

Modulatie en detectie

Enkele begrippen en notaties

Vectoren, signalen

Reële vectoren (1)

vectoren \mathbf{u} en \mathbf{v} :

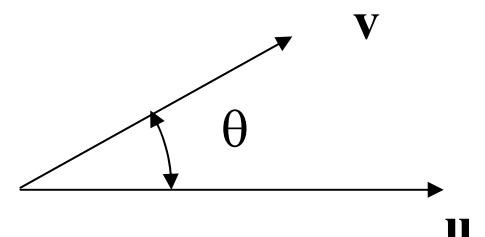
$$\mathbf{u} = \begin{pmatrix} u_1 \\ \vdots \\ u_M \end{pmatrix} \quad \mathbf{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_M \end{pmatrix} \quad \text{dimensie } M \times 1$$

getransponeerde : $\mathbf{u}^T = (u_1 \dots u_M)$ dimensie $1 \times M$

$$\mathbf{u}^T \mathbf{v} = \mathbf{v}^T \mathbf{u} = \sum_{m=1}^M u_m v_m \quad (\text{scalair product})$$

$$|\mathbf{u}|^2 = \mathbf{u}^T \mathbf{u} = \sum_{m=1}^M u_m^2 \quad (\text{kwadratische norm})$$

$$\mathbf{u}^T \mathbf{v} = \mathbf{v}^T \mathbf{u} = |\mathbf{u}| \cdot |\mathbf{v}| \cos(\theta) \quad \theta : \text{hoek tussen } \mathbf{u} \text{ en } \mathbf{v}$$

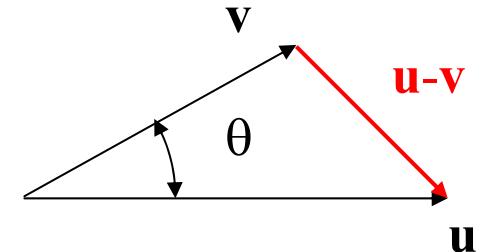


\mathbf{u} en \mathbf{v} orthogonaal $\Leftrightarrow \mathbf{u}^T \mathbf{v} = 0$

Reële vectoren (2)

Kwadratische norm van een verschil van 2 reële vectoren

$$\begin{aligned} |\mathbf{u} - \mathbf{v}|^2 &= (\mathbf{u} - \mathbf{v})^T (\mathbf{u} - \mathbf{v}) \\ &= (\mathbf{u}^T - \mathbf{v}^T)(\mathbf{u} - \mathbf{v}) \\ &= \mathbf{u}^T \mathbf{u} - \mathbf{u}^T \mathbf{v} - \mathbf{v}^T \mathbf{u} + \mathbf{v}^T \mathbf{v} \\ &= |\mathbf{u}|^2 + |\mathbf{v}|^2 - 2\mathbf{u}^T \mathbf{v} \\ &= |\mathbf{u}|^2 + |\mathbf{v}|^2 - 2\mathbf{v}^T \mathbf{u} \\ &= |\mathbf{u}|^2 + |\mathbf{v}|^2 - 2|\mathbf{u}| \cdot |\mathbf{v}| \cos(\theta) \end{aligned}$$



Complexe vectoren (1)

$$\mathbf{z} = \mathbf{x} + j\mathbf{y} = \begin{pmatrix} z_1 \\ \vdots \\ z_M \end{pmatrix} = \begin{pmatrix} x_1 + jy_1 \\ \vdots \\ x_M + jy_M \end{pmatrix} \quad \text{dimensie } M \times 1$$

complex toegevoegde : $\mathbf{z}^* = \mathbf{x} - j\mathbf{y} = \begin{pmatrix} z_1^* \\ \vdots \\ z_M^* \end{pmatrix} = \begin{pmatrix} x_1 - jy_1 \\ \vdots \\ x_M - jy_M \end{pmatrix} \quad \text{dimensie } M \times 1$

getransponeerde :

$$\mathbf{z}^T = \mathbf{x}^T + j\mathbf{y}^T = (z_1 \dots z_M) = (x_1 + jy_1 \dots x_M + jy_M) \quad \text{dimensie } 1 \times M$$

Hermitiaans toegevoegde :

$$\mathbf{z}^H = (\mathbf{z}^T)^* = (\mathbf{z}^*)^T = \mathbf{x}^T - j\mathbf{y}^T = (z_1^* \dots z_M^*) = (x_1 - jy_1 \dots x_M - jy_M) \quad \text{dimensie } 1 \times M$$

Complexe vectoren (2)

Kwadratische norm van \mathbf{z}

$$|\mathbf{z}|^2 = \mathbf{z}^H \mathbf{z} = \sum_{m=1}^M |z_m|^2 = \sum_{m=1}^M (x_m^2 + y_m^2) = |\mathbf{x}|^2 + |\mathbf{y}|^2$$

$$|z_m|^2 = x_m^2 + y_m^2$$

$$z_m^2 = (x_m + jy_m)^2 = (x_m^2 - y_m^2) + 2jx_my_m$$

$$|z_m|^2 \neq z_m^2$$

Scalair product $\mathbf{z}_1^H \mathbf{z}_2 = (\mathbf{x}_1^T - j\mathbf{y}_1^T)(\mathbf{x}_2 + j\mathbf{y}_2) = (\mathbf{x}_1^T \mathbf{x}_2 + \mathbf{y}_1^T \mathbf{y}_2) + j(\mathbf{x}_1^T \mathbf{y}_2 - \mathbf{y}_1^T \mathbf{x}_2)$

$$\mathbf{z}_1^H \mathbf{z}_2 = (\mathbf{z}_2^H \mathbf{z}_1)^*$$

$$\mathbf{z}_1^H \mathbf{z}_2 = 0 \iff \mathbf{z}_1 \text{ en } \mathbf{z}_2 \text{ orthogonaal}$$

Complexe vectoren (3)

Kwadratische norm van $\mathbf{z}_1 - \mathbf{z}_2$:

$$\begin{aligned} |\mathbf{z}_1 - \mathbf{z}_2|^2 &= (\mathbf{z}_1 - \mathbf{z}_2)^H (\mathbf{z}_1 - \mathbf{z}_2) \\ &= |\mathbf{z}_1|^2 + |\mathbf{z}_2|^2 - \mathbf{z}_1^H \mathbf{z}_2 - \mathbf{z}_2^H \mathbf{z}_1 \\ &= |\mathbf{z}_1|^2 + |\mathbf{z}_2|^2 - 2 \operatorname{Re}[\mathbf{z}_1^H \mathbf{z}_2] \\ &= |\mathbf{z}_1|^2 + |\mathbf{z}_2|^2 - 2 \operatorname{Re}[\mathbf{z}_2^H \mathbf{z}_1] \end{aligned}$$

Discrete-tijd signalen (1)

