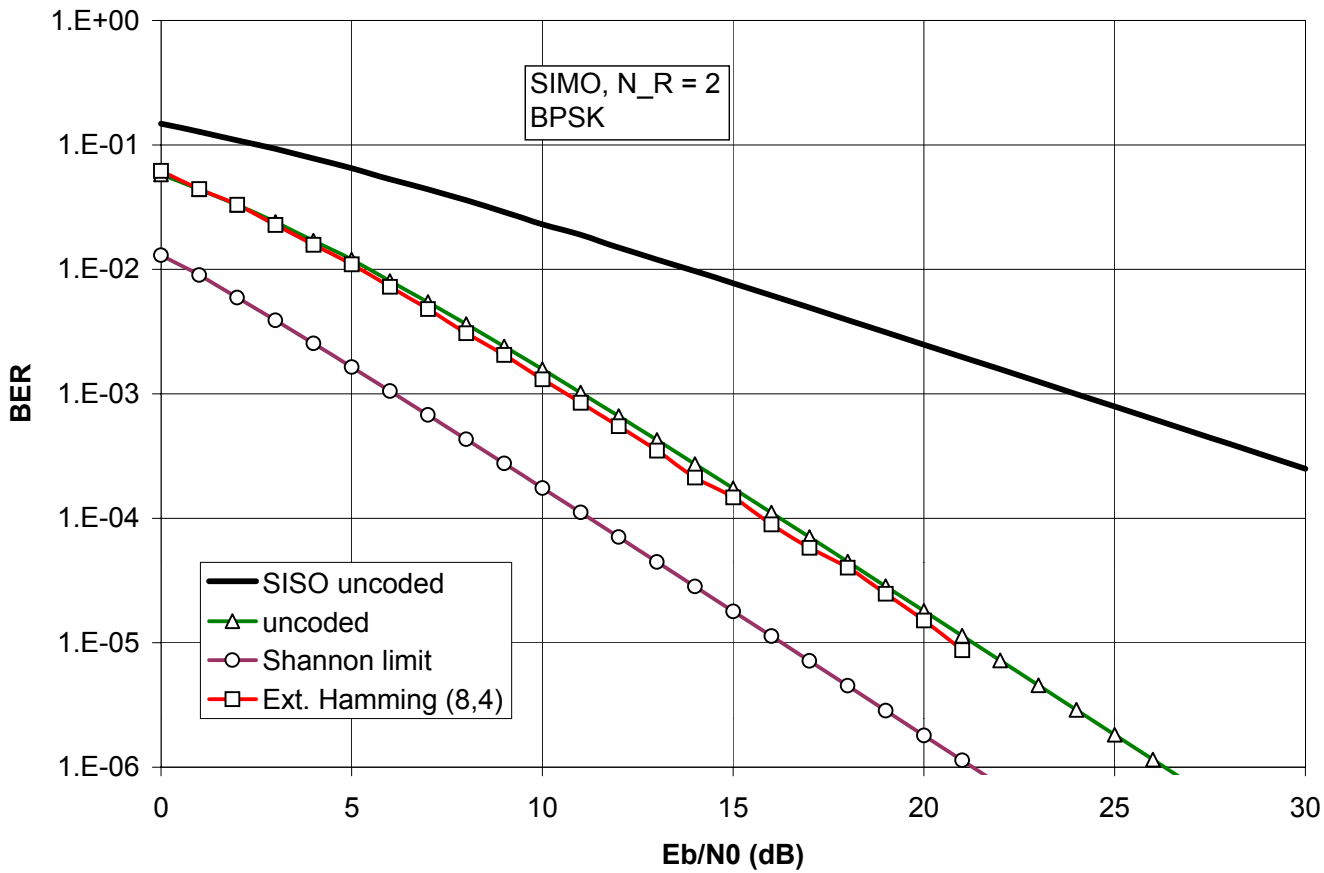
fading channel : array gain + diversity gain

SIMO : numerical results



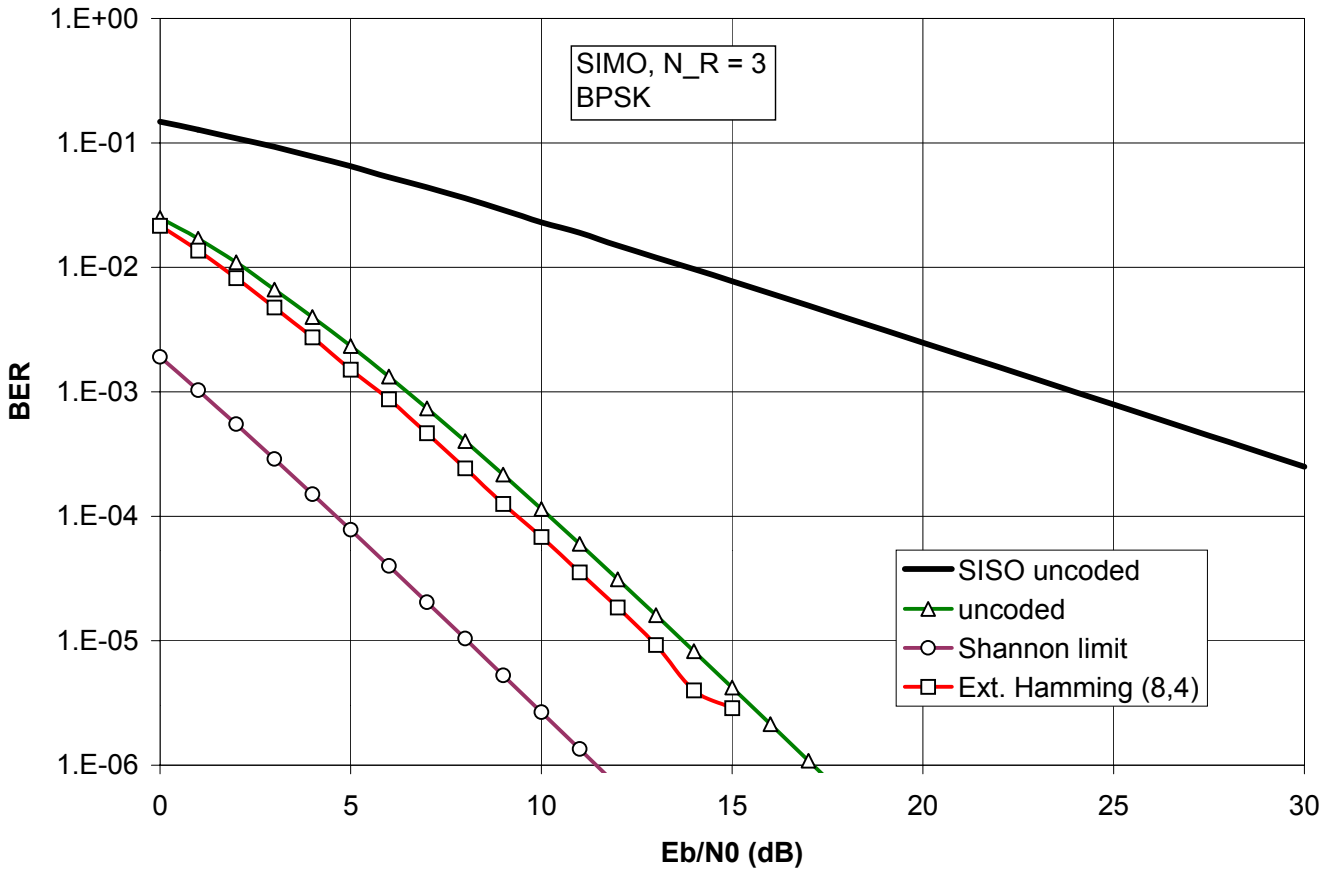
fading channel : array gain + diversity gain

