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# MMSE finite-length equalizers (1)

**The MMSE citerion**

The ZF-LE and ZF-DFE in general require an *infinite* number of taps to completely eliminate the ISI. As only finite-length equalizers can be implemented, taps are adjusted according to the minimum mean square error (MMSE) criterion instead of the ZF criterion.



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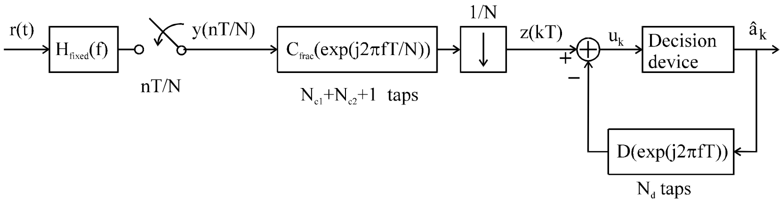
# MMSE finite-length equalizers (2)

**The MMSE citerion (cont.)**

$$C_{frac}\left(e^{j2\pi fT/N}\right) = \sum_{k=-N_{c1}}^{N_{c2}} c_{frac,k}\, e^{j2\pi fkT/N}$$

$$D\left(e^{j2\pi fT}\right) = \sum_{k=1}^{N_d} d_k\, e^{j2\pi fkT}$$



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## MMSE finite-length equalizers (3)

**The MMSE citerion (cont.)**

Equalizer error : $e_k = u_k - a_k$
Select equalizer taps $\{c_{frac,i}\}$ and $\{d_j\}$ such that $E[|e_k|^2]$ is minimum

$$e_k = u_k - a_k$$
$$= \sum_{n=-N_{c1}}^{N_{c2}} c_{frac,n} y\left(kT - \frac{nT}{N}\right) - \sum_{n=1}^{N_d} d_n a_{k-n} - a_k$$

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## MMSE finite-length equalizers (4)

**Equations for optimum taps**

$$\frac{\partial E\left[|e_k|^2\right]}{\partial c_{frac,i}} = 0 \Rightarrow E\left[e_k y^*\left(kT - \frac{iT}{N}\right)\right] = 0 \qquad i = -N_{c1}, \ldots, N_{c1}$$

$$\frac{\partial E\left[|e_k|^2\right]}{\partial d_j} = 0 \Rightarrow -E\left[e_k a^*_{k-j}\right] = 0 \qquad j = 1, \ldots, N_d$$

As $e_k$ is a linear function of the equalizer taps, the optimum taps satisfy a set of *linear* equations

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## MMSE finite-length equalizers (5)

**Iterative solution :**

(m: iteration index)

$$c_{frac,i}^{(m+1)} = c_{frac,i}^{(m)} - \mu E\left[e_k y^*\left(kT - \frac{iT}{N}\right)\right] \qquad i = -N_{c1}, \ldots, N_{c1}$$

$$d_j^{(m+1)} = d_j^{(m)} + \mu E\left[e_k a_{k-j}^*\right] \qquad j = 1, \ldots, N_d$$

For sufficiently small $\mu$, convergence to the optimum solution is achieved



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## MMSE finite-length equalizers (6)

$$E\left[e_k y^*\left(kT - \frac{iT}{N}\right)\right], \ E\left[e_k a_{k-j}^*\right]:$$

depend on channel transfer function (and on noise level); the linear equations can be solved only when these characteristics are known

## Adaptive equalization /channel estimation (1)

Channel transfer function not a priori known, and possibly time-varying (e.g. fading)

**Strategy 1 : adaptive equalization**
Equalizer taps are updated to minimize mean square error.

**Strategy 2 : adaptive channel estimation**
Receiver estimates impulse response $g_{rec}(t)$ of the “channel” (channel = $P(f)H_{ch}(f)\ H_{fixed}(f) = G_{rec}(f)$), and uses this estimate to compute equalizer taps. Channel estimates are updated to minimize mean square error.