$\{s(k)\}$, $s(k) = s_R(k) + j s_I(k)$, interval T tussen opeenvolgende samples

$$\text{DTFT : } S(e^{j2\pi fT}) = \sum_k s(k) e^{-j2\pi fkT}$$

$\{s(k)\}$ is te interpreteren als complexe vector \mathbf{s} , met $s_k = s(k)$

Kwadratische norm :

$$|\mathbf{s}|^2 = \sum_k |s(k)|^2 = \sum_k (s_R^2(k) + s_I^2(k)) = T \int_{1/T} |S(e^{j2\pi fT})|^2 df \quad (\text{Parseval})$$

$$\text{Scalair product : } \mathbf{s}_1^H \mathbf{s}_2 = \sum_k s_1^*(k) s_2(k) = T \int_{1/T} S_1^*(e^{j2\pi fT}) S_2(e^{j2\pi fT}) df \quad (\text{Parseval})$$

$$\sum_k s_1^*(k) s_2(k) = 0 \Leftrightarrow \{s_1(k)\} \text{ en } \{s_2(k)\} \text{ orthogonaal}$$

Discrete-tijd signalen (2)

Kwadratische norm van $\{s_1(k) - s_2(k)\}$

$$\begin{aligned}\sum_k |s_1(k) - s_2(k)|^2 &= \sum_k (|s_1(k)|^2 + |s_2(k)|^2 - 2 \operatorname{Re}[s_1^*(k)s_2(k)]) \\ &= \sum_k (|s_1(k)|^2 + |s_2(k)|^2 - 2 \operatorname{Re}[s_2^*(k)s_1(k)])\end{aligned}$$

Alternatieve berekening in frequentiedomein

$$\begin{aligned}\sum_k |s_1(k) - s_2(k)|^2 &= T \int_{1/T} |S_1(e^{j2\pi fT}) - S_2(e^{j2\pi fT})|^2 df \\ &= T \int_{1/T} (|S_1(e^{j2\pi fT})|^2 + |S_2(e^{j2\pi fT})|^2 - 2 \operatorname{Re}[S_1^*(e^{j2\pi fT})S_2(e^{j2\pi fT})]) df \\ &= T \int_{1/T} (|S_1(e^{j2\pi fT})|^2 + |S_2(e^{j2\pi fT})|^2 - 2 \operatorname{Re}[S_2^*(e^{j2\pi fT})S_1(e^{j2\pi fT})]) df\end{aligned}$$

Continue-tijd signalen (1)

$$s(t) = s_R(t) + j s_I(t)$$

$$\text{FT : } S(f) = \int s(t) e^{-j2\pi ft} dt$$

$s(t)$ is te interpreteren als complexe vector \mathbf{s} , met “continue” index t

Kwadratische norm (energie van $s(t)$) :

$$|\mathbf{s}|^2 = \int |s(t)|^2 dt = \int (s_R^2(t) + s_I^2(t)) dt = \int |S(f)|^2 df \quad (\text{Parseval})$$

Scalair product : $\mathbf{s}_1^* \mathbf{s}_2 = \int s_1^*(t) s_2(t) dt = \int S_1^*(f) S_2(f) df \quad (\text{Parseval})$

$$\int s_1^*(t) s_2(t) dt = 0 \Leftrightarrow s_1(t) \text{ en } s_2(t) \text{ orthogonaal}$$

Continue-tijd signalen (2)

Kwadratische norm van $s_1(t) - s_2(t)$

$$\begin{aligned}\int |s_1(t) - s_2(t)|^2 dt &= \int (|s_1(t)|^2 + |s_2(t)|^2 - 2 \operatorname{Re}[s_1^*(t)s_2(t)])dt \\ &= \int (|s_1(t)|^2 + |s_2(t)|^2 - 2 \operatorname{Re}[s_2^*(t)s_1(t)])dt\end{aligned}$$

Alternatieve berekening in frequentiedomein

$$\begin{aligned}\int |s_1(t) - s_2(t)|^2 dt &= \int |S_1(f) - S_2(f)|^2 df \\ &= \int (|S_1(f)|^2 + |S_2(f)|^2 - 2 \operatorname{Re}[S_1^*(f)S_2(f)])df \\ &= \int (|S_1(f)|^2 + |S_2(f)|^2 - 2 \operatorname{Re}[S_2^*(f)S_1(f)])df\end{aligned}$$

Lineair tijdsinvariant (LTI) filter

Discrete tijd : $y(k) = \sum_m h(m)x(k-m) = \sum_m h(k-m)x(m)$

voorstellen als matrixvermenigvuldiging : $\mathbf{y} = \mathbf{H}\mathbf{x}$, $y(k) = (\mathbf{y})_k$, $x(k) = (\mathbf{x})_k$

$$y(k) = \sum_m (\mathbf{H})_{k,m} x(m)$$
$$\Rightarrow (\mathbf{H})_{k,m} = h(k-m)$$
$$\begin{pmatrix} y(0) \\ y(1) \\ y(2) \\ y(3) \\ y(4) \end{pmatrix} = \begin{pmatrix} h(0) & 0 & 0 & 0 \\ h(1) & h(0) & 0 & 0 \\ 0 & h(1) & h(0) & 0 \\ 0 & 0 & h(1) & h(0) \\ 0 & 0 & 0 & h(1) \end{pmatrix} \begin{pmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \end{pmatrix}$$

Continue tijd :

$$y(t) = \int h(u)x(t-u)du = \int h(t-u)x(u)du \quad t, u : \text{"continue" indices}$$

voorstellen als $\mathbf{y} = \mathbf{H}\mathbf{x}$

Lineair tijdsvariabel (LTV) filter

Discrete tijd : $y(k) = \sum_m h(m;k)x(k-m) = \sum_m h(k-m;k)x(m)$

voorstellen als matrixvermenigvuldiging : $\mathbf{y} = \mathbf{H}\mathbf{x}$, $y(k) = (\mathbf{y})_k$, $x(k) = (\mathbf{x})_k$

$$y(k) = \sum_m (\mathbf{H})_{k,m} x(m)$$
$$\Rightarrow (\mathbf{H})_{k,m} = h(k-m;k)$$
$$\begin{pmatrix} y(0) \\ y(1) \\ y(2) \\ y(3) \\ y(4) \end{pmatrix} = \begin{pmatrix} h(0;0) & 0 & 0 & 0 \\ h(1;1) & h(0;1) & 0 & 0 \\ 0 & h(1;2) & h(0;2) & 0 \\ 0 & 0 & h(1;3) & h(0;3) \\ 0 & 0 & 0 & h(1;4) \end{pmatrix} \begin{pmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \end{pmatrix}$$

Continue tijd :

$$y(t) = \int h(u;t)x(t-u)du = \int h(t-u;t)x(u)du \quad t, u : \text{"continue" indices}$$

Draaggolfsystemen

- Draaggolfsignaal en complexe omhullende
- Banddoorlaatfilter en equivalent laagdoorlaatfilter
- Filteren van draaggolfsignaal en van complexe omhullende

Draaggolfsignaal en complexe omhullende

$x_{BP}(t)$: **reëel** draaggolfsignaal (*bandpass signal*)