SIMO : numerical results



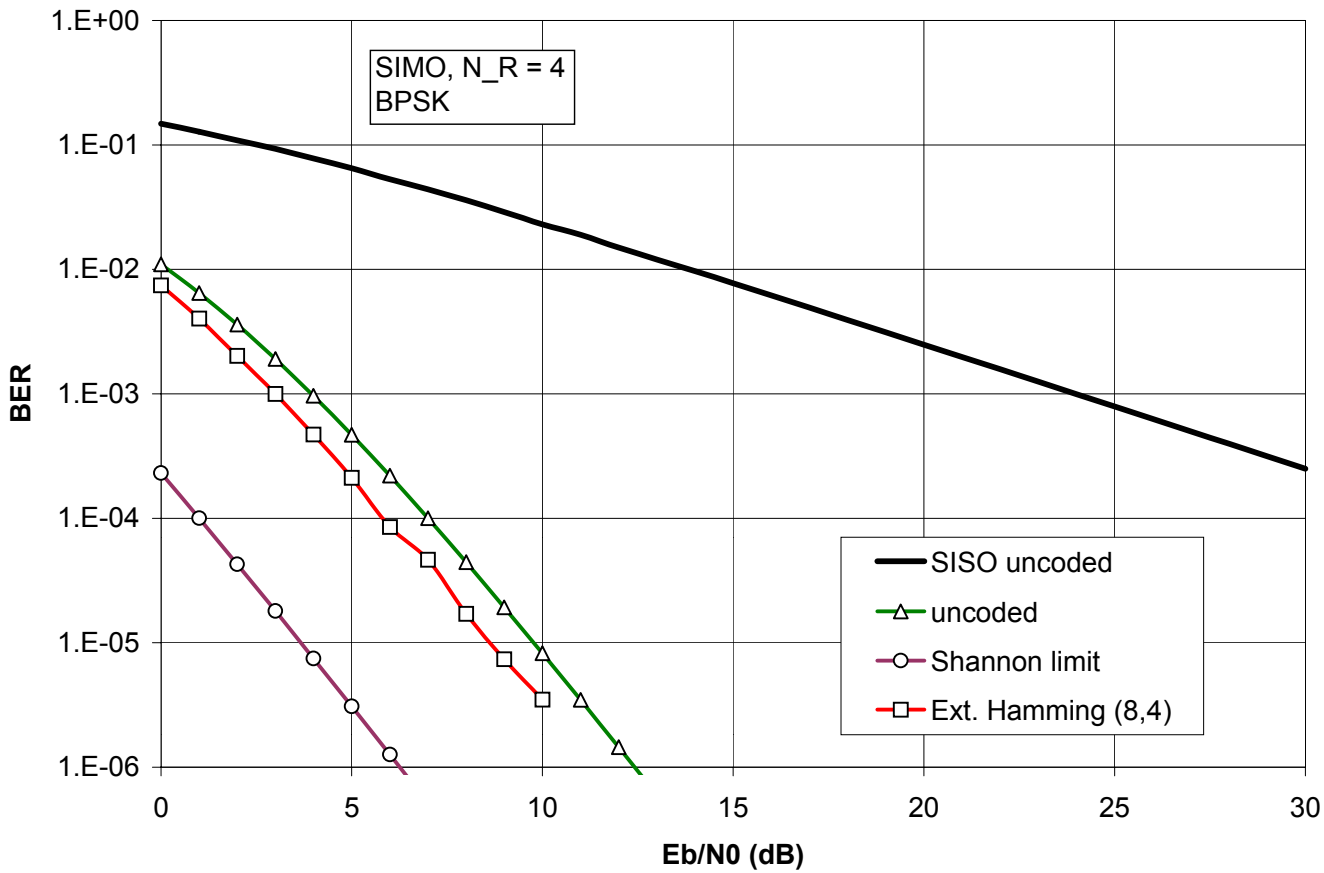
fading channel : coding gain increases with N_R

SIMO : numerical results



fading channel : coding gain increases with N_R

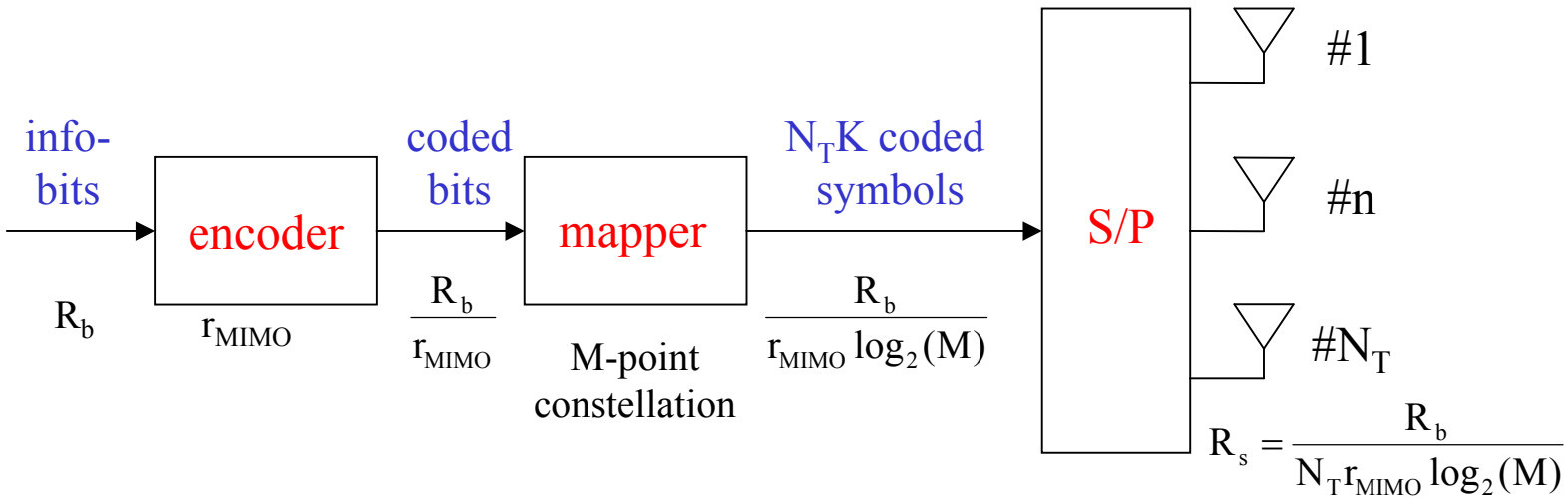
SIMO : numerical results



fading channel : coding gain increases with N_R

Multiple-input multiple-output (MIMO) systems

MIMO : observation model



$$r_m(k) = \sum_{n=1}^{N_T} h_{m,n} a_n(k) + w_m(k) \qquad (m = 1, \ldots, N_R;\ k = 1, \ldots, K)$$

$$\mathbf{R} = \mathbf{H}\cdot\mathbf{A} + \mathbf{W}$$

at any receive antenna, symbols from different transmit antennas interfere

Bandwidth efficiency : $R_b/R_s = N_T r_{MIMO}\cdot\log_2(M)$ (bit/s/Hz)

MIMO : ML detection

ML detector : $\hat{\mathbf{A}} = \arg\min_{\mathbf{A}} \| \mathbf{R} - \mathbf{H} \cdot \mathbf{A} \|^2$

different symbol matrices $\mathbf{A} = 2^{N_T \cdot r_{MIMO} \cdot \log_2(M) \cdot K} = M^{N_T \cdot r_{MIMO} \cdot K}$

$\Rightarrow$ detector complexity in general increases exponentially with $N_T r_{MIMO}$ and K

MIMO : PEP computation

PEP, Rayleigh fading :

$$\mathrm{PEP}^{(i,0)} \le \frac{1}{2}\left(\det\left[\mathbf{I}_{N_T} + \frac{\Delta^{(i,0)} \cdot \Delta^{(i,0)H}}{4N_0}\right]\right)^{-N_R} = \frac{1}{2}\left(\prod_{n=1}^{N_T}\left(1 + \frac{\lambda_n^{(i,0)}}{4N_0}\right)\right)^{-N_R}$$

nonzero eigenvalues equals rank($\Delta^{(i,0)}$), with $1 \le \mathrm{rank}(\Delta^{(i,0)}) \le N_T$