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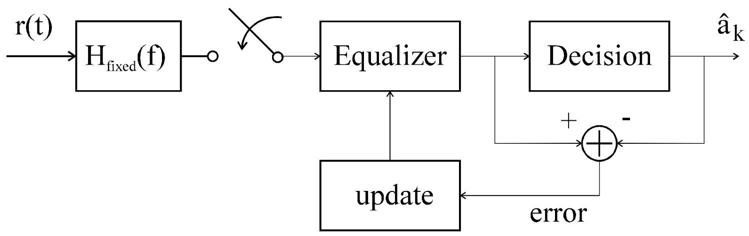
## Adaptive equalization /channel estimation (2)

**Adaptive equalization block diagram**

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## Adaptive equalization /channel estimation (3)

**Adaptive equalization algorithm**

Least mean square (LMS) adaptation (k : symbol index)

$$c_{frac,i}^{(k+1)} = c_{frac,i}^{(k)} - \mu e_k y^*\left(kT - \frac{iT}{N}\right) \qquad i = -N_{c1}, \ldots, N_{c1}$$

$$d_j^{(k+1)} = d_j^{(k)} + \mu e_k \hat{a}_{k-j}^* \qquad j = 1, \ldots, N_d$$

$\mu$ : step size

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## Adaptive equalization /channel estimation (4)

**Adaptive equalization algorithm (cont.)**

Assume correct decisions. Note that

$$e_k y^*\left(kT - \frac{iT}{N}\right) = E\left[e_k y^*\left(kT - \frac{iT}{N}\right)\right] + \text{statistical fluctuation}$$

$$(i = -N_{c1}, \ldots, N_{c1})$$

$$e_k a_{k-j}^* = E\left[e_k a_{k-j}^*\right] + \text{statistical fluctuation}$$

$$(j = 1, \ldots, N_d)$$

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## Adaptive equalization /channel estimation (5)

**Adaptive equalization algorithm (cont.)**

The statistical fluctuations are caused by the noise and the random data sequence.
The equalizer taps are “noisy” solutions of the set of MMSE equations; in the steady state, the taps fluctuate around the optimum values. The fluctuations of the taps yield an increase of $E[|e_k|^2]$, as compared to the optimum tap values.

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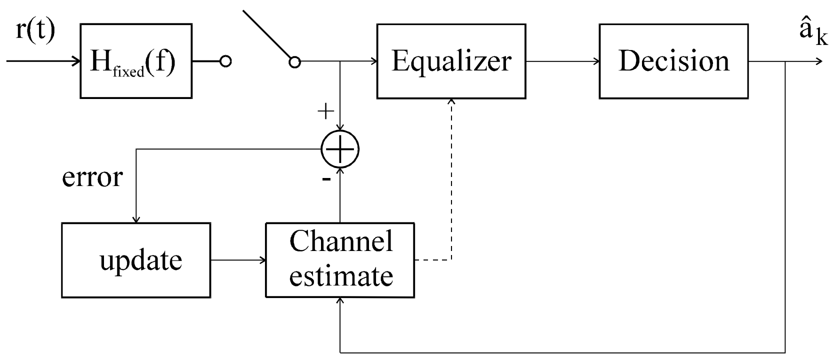
## Adaptive equalization /channel estimation (6)

**Adaptive channel estimation block diagram**

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## Adaptive equalization /channel estimation (7)

**Adaptive channel estimation algorithm**

sampled output (rate 1/T) of receive filter $H_{fixed}(f)$ :

$$y(kT) = \sqrt{E_s}\sum_m a_{k-m} g_{rec}(mT) + n(kT)$$

(can be extended to sampling rate N/T)

$\hat{g}_{rec}(mT)$ : estimate of $g_{rec}(mT)$

error signal :

$$e_k = y(kT) - \sqrt{E_s}\sum_{m=-M_1}^{M_2} a_{k-m}\hat{g}_{rec}(mT)$$

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## Adaptive equalization /channel estimation (8)

**Adaptive channel estimation algorithm (cont.)**

MMSE estimation

$$E\left[|e_k|^2\right] \text{ is minimum for } \hat{g}_{rec}(mT) = g_{rec}(mT)$$

MMSE equation

$$\hat{g}_{rec}(mT) : \text{ solution of } \frac{d\,E\left[|e_k|^2\right]}{d\,\hat{g}_{rec}(mT)} = 0 \Rightarrow E\left[e_k a^*_{k-m}\right] = 0$$

$$m = -M_1, \ldots, M_2$$

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## Adaptive equalization /channel estimation (9)