$X_{BP}(f) = 0$ voor $|f-f_0| > B$, $f_0 > B$

$x_{LP}(t)$: **complexe** omhullende (*complex baseband representation*) van $x_{BP}(t)$

$$x_{LP}(t) = x_R(t) + jx_I(t)$$

$x_R(t)$: in-fase component

$x_I(t)$: kwadratuurcomponent

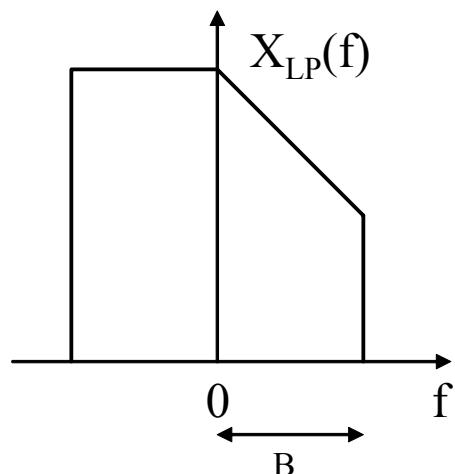
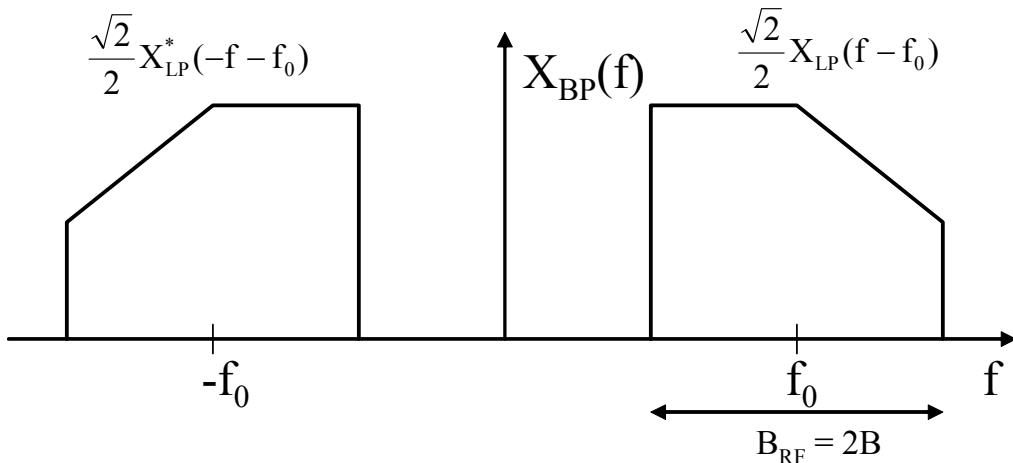
$$x_{BP}(t) = \sqrt{2} \operatorname{Re}[x_{LP}(t) \exp(j2\pi f_0 t)] \Leftrightarrow X_{BP}(f) = \frac{\sqrt{2}}{2} X_{LP}(f - f_0) + \frac{\sqrt{2}}{2} X_{LP}^*(-f - f_0)$$



$$x_{LP}(t) = \{\sqrt{2}x_{BP}(t) \exp(-j2\pi f_0 t)\}_{LP} \Leftrightarrow X_{LP}(f) = \sqrt{2}X_{BP}(f + f_0)\Pi_{LP}(f)$$

$$\Pi_{LP}(f) = \begin{cases} 1 & |f| < B \\ 0 & \text{otherwise} \end{cases}$$

Draaggolfsignaal en complexe omhullende



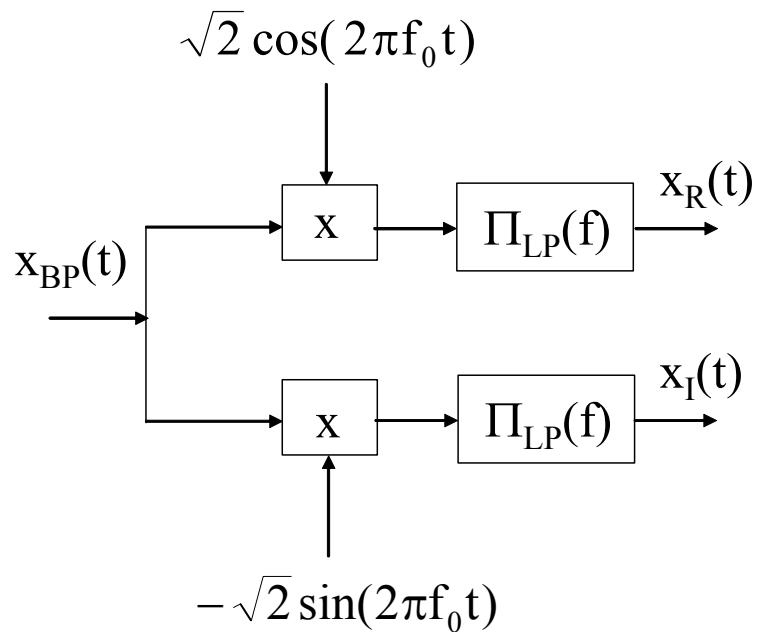
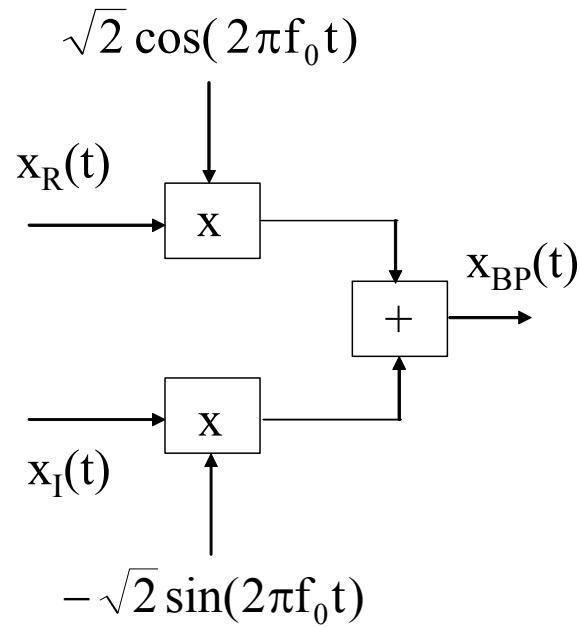
$$X_{BP}(f) = \frac{\sqrt{2}}{2} X_{LP}(f - f_0) + \frac{\sqrt{2}}{2} X_{LP}^*(-f - f_0)$$

$$\Updownarrow$$

$$X_{LP}(f) = \sqrt{2} X_{BP}(f + f_0) \Pi_{LP}(f)$$

$$\Pi_{LP}(f) = \begin{cases} 1 & |f| < B \\ 0 & \text{otherwise} \end{cases}$$

Draaggolfsignaal en complexe omhullende



$$x_{BP}(t) = \sqrt{2} \operatorname{Re}[x_{LP}(t) \exp(j2\pi f_0 t)] = \sqrt{2} x_R(t) \cos(2\pi f_0 t) - \sqrt{2} x_I(t) \sin(2\pi f_0 t)$$

\Updownarrow

$$x_{LP}(t) = \{\sqrt{2} x_{BP}(t) \exp(-j2\pi f_0 t)\}_{LP} \Rightarrow \begin{aligned} x_R(t) &= \{\sqrt{2} x_{BP}(t) \cos(2\pi f_0 t)\}_{LP} \\ x_I(t) &= \{-\sqrt{2} x_{BP}(t) \sin(2\pi f_0 t)\}_{LP} \end{aligned}$$