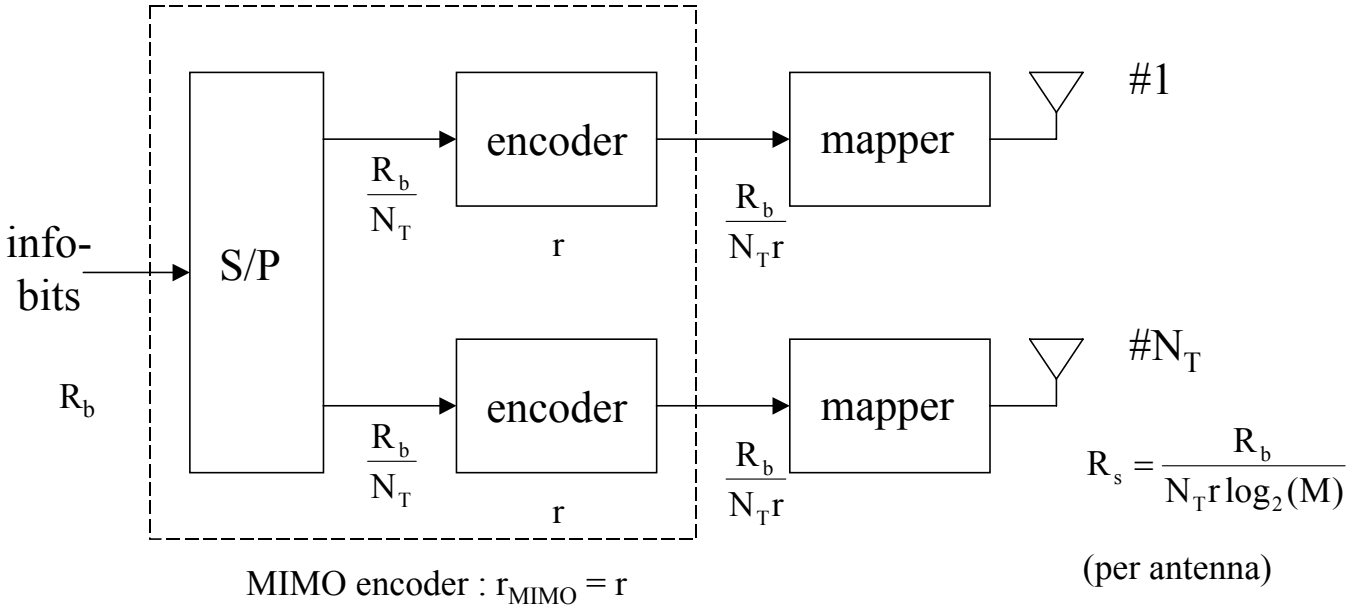
$\mathrm{PEP}^{(i,0)}$ inversely proportional to (E_b/N_0)^($N_R \cdot \mathrm{rank}(\Delta^{(i,0)})$)

PEPs that dominate BER are those for which $\Delta^{(i,0)} = \mathbf{A}^{(0)} - \mathbf{A}^{(i)}$ has minimum rank.

$$\Rightarrow \quad \text{diversity order} = N_R \cdot \left(\min_{(i,0)} \mathrm{rank}(\Delta^{(i,0)})\right)$$

MIMO : spatial multiplexing

Spatial multiplexing



$$\mathbf{A} = \begin{pmatrix} \mathbf{a}_1^T \\ \vdots \\ \mathbf{a}_{N_T}^T \end{pmatrix}$$

$$\mathbf{a}_n^T = (a_n(1), \cdots, a_n(k), \cdots, a_n(N))$$

codeword transmitted by antenna #n

codewords transmitted by different antennas are statistically independent

Aim : to increase bandwidth efficiency by a factor N_T as compared to SISO/SIMO

Bandwidth efficiency : $R_b/R_s = N_T \cdot r \cdot \log_2(M)$

Spatial multiplexing : ML detection

ML detector : $\hat{\mathbf{A}} = \arg\min_{\mathbf{A}} \| \mathbf{R} - \mathbf{H} \cdot \mathbf{A} \|^2$

$$= \arg\min_{\mathbf{A}} \left(\sum_{n,n'=1}^{N_T} (\mathbf{H}^H \mathbf{H})_{n,n'} (\mathbf{a}_n^H \mathbf{a}_{n'}) - 2\,\mathrm{Re}\left[\sum_{n=1}^{N_T} \mathbf{a}_n^H \mathbf{u}_n \right] \right)$$

$(\mathbf{u}_n)^T$: n-th row of $\mathbf{H}^H\mathbf{R}$

As $\mathbf{H}^H\mathbf{H}$ is nondiagonal, codewords on different rows must be detected *jointly* instead of individually

different symbol matrices $\mathbf{A} = 2^{N_T \cdot r \cdot \log_2(M) \cdot K} = M^{N_T \cdot r \cdot K}$

$\Rightarrow$ ML detector complexity increases exponentially with N_T as compared to SISO/SIMO

Spatial multiplexing : PEP computation

$\Delta^{(i,0)}$ nonzero $\Rightarrow$ rank($\Delta^{(i,0)}$) ≥ 1

rank($\Delta^{(i,0)}$) = 1 when all rows of $\Delta^{(i,0)}$ are proportional to a common row vector $\delta^T = (\delta(1), ..., \delta(k), ..., \delta(K))$.

Example : $\Delta^{(i,0)}$ has only one nonzero row; this corresponds to a detection error in only one row of **A**

$$\Rightarrow \min_{i \neq 0} \operatorname{rank}(\Delta^{(i,0)}) = 1$$

Denoting the n-th row of a rank-1 error matrix $\Delta^{(i,0)}$ by $\alpha_n \delta^T$, the bound on the corresponding PEP is

$$\mathrm{PEP}^{(i,0)} \leq \frac{1}{2}\left(1 + \frac{|\boldsymbol{\delta}|^2}{4N_0}\sum_n |\alpha_n|^2\right)^{-N_R}$$

dominating PEPs are inversely proportional to (E_b/N_0)^N_R
$\Rightarrow$ diversity order N_R (same as for SIMO)

Spatial multiplexing : zero-forcing detection

Complexity of ML detection is exponential in N_T

Simpler *sub-optimum* detection : zero-forcing (ZF) detection

linear combination of received antenna signals $r_m(k)$ to eliminate mutual interference between symbols $a_n(k)$ from different transmit antennas and to minimize resulting noise variance

Condition for ZF solution to exist : rank($\mathbf{H}$) = N_T (this requires $N_R \geq N_T$)

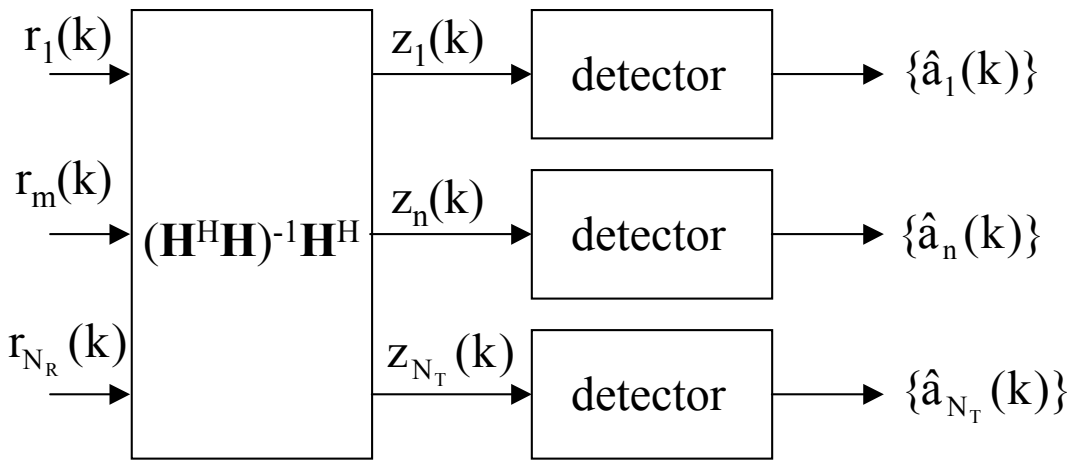
$\mathbf{r}(k) = \mathbf{H}\mathbf{a}(k) + \mathbf{w}(k)$ rank($\mathbf{H}$) = $N_T \Rightarrow (\mathbf{H}^H\mathbf{H})^{-1}$ exists

$\mathbf{z}(k) = (\mathbf{H}^H\mathbf{H})^{-1}\mathbf{H}^H\mathbf{r}(k) = \mathbf{a}(k) + \mathbf{n}(k)$ $\mathbf{n}(k) = (\mathbf{H}^H\mathbf{H})^{-1}\mathbf{H}^H\mathbf{w}(k)$

$\mathbf{z}(k)$: no interference between symbols from different transmit antennas

$E[\mathbf{n}(k)\mathbf{n}^H(k)] = N_0(\mathbf{H}^H\mathbf{H})^{-1}$: $n_i(k)$ and $n_j(k)$ correlated when $i \neq j$