**Adaptive channel estimation algorithm (cont.)**

LMS adaptation (k : symbol index)

$$\hat{g}_{rec}^{(k+1)}(mT) = \hat{g}_{rec}^{(k)}(mT) - \mu e_k \hat{a}_{k-m}^{*}$$

Assume no decision errors :

$$e_k a_{k-m}^{*} = E\left[e_k a_{k-m}^{*}\right] + \text{ statistical fluctuation}$$

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## Adaptive equalization /channel estimation (10)

**Adaptive channel estimation algorithm (cont.)**

The statistical fluctuations are caused by the noise and the random data sequence.
The filter taps are “noisy” solutions of the set of MMSE equations; in the steady state, the taps fluctuate around the optimum values. The fluctuations of the taps yield an increase of $E[|e_k|^2]$, as compared to the optimum tap values.

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## Adaptive equalization /channel estimation (11)

**Performance of adaptive channel estimation**

dynamics of instantaneous MSE :

$$E\left[|e_{k+1}|^2\right] \cong \gamma E\left[|e_k|^2\right] + \left(1-\gamma+E_s\mu^2 N_t\right)\text{MMSE}$$

$\gamma = 1 - 2\sqrt{E_s}\mu + E_s\mu^2 N_t$

$N_t = M_1 + M_2 - 1$ (number of taps)

$$\text{MMSE} = E\left[|n_k|^2\right] + \sum_{m \notin (-M_1,\dots,M_2)} |g_{rec}(mT)|^2 E_s = E\left[|e_k|^2\right]\Big|_{\text{optimum taps}}$$

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## Adaptive equalization /channel estimation (12)

**Performance of adaptive channel estimation (cont.)**

Stability :

$|\gamma| > 1$ : divergence (unstable)

$|\gamma| < 1$ : convergence (stable)

transients decay like $|\gamma|^k$

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## Adaptive equalization /channel estimation (13)

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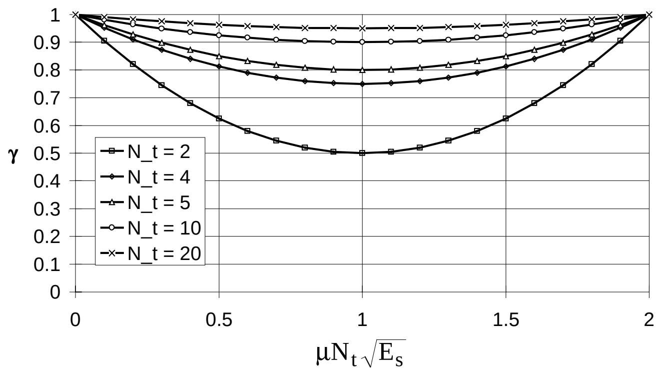
**Performance of adaptive channel estimation (cont.)**

Convergence parameter $\gamma$

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## Adaptive equalization /channel estimation (14)

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**Performance of adaptive channel estimation (cont.)**

$$|\gamma| < 1 \Leftrightarrow 0 < \mu < \frac{2}{N_t \sqrt{E_s}}$$

Fastest convergence when $\gamma$ is minimum

$$\mu = \frac{1}{N_t \sqrt{E_s}} \Rightarrow \gamma = \gamma_{min} = 1 - \frac{1}{N_t}$$

convergence speed $\downarrow$ when $N_t \uparrow$

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## Adaptive equalization /channel estimation (15)

**Performance of adaptive channel estimation (cont.)**

Steady-state MSE

$$E\left[|e_\infty|^2\right] = \lim_{k\to\infty} E\left[|e_k|^2\right] = \text{MMSE}\left(1 + \frac{\sqrt{E_s}\mu N_t}{2-\sqrt{E_s}\mu N_t}\right)$$