Banddoorlaatfilter en equivalent laagdoorlaatfilter

$h_{BP}(t)$: **reëel** impulsantwoord banddoorlaatfilter

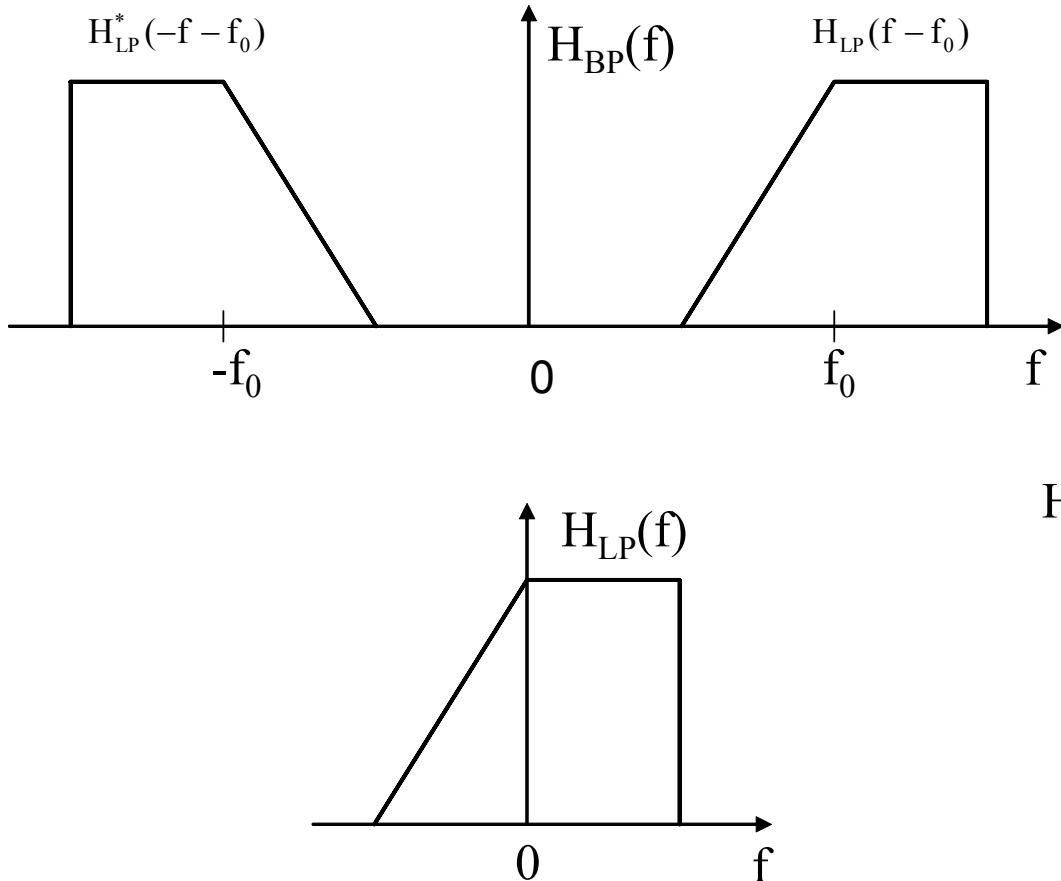
$h_{LP}(t)$: **complex** impulsantwoord equivalent laagdoorlaatfilter

$$h_{BP}(t) = 2 \operatorname{Re}[h_{LP}(t) \exp(j2\pi f_0 t)] \Leftrightarrow H_{BP}(f) = H_{LP}(f - f_0) + H_{LP}^*(-f - f_0)$$

\Downarrow \Downarrow

$$h_{LP}(t) = \{h_{BP}(t) \exp(-j2\pi f_0 t)\}_{LP} \Leftrightarrow H_{LP}(f) = H_{BP}(f + f_0) \Pi_{LP}(f)$$

Banddoorlaatfilter en equivalent laagdoorlaatfilter



$$H_{BP}(f) = H_{LP}(f - f_0) + H_{LP}^*(-f - f_0)$$

\Updownarrow

$$H_{LP}(f) = H_{BP}(f + f_0)\Pi_{LP}(f)$$

Filteren van draaggolfsignaal en van complexe omhullende

$$x_{BP}(t) \xrightarrow{H_{BP}(f)} y_{BP}(t) \quad \equiv \quad x_{LP}(t) \xrightarrow{H_{LP}(f)} y_{LP}(t)$$

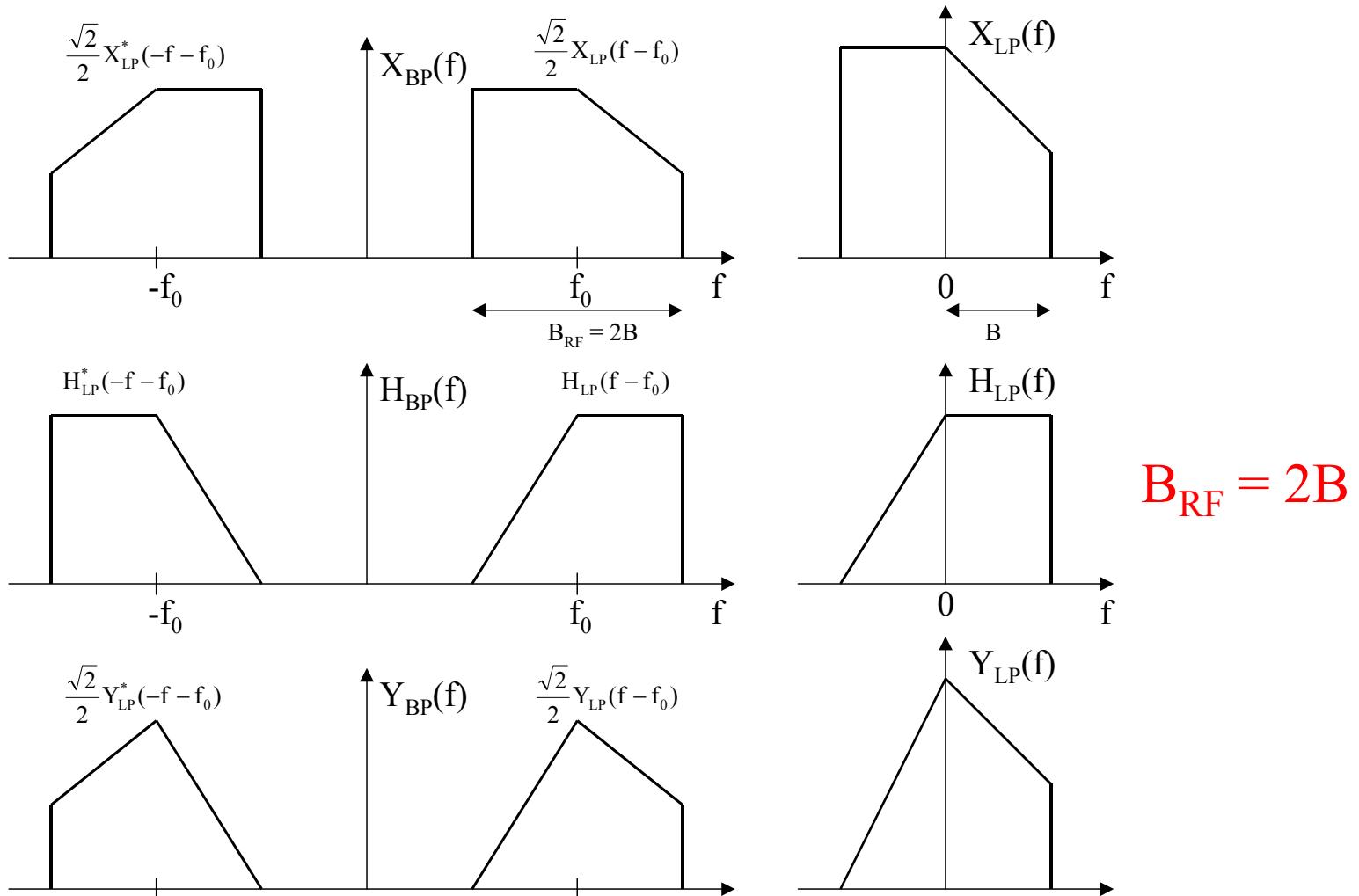
$$y_{BP}(t) = \int h_{BP}(u)x_{BP}(t-u)du \quad \Leftrightarrow \quad Y_{BP}(f) = H_{BP}(f)X_{BP}(f)$$