Spatial multiplexing : zero-forcing detection



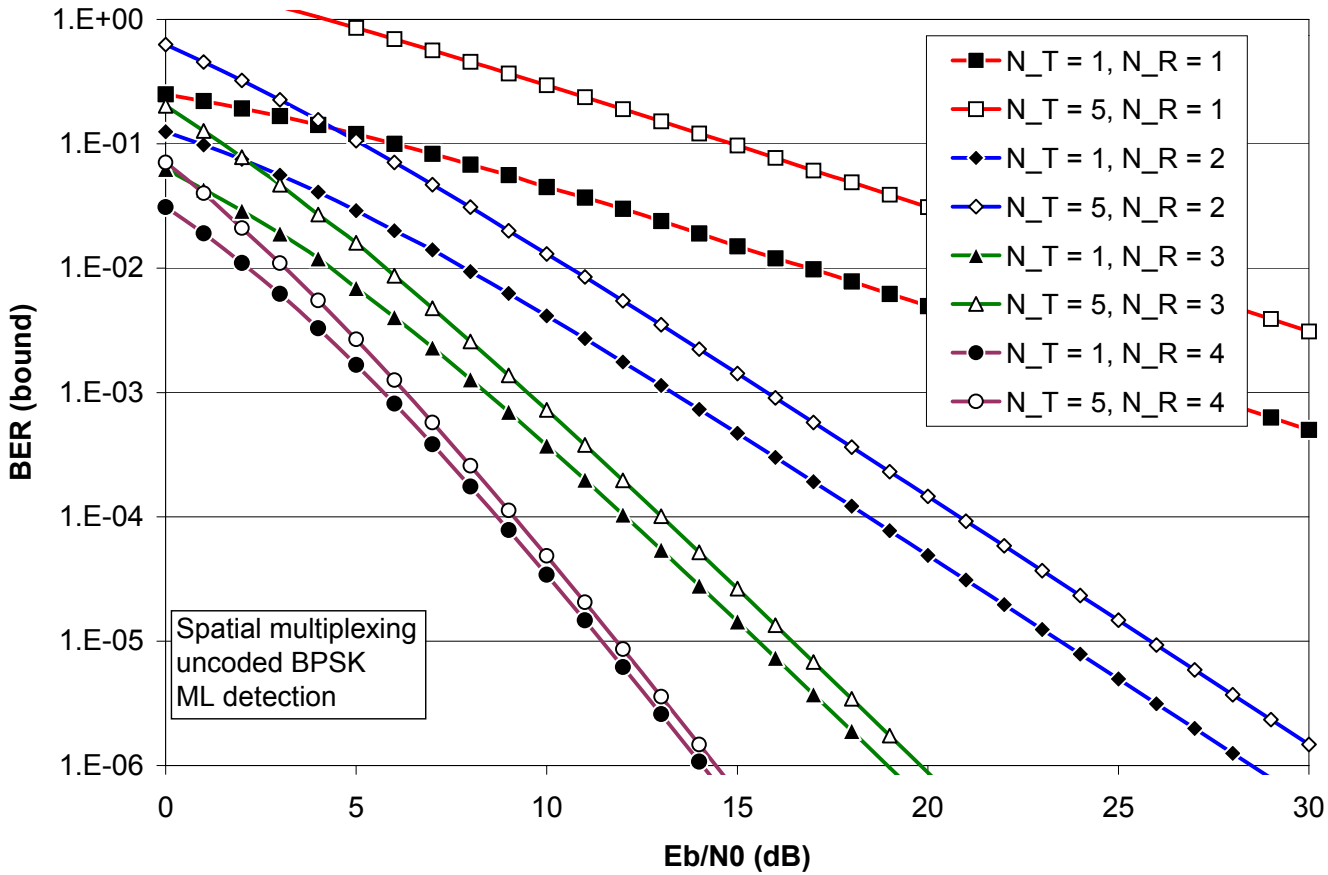
Sequences $\{a_n(k)\}$ are detected *independently* instead of jointly, ignoring the correlation of the noise on different outputs :

$$\{\hat{a}_n(k)\} = \arg\min_{a_n(k)} \sum_k |z_n(k) - a_n(k)|^2$$

Complexity of ZF detector is linear (instead of exponential) in N_T

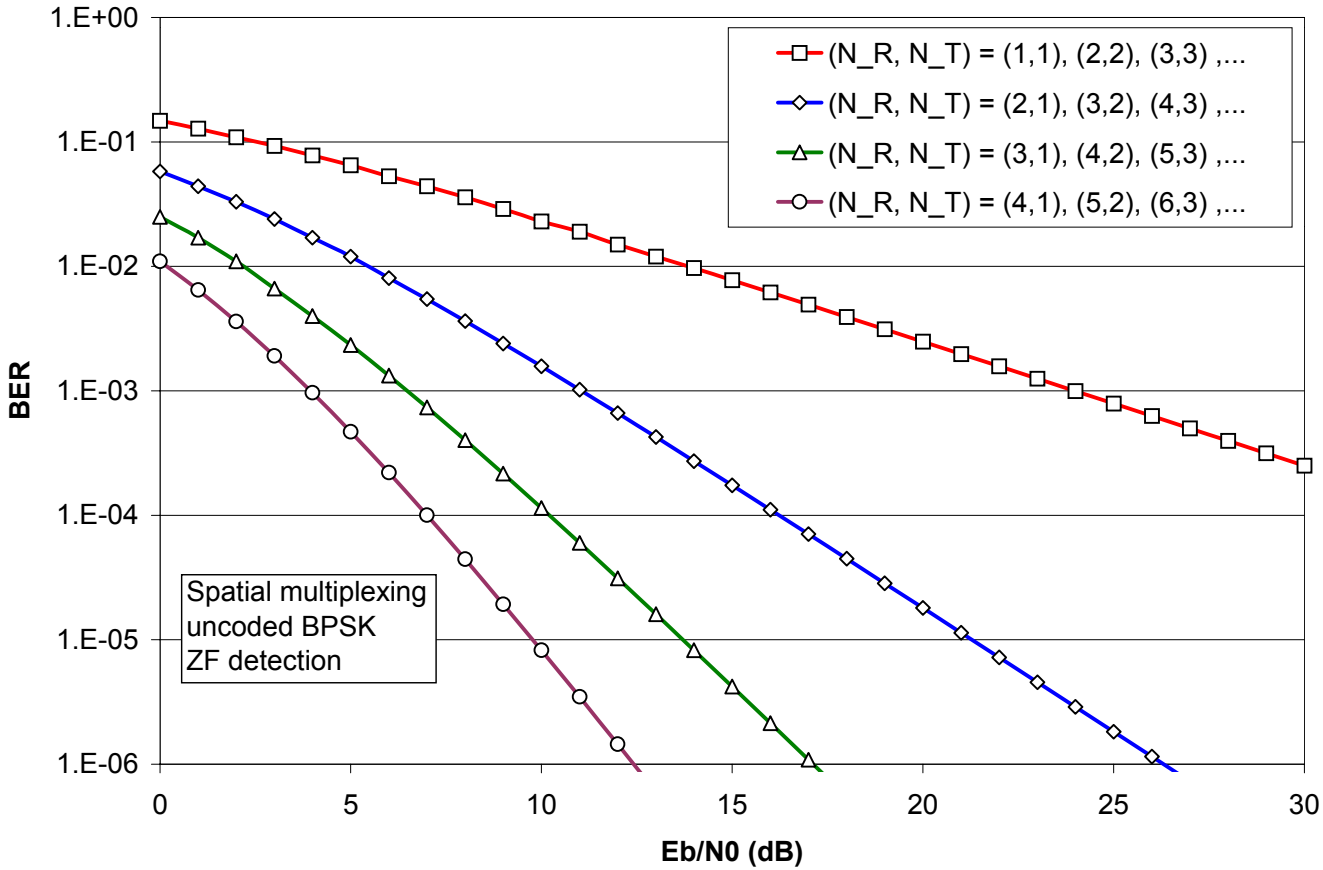
Performance penalty : diversity equals N_R-N_T+1 (instead of N_T)