Steady-state MSE ↓ when $\mu$ ↓

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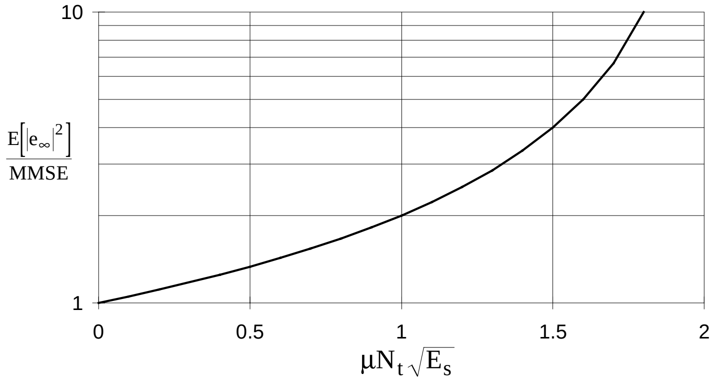
## Adaptive equalization /channel estimation (16)

**Performance of adaptive channel estimation (cont.)**

Steady-state MSE

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## Adaptive equalization /channel estimation (17)

**Performance of adaptive channel estimation (cont.)**

Effect of number of taps

For given value of $\mu N_t$ :

$N_t \uparrow \Rightarrow$ MMSE $\downarrow \Rightarrow$ steady state MSE $\downarrow$

$N_t \uparrow \Rightarrow$ convergence speed $\downarrow$

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## Adaptive equalization /channel estimation (18)

**Performance of adaptive equalization**

Similar conclusions as for adaptive channel estimation

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## Adaptive equalization /channel estimation (19)

**Example : adaptive channel estimation**

$$g_{rec}(mT) = \begin{cases} \sqrt{\dfrac{1-\alpha^2}{1-\alpha^{20}}}\,\alpha^m & m = 0, 1, \ldots, 9 \\ 0 & \text{otherwise} \end{cases}$$

(energy of $\{g_{rec}(mT)\}$ is normalized to 1)

$E_s = 1$

$$SNR = -10\log\left(E\left[|n_k|^2\right]\right) \qquad [dB]$$



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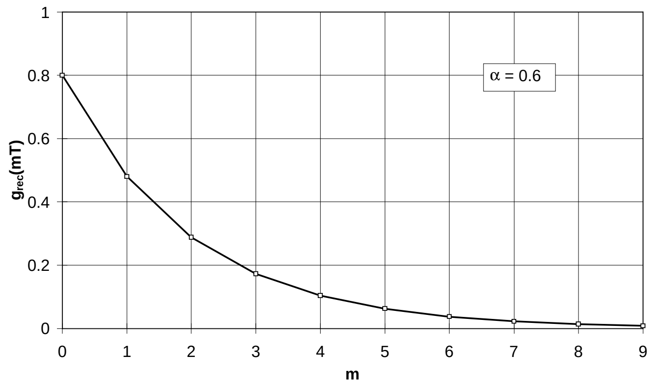
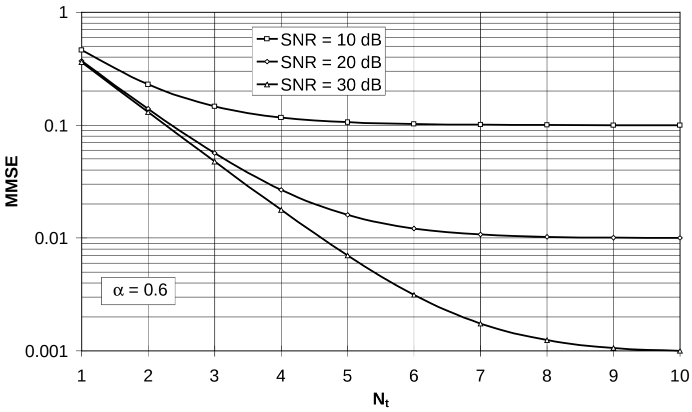
## Adaptive equalization /channel estimation (20)

**Example (cont.)**



## Adaptive equalization /channel estimation (21)

**Example (cont.)**

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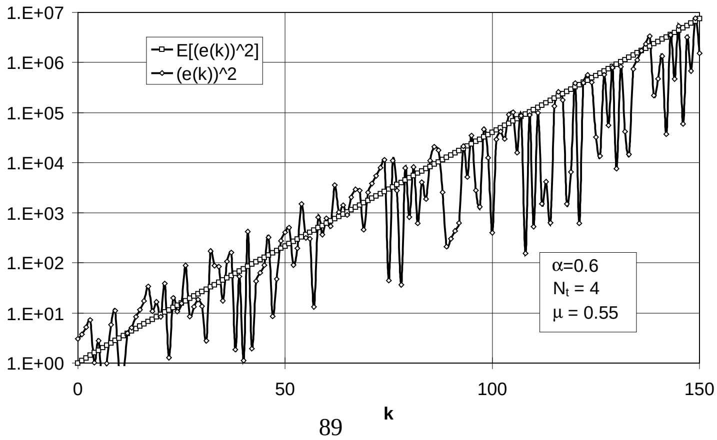
## Adaptive equalization /channel estimation (22)