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$$y_{LP}(t) = \int h_{LP}(u)x_{LP}(t-u)du \quad \Leftrightarrow \quad Y_{LP}(f) = H_{LP}(f)X_{LP}(f)$$

Filteren van draaggolfsignaal en van complexe omhullende



Stationnaire TPn

Continue-tijd stationaire TPn

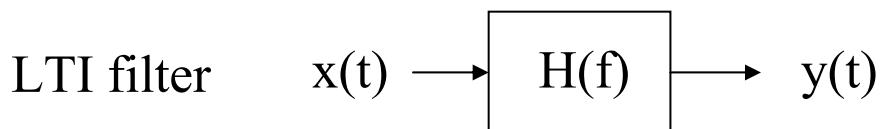
$x(t)$ stationair $\Leftrightarrow x(t)$ en $x(t-\tau)$ zelfde statistiek, ongeacht τ

Gevolg : $E[x(t+u)x^*(t)] = E[x(u)x^*(0)] = R_x(u)$
autocorrelatiefunctie $E[x(t+u)x^*(t)]$ hangt niet af van t

Vermogen $P_x = E[|x(t)|^2] = R_x(0)$

Vermogenspectrum $S_x(f)$ is FT van $R_x(u)$,
 $R_x(u)$ is IFT van $S_x(f)$

$$E[|x(t)|^2] = \int_{-\infty}^{+\infty} S_x(f) df$$



$$S_y(f) = |H(f)|^2 S_x(f)$$

$$E[|y(t)|^2] = \int_{-\infty}^{+\infty} S_x(f) |H(f)|^2 df$$

Discrete-tijd stationaire TPn

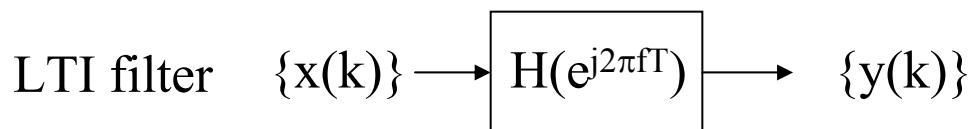
$\{x(k)\}$ stationair $\Leftrightarrow \{x(k)\}$ en $\{x(k-k_0)\}$ zelfde statistiek, ongeacht k_0

Gevolg : $E[x(k+m)x^*(k)] = E[x(m)x^*(0)] = R_x(m)$
autocorrelatiefunctie $E[x(k+m)x^*(k)]$ hangt niet af van k

$$\text{Vermogen } P_x = E[|x(k)|^2] = R_x(0)$$

Vermogenspectrum $S_x(e^{j2\pi fT})$ is DFT van $R_x(m)$,
 $R_x(m)$ is IDFT van $S_x(e^{j2\pi fT})$

$$E[|x(k)|^2] = T \int_{1/T} S_x(e^{j2\pi fT}) df$$

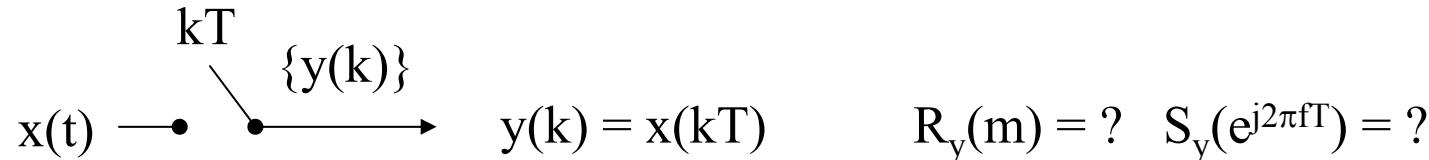


$$S_y(e^{j2\pi fT}) = |H(e^{j2\pi fT})|^2 S_x(e^{j2\pi fT})$$

$$E[|y(k)|^2] = T \int_{1/T} S_x(e^{j2\pi fT}) |H(e^{j2\pi fT})|^2 df$$

Bemonstering van stationair TP

$x(t)$ stationair, autocorrelatiefunctie $R_x(u)$, vermogenspectrum $S_x(f)$



$$R_y(m) = E[y(k+m)y^*(k)] = E[x(kT+mT)x^*(kT)] = R_x(mT)$$

$$\begin{aligned} S_y(e^{j2\pi fT}) &= \sum_{m=-\infty}^{+\infty} R_y(m) e^{-j2\pi fmT} = \sum_{m=-\infty}^{+\infty} R_x(mT) e^{-j2\pi fmT} \\ &= \frac{1}{T} \sum_{i=-\infty}^{+\infty} S_x\left(f - \frac{i}{T}\right) = \{S_x(f); 1/T\}_{\text{fld}} \end{aligned}$$

periodieke uitbreiding
van $S_x(f)/T$, periode $1/T$

$$E[|y(k)|^2] = T \int_{1/T} \left(\frac{1}{T} \sum_{i=-\infty}^{+\infty} S_x\left(f - \frac{i}{T}\right) \right) df = \int_{-\infty}^{+\infty} S_x(f) df = E[|x(t)|^2]$$

Reële Gaussiaanse TGn en TPn

Reële Gaussiaanse TGn

\mathbf{x} (dim. $M \times 1$) is Gauss. met $E[\mathbf{x}] = 0$, $E[\mathbf{x}\mathbf{x}^T] = \mathbf{R}$ (\mathbf{R} : covariantiematrix)

$$(\mathbf{R})_{m,n} = E[x_m x_n]$$

\mathbf{R} is symmetrische matrix : $(\mathbf{R})_{m,n} = (\mathbf{R})_{n,m}$

statistiek van \mathbf{x} volledig bepaald door \mathbf{R}

$$p_{\mathbf{x}}(\mathbf{x}) = \frac{e^{\frac{-1}{2}\mathbf{x}^T \mathbf{R}^{-1} \mathbf{x}}}{(2\pi)^{\frac{M}{2}} (\det(\mathbf{R}))^{\frac{1}{2}}}$$

Notatie : $\mathbf{x} \sim N(0, \mathbf{R})$

Eigenschap

$$\mathbf{x}_1 \sim N(0, \mathbf{R}_1) \text{ en } \mathbf{x}_2 = \mathbf{M}\mathbf{x}_1 \Rightarrow \mathbf{x}_2 \sim N(0, \mathbf{R}_2), \mathbf{R}_2 = \mathbf{M}\mathbf{R}_1\mathbf{M}^T$$

Gaussiaans karakter blijft behouden na lineaire transformatie

Reële Gaussiaanse TPn

Discrete tijd

$$\{x(k)\} \sim N(0, R(m, k))$$

$\Leftrightarrow \{x(k)\}$ is Gauss. met $E[x(k)] = 0$, covariantiefunctie $E[x(k+m)x(k)] = R(m, k)$