Spatial multiplexing : numerical results



ML detection : penalty w.r.t. SIMO decreases with increasing N_R

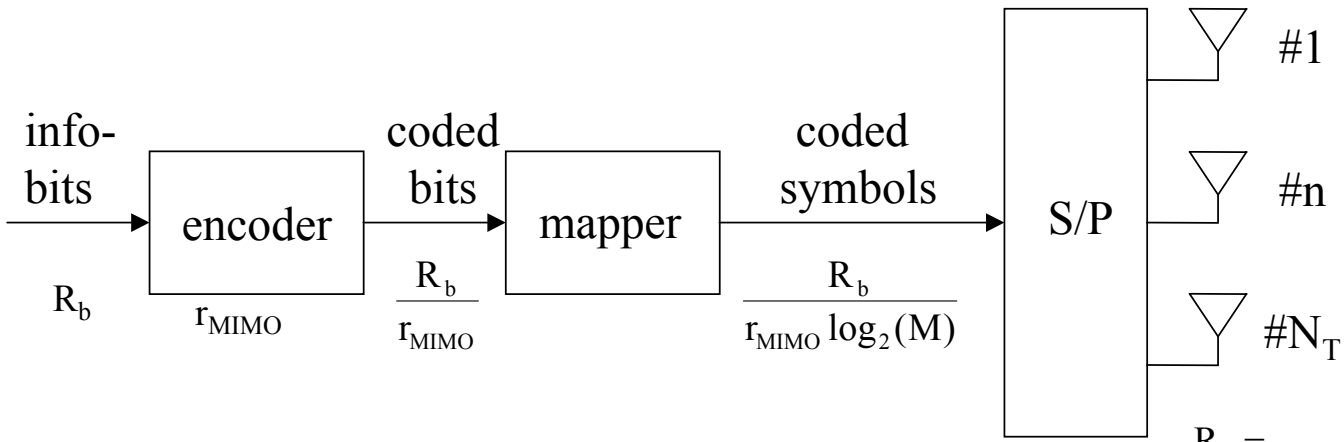
Spatial multiplexing : numerical results



ZF detection : same BER as SIMO with N_R-N_T+1 receive antennas

MIMO : space-time coding

Space-time coding



$$R_s = \frac{R_b}{N_T r_{MIMO} \log_2(M)}$$

(per antenna)

Aim : to achieve higher diversity order than with SIMO

$\Rightarrow$ any $\Delta^{(i,0)}$ must have rank larger than 1

(max. rank is N_T, max. diversity order is $N_T N_R$)

$\Rightarrow$ coded symbols must be distributed over different transmit antennas and different symbol intervals ("space-time" coding)

Property : $r_{MIMO} \leq 1/N_T$ is necessary condition to have rank N_T for all $\Delta^{(i,0)}$

$\Rightarrow$ at max. diversity, $R_b/R_s \leq \log_2(M)$

R_b/R_s limited by bandwidth efficiency of uncoded SIMO/SISO

Three classes of space-time coding :

- space-time trellis codes (STTrC)
- delay diversity
- space-time block codes (STBC)

STTrC : trellis representation

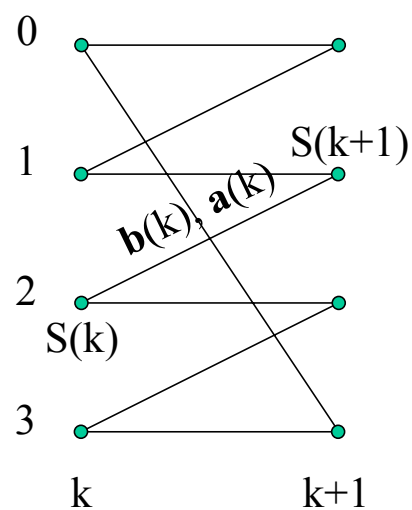
$\mathbf{A} = (\mathbf{a}(1), \cdots, \mathbf{a}(k), \cdots, \mathbf{a}(K))$ $\qquad$ $\mathbf{a}^T(k) = (a_1(k), \cdots, a_n(k), \cdots, a_{N_T}(k))$

$\mathbf{b}(k) = (b_1(k), \cdots, b_L(k))$ $\qquad$ L input bits at beginning of k-th symbol interval

S(k) : state at beginning of k-th symbol interval

state transitions : (S(k+1), **a**(k)) determined by (S(k), **b**(k))

trellis diagram :
(L=1)



2^L branches leaving from each state

2^L branches entering each state

STTrC : bandwidth efficiency

Bandwidth efficiency :

$$r_{MIMO} = L/(N_T \log_2(M)) \qquad \Rightarrow \quad R_b/R_s = L \ \text{(bit/s/Hz)}$$

at max. diversity : $r_{MIMO} \leq 1/N_T \Rightarrow L \leq \log_2(M)$

$$\#\text{states} \geq M^{N_T - 1} \qquad \text{(exponential in } N_T\text{)}$$

ML decoding by means of Viterbi algorithm :

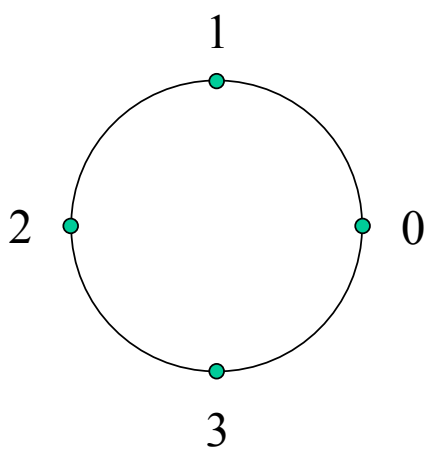
decoding complexity linear (instead of exponential) in K
but still exponential in N_T (at max. diversity)

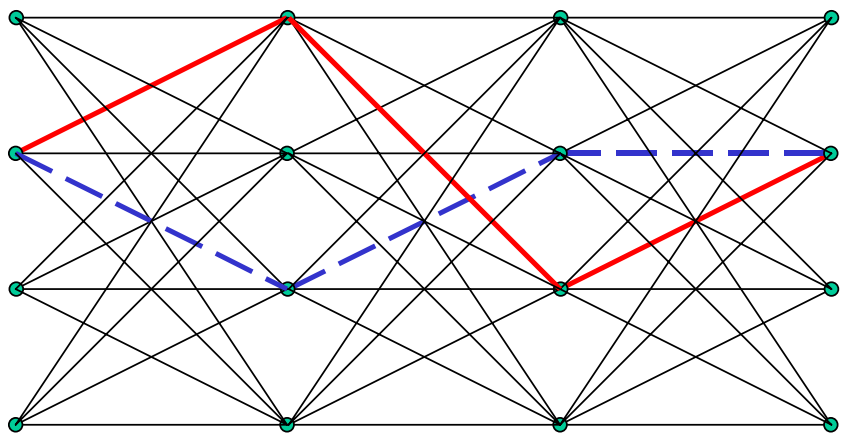
STTrC : example

Example for $N_T = 2$, 4 states, 4-PSK, L = 2 infobits per coded symbol pair

	0	**1**	**2**	**3**
0	0,0	0,1	0,2	0,3
1	1,0	1,1	1,2	1,3
2	2,0	2,1	2,2	2,3
3	3,0	3,1	3,2	3,3

$\text{entry}_{i,j}$: 2 data symbols corresponding to transition from state i to state j

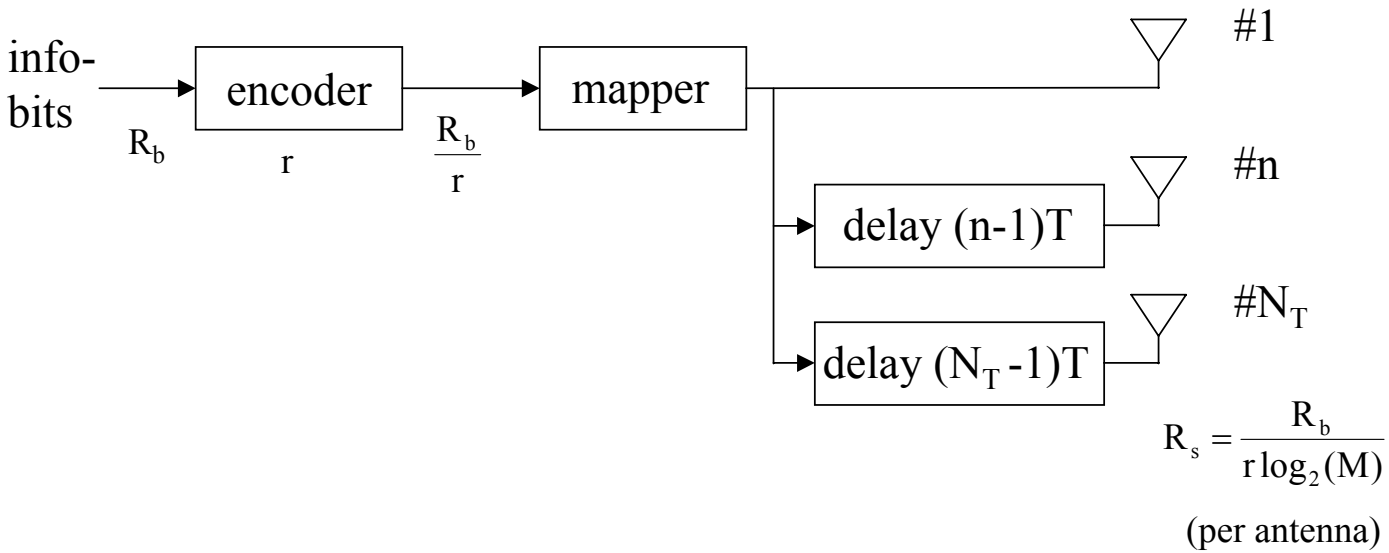




any $\Delta^{(i,0)}$ has columns $(0, \delta_1)^T$, $(\delta_2, 0)^T$
with nonzero δ_1 and δ_2
$\Rightarrow$ rank = 2, i.e., max. rank

max. diversity ($N_T N_R = 2N_R$),
max. bandwidth efficiency ($L = \log_2(M)$),
corresponding to max. diversity

Delay diversity : transmitter



Example : $N_T = 3$

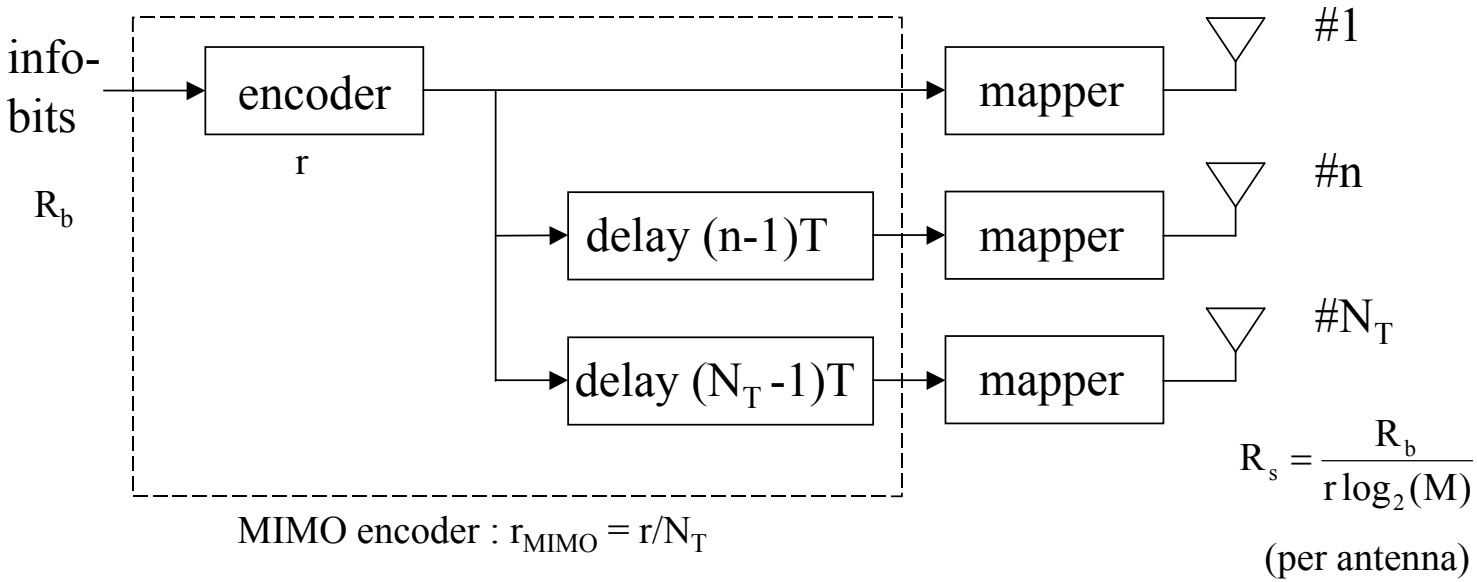
$$\mathbf{A} = \begin{pmatrix} a(1) & a(2) & a(3) & a(4) & \cdots & a(K-3) & a(K-2) & 0 & 0 \\ 0 & a(1) & a(2) & a(3) & \cdots & a(K-4) & a(K-3) & a(K-2) & 0 \\ 0 & 0 & a(1) & a(2) & \cdots & a(K-5) & a(K-4) & a(K-3) & a(K-2) \end{pmatrix}$$

$\Rightarrow \text{rank}(\Delta^{(i,0)}) = N_T \Rightarrow$ diversity order = $N_T N_R$

maximum possible diversity order !

Delay diversity : bandwidth efficiency

Equivalent configuration :



Bandwidth efficiency (for K >> 1) : $R_b/R_s = N_T r_{MIMO} \log_2(M) = r \cdot \log_2(M)$

$\Rightarrow$ same bandwidth efficiency as SIMO/SISO with code rate r

Max. bandwidth efficiency (at max. diversity) of $\log_2(M)$
achieved for r = 1 (uncoded delay diversity)

Delay diversity : trellis representation

Uncoded delay diversity (r = 1) can be interpreted as space-time trellis code :

state $S(k) = (a(k-1), a(k-2), ..., a(k-N_T+1))$ #states = M^{N_T-1}

input at time k : a(k)
output during k-th symbolinterval : $(a(k), a(k-1), ..., a(k-N_T+1))$

decoding complexity : linear in K, exponential in N_T

Space-time block codes

ML decoding of space-time code :
minimizing $||\mathbf{R} - \mathbf{HA}||^2$ over all possible codewords $\mathbf{A}$

$\mathbf{A}$ contains $N_T K$ coded symbols, which corresponds to $N_T K r_{MIMO} \log_2(M)$ infobits

In general, ML decoding complexity is exponential in $N_T K$

Space-time block codes can be designed to yield maximum diversity and to reduce ML decoding to *simple symbol-by-symbol decisions*

$\Rightarrow$ decoding complexity linear (instead of exponential) in $N_T K$

STBC : example 1

Example 1 : $N_T = 2$ (Alamouti code)

$$\mathbf{A} = \begin{pmatrix} a_1 & -a_2^* \\ a_2 & a_1^* \end{pmatrix} \quad \Rightarrow \quad \mathbf{A}\mathbf{A}^H = (|a_1|^2 + |a_2|^2)\mathbf{I}_2$$

(**A** has orthogonal rows, irrespective of the values of a_1 and a_2)

ML decoding ($\mathbf{H} = (\mathbf{h}_1, \mathbf{h}_2)$, $\mathbf{R} = (\mathbf{r}_1, \mathbf{r}_2)$, a_1 and a_2 i.i.d. symbols)

$$(\hat{a}_1, \hat{a}_1) = \arg\min_{a_1, a_2} \| \mathbf{R} - \mathbf{H}\mathbf{A} \|^2 = \arg\min_{a_1, a_2} \left(-2\,\mathrm{Re}[\mathrm{Tr}[\mathbf{A}^H \mathbf{H}^H \mathbf{R}]] + \mathrm{Tr}[(\mathbf{H}^H\mathbf{H})(\mathbf{A}\mathbf{A}^H)] \right)$$

$$= \arg\min_{a_1, a_2} \left(-2\,\mathrm{Re}[a_1^*(\mathbf{h}_1^H \mathbf{r}_1 + \mathbf{h}_1^T \mathbf{r}_2^*) + a_2^*(\mathbf{h}_2^H \mathbf{r}_2 - \mathbf{h}_1^T \mathbf{r}_2^*)] + (|a_1|^2 + |a_2|^2)(|\mathbf{h}_1|^2 + |\mathbf{h}_2|^2) \right)$$

no cross-terms involving both a_1 and a_2 $\Rightarrow$ symbol-by-symbol detection

$$\hat{a}_1 = \arg\min_{a_1} |u_1 - a_1|^2 \qquad \hat{a}_2 = \arg\min_{a_2} |u_2 - a_2|^2$$

$$u_1 = \frac{\mathbf{h}_1^H \mathbf{r}_1 + \mathbf{h}_2^T \mathbf{r}_2^*}{|\mathbf{h}_1|^2 + |\mathbf{h}_2|^2} \qquad u_2 = \frac{\mathbf{h}_2^H \mathbf{r}_1 - \mathbf{h}_1^T \mathbf{r}_2^*}{|\mathbf{h}_1|^2 + |\mathbf{h}_2|^2}$$