Indien **stationariteit** : $E[x(k+m)x(k)] = R(m)$, hangt niet af van k

voorbeeld : stationaire witte ruis, discrete tijd : $R(m, k) = \sigma^2 \delta(m)$ (Kronecker)

Continue tijd

$$x(t) \sim N(0, R(u, t))$$

$\Leftrightarrow x(t)$ is Gauss. met $E[x(t)] = 0$, covariantiefunctie $E[x(t+u)x(t)] = R(u, t)$

Indien **stationariteit** : $E[x(t+u)x(t)] = R(u)$, hangt niet af van t

voorbeeld : stationaire witte ruis, continue tijd : $R(u, t) = (N_0/2)\delta(u)$ (Dirac)

Complexe Gaussiaanse TGn en TPn

Complexe omhullende van stationair re el Gaussiaans TP

$x_{BP}(t)$: stationair re el Gaussiaans TP, banddoorlaat

$x_{LP}(t)$: complexe omhullende van $x_{BP}(t)$, dus ook Gaussiaans
(wgens lineaire transformatie van $x_{BP}(t)$)

$x_{LP}(t)e^{j\theta}$, met θ willekeurig : complexe omhullende van $x_{BP,1}(t)$

$x_{LP}(t-\tau)$: complexe omhullende van $x_{BP,2}(t)$

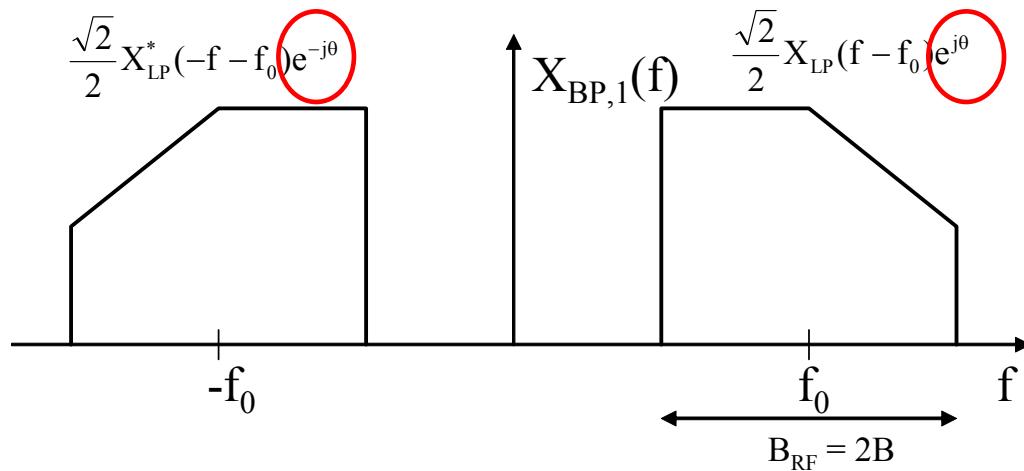
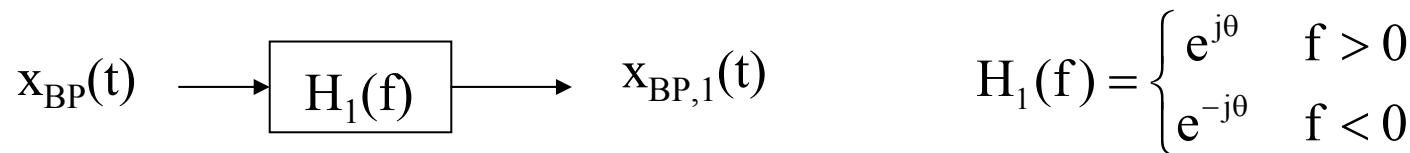
We tonen aan dat $x_{BP}(t)$ en $x_{BP,1}(t)$ dezelfde statistische eigenschappen hebben.
Hieruit volgt dan dat $x_{LP}(t)$ en $x_{LP}(t)e^{j\theta}$ dezelfde statistische eigenschappen hebben,
m.a.w. $x_{LP}(t)$ is **circulair symmetrisch**.

We tonen aan dat $x_{BP}(t)$ en $x_{BP,2}(t)$ dezelfde statistische eigenschappen hebben.
Hieruit volgt dan dat $x_{LP}(t)$ en $x_{LP}(t-\tau)$ dezelfde statistische eigenschappen hebben,
m.a.w. $x_{LP}(t)$ is **stationair**.

Complexe omhullende van stationair reëel Gaussiaans TP

$$X_{BP}(f) = \frac{\sqrt{2}}{2} (X_{LP}(f - f_0) + X_{LP}^*(-f - f_0))$$

$$X_{BP,1}(f) = \frac{\sqrt{2}}{2} (X_{LP}(f - f_0)e^{j\theta} + X_{LP}^*(-f - f_0)e^{-j\theta})$$



Complexe omhullende van stationair reëel Gaussiaans TP

$$x_{BP}(t) \rightarrow \boxed{H_1(f)} \rightarrow x_{BP,1}(t)$$
$$H_1(f) = \begin{cases} e^{j\theta} & f > 0 \\ e^{-j\theta} & f < 0 \end{cases}$$

$x_{BP}(t)$ stationair en Gaussiaans $\Rightarrow x_{BP,1}(t)$ stationair en Gaussiaans (want $H_1(f)$ is LTI)

Statistiek bepaald door autocorrelatiefuncties, dus door vermogensspectra

$$R_{x_{BP}}(u) = \text{IFT}[S_{x_{BP}}(f)](u) \quad R_{x_{BP,1}}(u) = \text{IFT}[S_{x_{BP,1}}(f)](u)$$

$$S_{x_{BP,1}}(f) = \underbrace{|H_1(f)|^2}_{=1} S_{x_{BP}}(f) = S_{x_{BP}}(f)$$

$x_{BP}(t)$ en $x_{BP,1}(t)$: identieke vermogensspectra, identieke autocorrelatiefuncties, dus identieke statistiek (wegen Gaussiaans)

Complexe omhullende van stationair reëel Gaussiaans TP

$$X_{BP}(f) = \frac{\sqrt{2}}{2} (X_{LP}(f - f_0) + X_{LP}^*(-f - f_0))$$

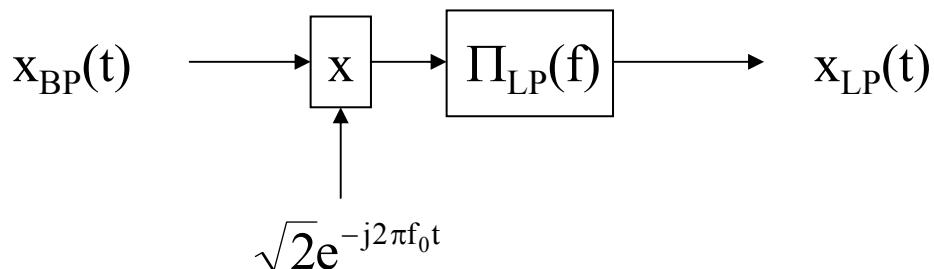
$$X_{BP,2}(f) = \frac{\sqrt{2}}{2} (X_{LP}(f - f_0)e^{-j2\pi(f-f_0)\tau} + X_{LP}^*(-f - f_0)e^{-j2\pi(f+f_0)\tau})$$

$$x_{BP}(t) \longrightarrow \boxed{H_2(f)} \longrightarrow x_{BP,2}(t) \quad H_2(f) = \begin{cases} e^{-j2\pi(f-f_0)\tau} & f > 0 \\ e^{-j2\pi(f+f_0)\tau} & f < 0 \end{cases}$$