STBC : example 1

$$\mathbf{A} = \begin{pmatrix} a_1 & -a_2^* \\ a_2 & a_1^* \end{pmatrix} \quad \Rightarrow \quad \mathbf{\Delta}^{(i,0)} = \begin{pmatrix} a_1^{(0)} - a_1^{(i)} & -(a_2^{(0)} - a_2^{(i)})^* \\ a_2^{(0)} - a_2^{(i)} & (a_1^{(0)} - a_1^{(i)})^* \end{pmatrix}$$

orthogonal rows
$\Rightarrow$ rank = 2
(= max. rank for $N_T = 2$)

Assume M-point constellation
$\Rightarrow$ **A** contains 4 symbols, i.e. $4 \cdot \log_2(M) r_{MIMO} = 2 \cdot \log_2(M)$ infobits
$\Rightarrow r_{MIMO} = 1/2$ (= $1/N_T$) : highest possible rate for max. diversity

Alamouti space-time block code yields
- maximum diversity (i.e., $2N_R$)
- maximum corresponding bandwidth efficiency (i.e., $R_b/R_s = \log_2(M)$)
- small ML decoding complexity

STBC : example 2

Example 2 : $N_T = 3$

$$\mathbf{A} = \begin{pmatrix} a_1 & -a_2 & -a_3 & -a_4 & a_1^* & -a_2^* & -a_3^* & -a_4^* \\ a_2 & a_1 & a_4 & -a_3 & a_2^* & a_1^* & a_4^* & -a_3^* \\ a_3 & -a_4 & a_1 & a_2 & a_3^* & -a_4^* & a_1^* & a_2^* \end{pmatrix} \qquad \mathbf{A}\mathbf{A}^H = 2(|a_1|^2 + |a_2|^2 + |a_3|^2 + |a_4|^2)\mathbf{I}_3$$

rows of **A** are orthogonal $\Rightarrow$ any $\Delta^{(i,0)}$ has rank = 3

$\Rightarrow$ max. diversity (i.e., $3N_R$)

ML detection of (a_1, a_2, a_3, a_4) : symbol-by-symbol detection (no cross-terms)

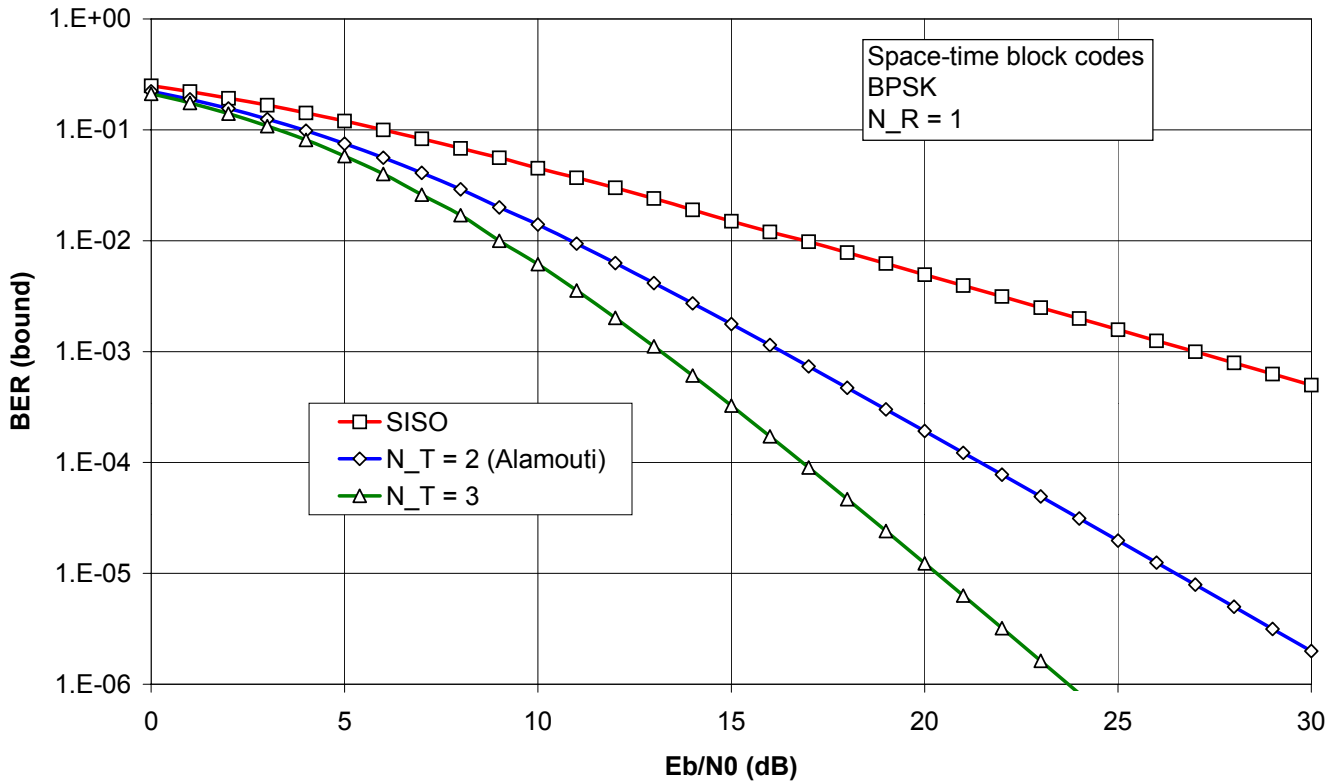
$r_{MIMO} = 1/6 \qquad (1/N_T = 1/3)$

Bandwidth efficiency : $R_b/R_s = 4 \cdot \log_2(M)/8 = \log_2(M)/2$

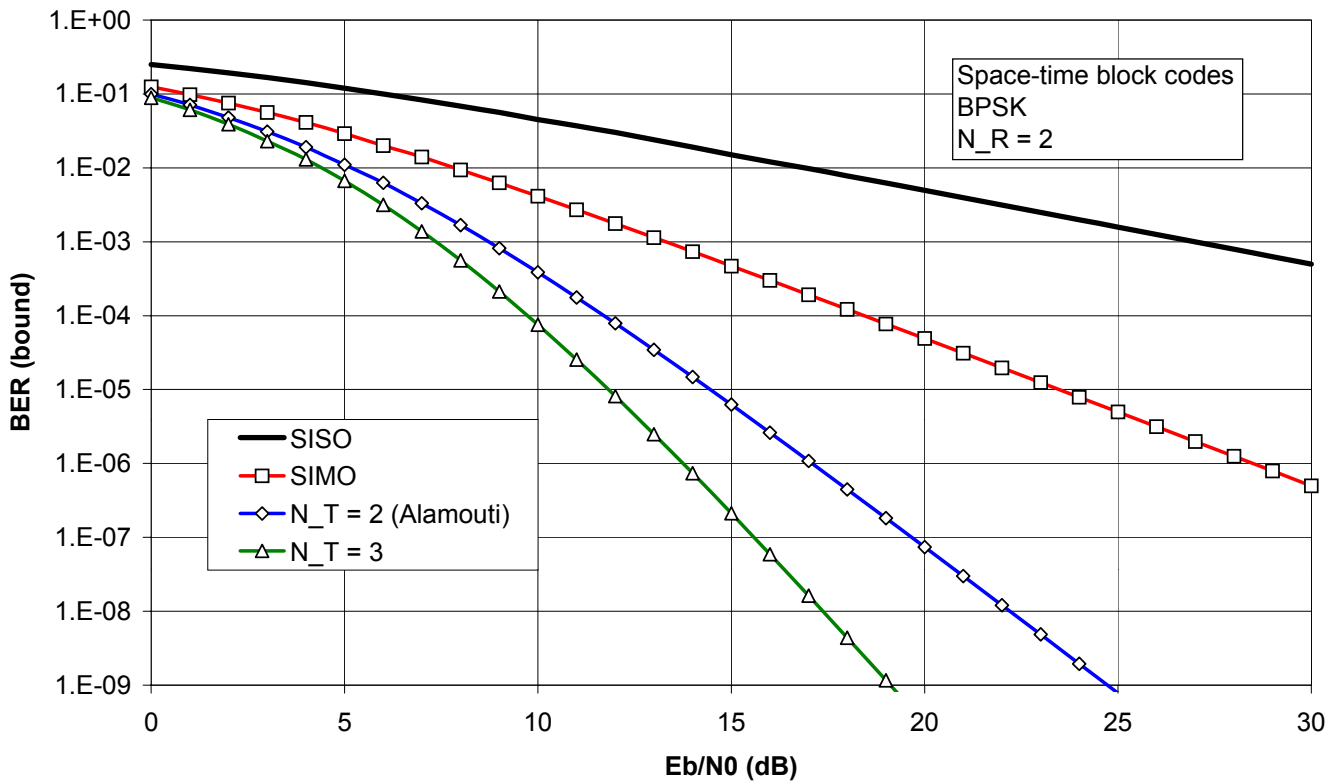
i.e., only half of the max. possible value for max. diversity

max. bandwidth efficiency (under restriction of max. diversity and symbol-by-symbol detection) cannot be achieved for all values of N_T

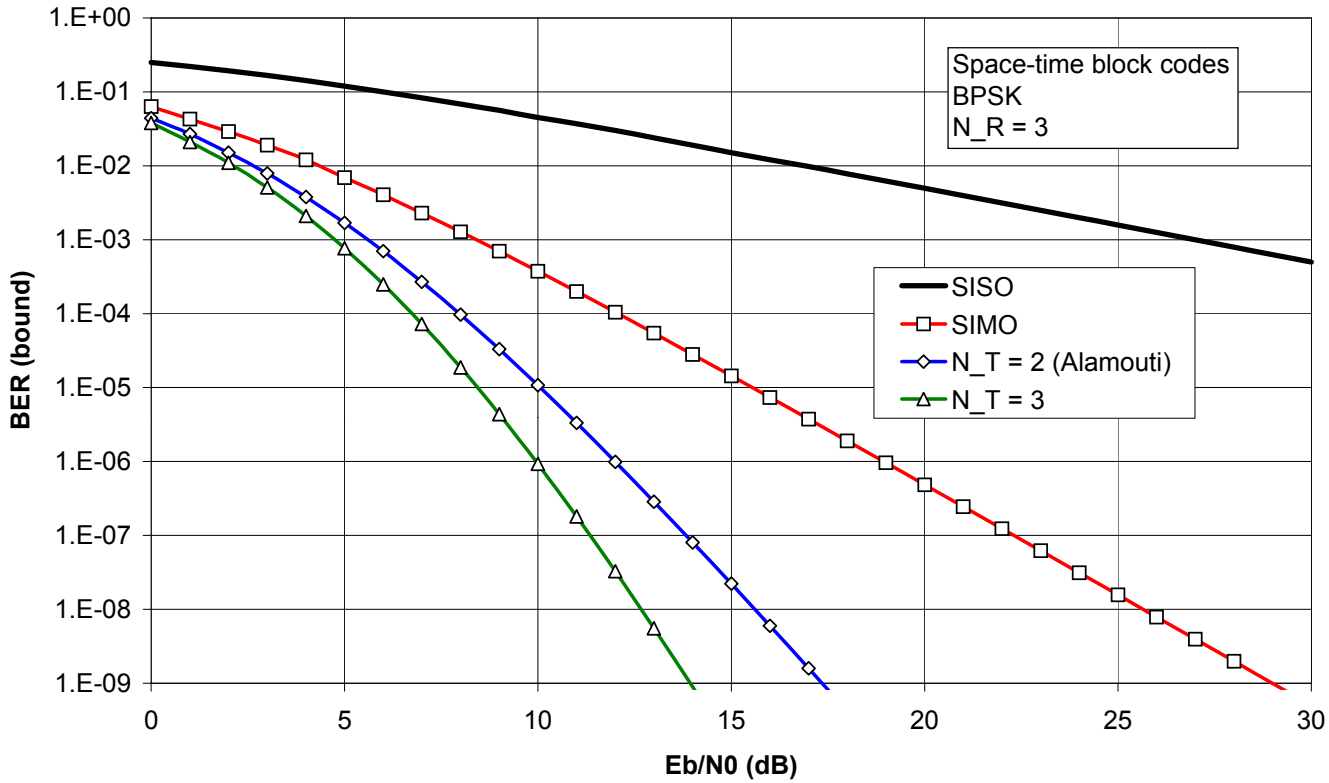
STBC : numerical results



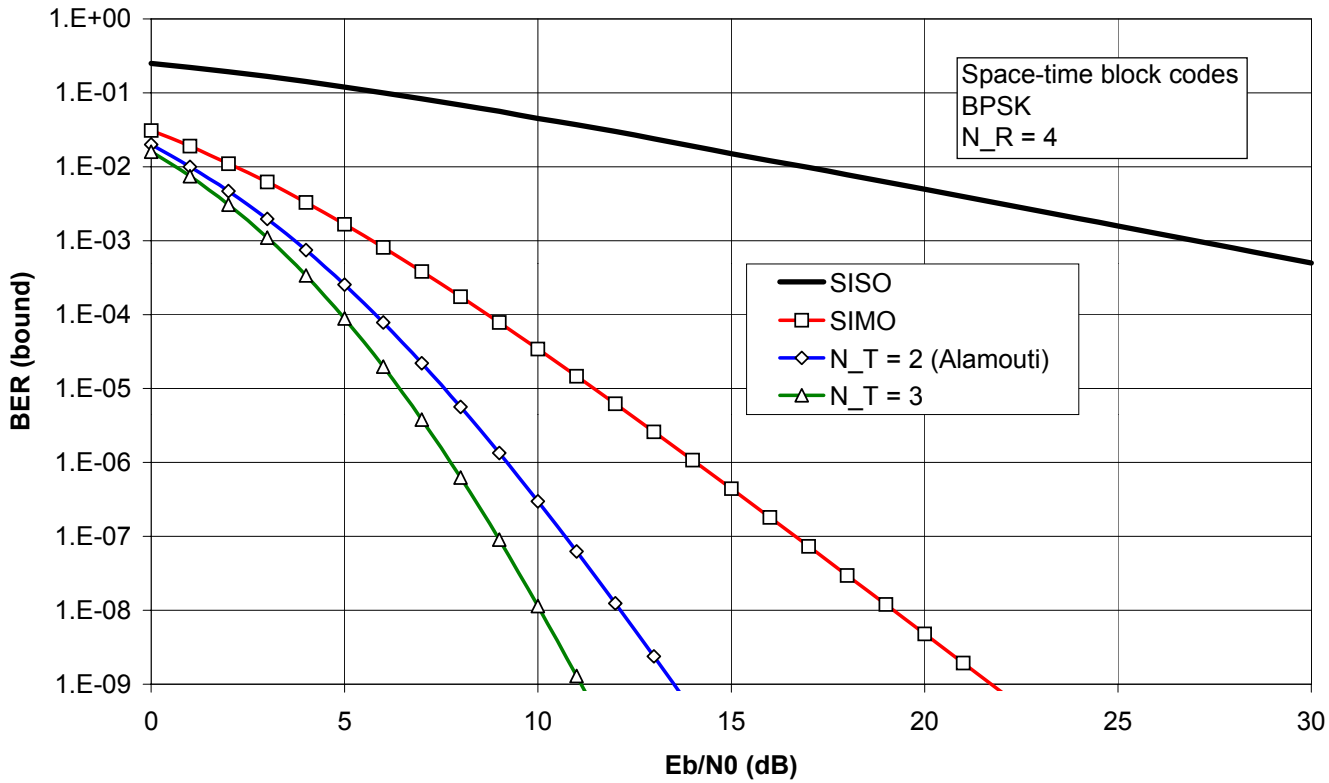
STBC : numerical results



STBC : numerical results



STBC : numerical results



Summary : SISO/SIMO/MIMO

	Bandwidth efficiency	**Diversity order (ML detection)**	**ML detector complexity**
SISO	$r_{SISO}\log_2(M)$	1	in general : expon. in K (trellis : linear in K)
SIMO	$r_{SIMO}\log_2(M)$	N_R	in general : expon. in K (trellis : linear in K)
MIMO	$N_T r_{MIMO}\log_2(M)$	$N_R \cdot \left(\max_{(i,0)} \operatorname{rank}(\Delta^{(i,0)}) \right)$	in general : expon. in $N_T K$

Summary : MIMO

	Bandwidth efficiency	Diversity order (ML detection)	ML detector complexity
spatial multiplexing	$N_T r \cdot \log_2(M)$	N_R	in general : expon. in $N_T K$ (trellis : linear in K, expon. in N_T)
STTrC	$\log_2(M)$ (*)	$N_T N_R$	expon. in N_T, linear in K
delay diversity	$r \cdot \log_2(M)$	$N_T N_R$	if uncoded (r = 1) : expon. in N_T, linear in K
STBC	$\leq \log_2(M)$ (*)	$N_T N_R$	symbol-by-symbol decision

(*) assuming max. diversity $N_T N_R$ is achieved

Remark : spatial multiplexing with ZF detection
- complexity linear in N_T
- diversity order is $N_R - N_T + 1$