$x_{BP}(t)$ en $x_{BP,2}(t)$: identieke vermogenspectra, identieke autocorrelatiefuncties,
dus identieke statistiek (wgens Gaussiaans)

Complexe omhullende van stationair re el Gaussiaans TP

Samenvatting



$$\Pi_{LP}(f) = \begin{cases} 1 & |f| < B \\ 0 & \text{otherwise} \end{cases}$$

$$f_0 > B$$

$x_{BP}(t)$ is stationair re el Gaussiaans TP, met vermogenspectrum $S_{BP}(f)$

\Rightarrow complexe omhullende $x_{LP}(t)$:

- is stationair, Gaussiaans en circulair symmetrisch
- heeft vermogenspectrum $2S_{BP}(f+f_0)\Pi_{LP}(f)$

Complexe Gaussiaanse TGn

Beschouw $\mathbf{z} = \mathbf{x} + j\mathbf{y}$ met \mathbf{x} en \mathbf{y} gezamenlijk Gaussiaans en $E[\mathbf{x}] = E[\mathbf{y}] = 0$

$$\mathbf{x} = (\mathbf{z} + \mathbf{z}^*)/2 \quad \mathbf{y} = (\mathbf{z} - \mathbf{z}^*)/(2j)$$

\mathbf{z} bevat dezelfde
informatie als \mathbf{v} , met

$$\mathbf{v} = \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix} = \mathbf{T} \begin{pmatrix} \mathbf{z} \\ \mathbf{z}^* \end{pmatrix} \quad \mathbf{T} = \frac{1}{2} \begin{pmatrix} \mathbf{I} & \mathbf{I} \\ -j\mathbf{I} & j\mathbf{I} \end{pmatrix} \quad \mathbf{T}^{-1} = \begin{pmatrix} \mathbf{I} & j\mathbf{I} \\ \mathbf{I} & -j\mathbf{I} \end{pmatrix}$$

Statistiek van \mathbf{z} volledig bepaald door statistiek van \mathbf{v} , dus door $\mathbf{R}_v = E[\mathbf{v}\mathbf{v}^T]$

$$\mathbf{R}_v = E[\mathbf{v}\mathbf{v}^T] = \begin{pmatrix} E[\mathbf{x}\mathbf{x}^T] & E[\mathbf{x}\mathbf{y}^T] \\ E[\mathbf{y}\mathbf{x}^T] & E[\mathbf{y}\mathbf{y}^T] \end{pmatrix} = \mathbf{T} \begin{pmatrix} E[\mathbf{z}\mathbf{z}^H] & E[\mathbf{z}\mathbf{z}^T] \\ (E[\mathbf{z}\mathbf{z}^T])^* & (E[\mathbf{z}\mathbf{z}^H])^* \end{pmatrix} \mathbf{T}^H$$

\mathbf{R}_v volledig bepaald door $E[\mathbf{z}\mathbf{z}^H]$ en $E[\mathbf{z}\mathbf{z}^T]$

Circulair symmetrische complexe Gaussiaanse TГn

\mathbf{z} Gaussiaans en **circulair symmetrisch** $\Leftrightarrow \mathbf{z}$ en $\mathbf{z}' = \mathbf{z}e^{j\theta}$ zelfde statistiek, ongeacht θ
 $\Leftrightarrow \mathbf{v}$ en \mathbf{v}' dezelfde covariantiematrix

$$\mathbf{v} = \mathbf{T} \begin{pmatrix} \mathbf{z} \\ \mathbf{z}^* \end{pmatrix}$$

$$\mathbf{R}_v = \mathbf{T} \begin{pmatrix} E[\mathbf{z}\mathbf{z}^H] & E[\mathbf{z}\mathbf{z}^T] \\ (E[\mathbf{z}\mathbf{z}^T])^* & (E[\mathbf{z}\mathbf{z}^H])^* \end{pmatrix} \mathbf{T}^H$$

$$\mathbf{v}' = \mathbf{T} \begin{pmatrix} \mathbf{z}e^{j\theta} \\ \mathbf{z}^* e^{-j\theta} \end{pmatrix}$$

$$\mathbf{R}_{v'} = \mathbf{T} \begin{pmatrix} E[\mathbf{z}\mathbf{z}^H] & E[\mathbf{z}\mathbf{z}^T]e^{2j\theta} \\ (E[\mathbf{z}\mathbf{z}^T])^* e^{-2j\theta} & (E[\mathbf{z}\mathbf{z}^H])^* \end{pmatrix} \mathbf{T}^H$$

$$\mathbf{R}_v = \mathbf{R}_{v'} \Leftrightarrow E[\mathbf{z}\mathbf{z}^T] = 0 \text{ (wegens } \mathbf{T} \text{ inverteerbaar)}$$

Circulair symmetrische complex Gaussiaanse TGN

$$E[\mathbf{z}\mathbf{z}^T] = 0 \Rightarrow \mathbf{R}_v = \mathbf{T} \begin{pmatrix} E[\mathbf{z}\mathbf{z}^H] & 0 \\ 0 & (E[\mathbf{z}\mathbf{z}^H])^* \end{pmatrix} \mathbf{T}^H = \frac{1}{2} \begin{pmatrix} \operatorname{Re}[E[\mathbf{z}\mathbf{z}^H]] & -\operatorname{Im}[E[\mathbf{z}\mathbf{z}^H]] \\ -\operatorname{Im}[E[\mathbf{z}\mathbf{z}^H]] & \operatorname{Re}[E[\mathbf{z}\mathbf{z}^H]] \end{pmatrix}$$

statistiek van \mathbf{z} enkel door \mathbf{R}_z bepaald, met $\mathbf{R}_z = E[\mathbf{z}\mathbf{z}^H]$ Notatie : $\mathbf{z} \sim N_c(0, \mathbf{R}_z)$,

Statistiek van (x_k, y_k) , met $z_k = x_k + jy_k$

$$E[x_k^2] = E[y_k^2] = (E[\mathbf{x}\mathbf{x}^T])_{k,k} = \frac{1}{2} \operatorname{Re}[(E[\mathbf{z}\mathbf{z}^H])_{k,k}] = \frac{1}{2} \operatorname{Re}[(E[|z_k|^2])] = \frac{1}{2} E[|z_k|^2]$$

$$E[x_k y_k] = (E[\mathbf{x}\mathbf{y}^T])_{k,k} = \frac{-1}{2} \operatorname{Im}[(E[\mathbf{z}\mathbf{z}^H])_{k,k}] = \frac{-1}{2} \operatorname{Im}[(E[|z_k|^2])] = 0$$

x_k en y_k i.i.d., $\sim N_c(0, E[|z_k|^2]/2)$

Complexe circulaire Gaussiaanse TGn

Lineaire transformatie

$$\mathbf{z}_2 = \mathbf{M}\mathbf{z}_1, \mathbf{z}_1 \sim N_c(0, \mathbf{R}_1)$$

$$\mathbf{R}_2 = E[\mathbf{z}_2\mathbf{z}_2^H] = \mathbf{M}E[\mathbf{z}_1\mathbf{z}_1^H]\mathbf{M}^H = \mathbf{M}\mathbf{R}_1\mathbf{M}^H$$

$$E[\mathbf{z}_2\mathbf{z}_2^T] = \mathbf{M}E[\mathbf{z}_1\mathbf{z}_1^T]\mathbf{M}^T = 0$$

$$\Rightarrow \mathbf{z}_2 \sim N_c(0, \mathbf{M}\mathbf{R}_1\mathbf{M}^H)$$

circulair Gaussiaans karakter blijft behouden na lineaire transformatie

Complexe circulaire Gaussiaanse TPn

Discrete tijd $\{z(k)\} \sim N_c(0, R_z(m,k))$ met $E[z(k+m)z^*(k)] = R_z(m,k)$
 $E[z(k+m)z(k)] = 0$

voorbeeld : discrete-tijd complexe stationaire witte ruis, $w(k) = w_R(k) + jw_I(k)$

$$w_R(k) \sim N(0, \sigma^2 \delta(i)), w_I(t) \sim N(0, \sigma^2 \delta(i)), E[w_R(k)w_I(k')] = 0$$

$$\Rightarrow w(k) \sim N_c(0, 2\sigma^2 \delta(m))$$

Continue tijd $z(t) \sim N_c(0, R_z(u, t))$ met $E[z(t+u)z^*(t)] = R_z(u, t)$
 $E[z(t+u)z(t)] = 0$

voorbeeld : continue-tijd complexe stationaire witte ruis, $w(t) = w_R(t) + jw_I(t)$

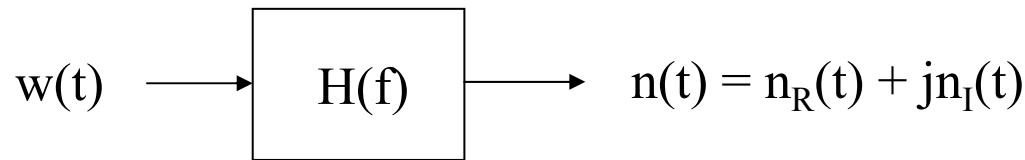
$$w_R(t) \sim N(0, (N_0/2)\delta(u)), w_I(t) \sim N(0, (N_0/2)\delta(u)), E[w_R(t)w_I(t')] = 0$$

$$\Rightarrow w(t) \sim N_c(0, N_0\delta(u))$$

Complexe circulaire Gaussiaanse TPn

Continue tijd

Filteren van complexe stationaire circulaire witte ruis $w(t) \sim N_c(0, N_0\delta(u))$



$w(t)$ circulair Gauss. $\Rightarrow n(t)$ circulair Gauss.

$n(t) \sim N_c(0, R_n(u))$ $R_n(u)$ is IFT van $N_0|H(f)|^2$

$n_R(t_0)$ en $n_I(t_0)$ zijn i.i.d., met

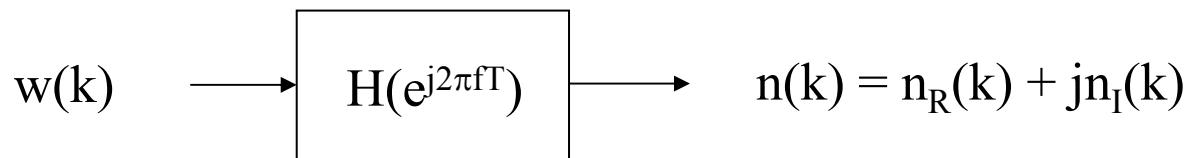
$$E[n_R^2(t_0)] = E[n_I^2(t_0)] = \frac{1}{2} E[|n(t_0)|^2] = \frac{N_0}{2} \int |H(f)|^2 df = \frac{N_0}{2} \int |h(t)|^2 dt$$

$$E[n_R(t_0)n_I(t_0)] = 0$$

Complexe circulaire Gaussiaanse TPn

Discrete tijd

Filteren van complexe stationaire circulaire witte ruis $w(k) \sim N_c(0, N_0\delta(m))$



$w(k)$ circulair Gauss. $\Rightarrow n(k)$ circulair Gauss.

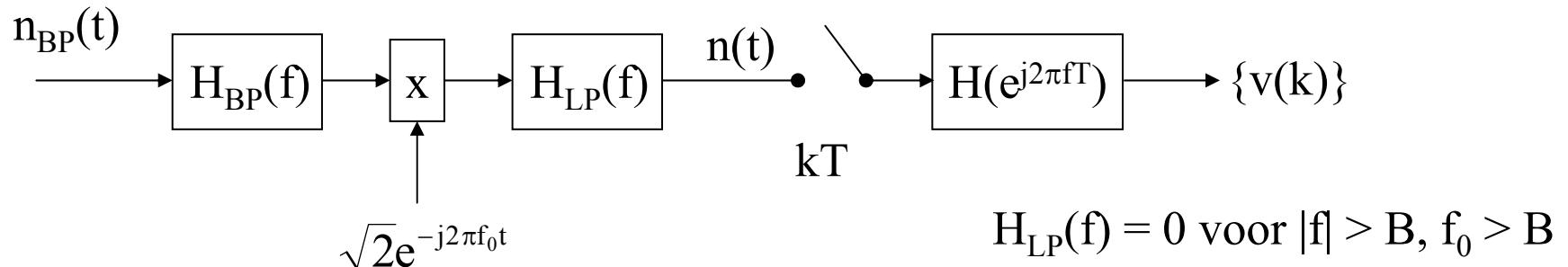
$n(k) \sim N_c(0, R_n(m))$ $R_n(m)$ is IDTFT van $N_0|H(e^{j2\pi fT})|^2$

$n_R(k_0)$ en $n_I(k_0)$ zijn i.i.d., met

$$E[n_R^2(k_0)] = E[n_I^2(k_0)] = \frac{1}{2} E[|n(k_0)|^2] = \frac{N_0}{2} T \int_{1/T} |H(e^{j2\pi fT})|^2 df = \frac{N_0}{2} \sum_k |h(k)|^2$$

$$E[n_R(k_0)n_I(k_0)] = 0$$

Effect van stationaire banddoorlaatruis



$$n_{BP}(t) \sim N(0, R_{BP}(u)) \quad R_{BP}(u) \text{ is IFT van } S_{BP}(f)$$

$$n(t) \sim N_c(0, R_n(u)) \quad R_n(u) \text{ is IFT van } S_n(f) = 2|H_{LP}(f)|^2|H_{BP}(f+f_0)|^2|S_{BP}(f+f_0)|$$

$$\{v(k)\} \sim N_c(0, R_v(m)) \quad R_v(m) \text{ is IDFT van } S_v(e^{j2\pi f T}) = |H(e^{j2\pi f T})|^2 \{S_n(f); 1/T\}_{\text{fld}}$$

$$v(k) = x(k) + jy(k)$$

$x(k)$ en $y(k)$ i.i.d., met

$$E[x^2(k)] = E[y^2(k)] = \frac{R_v(0)}{2} = \frac{1}{2} \int_{-\infty}^{+\infty} |H(e^{j2\pi f T})|^2 S_n(f) df$$

$$E[x(k)y(k)] = 0$$