**Example (cont.)** $\mu > 2/N_t \Rightarrow$ unstable

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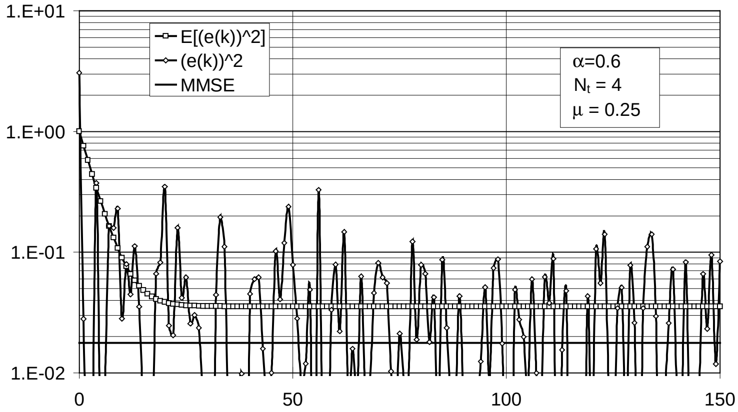
## Adaptive equalization /channel estimation (23)

**Example (cont.)** $\mu = 1/N_t \Rightarrow$ fastest convergence

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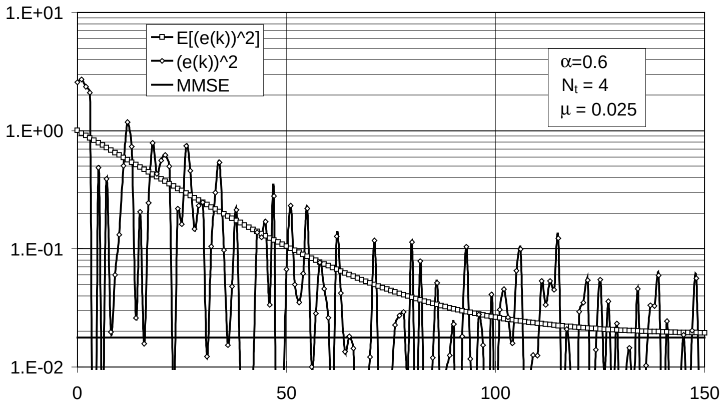
## Adaptive equalization /channel estimation (24)

**Example (cont.)** $\mu < 1/N_t \Rightarrow$ slow convergence

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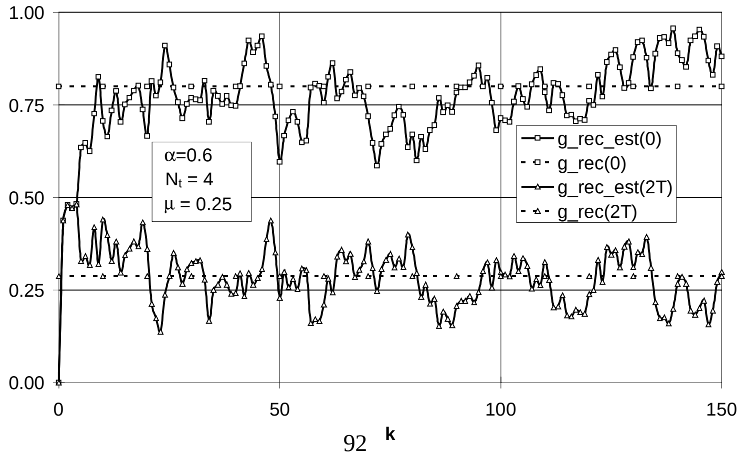
# Adaptive equalization /channel estimation (25)

**Example (cont.)** Evolution of filter taps

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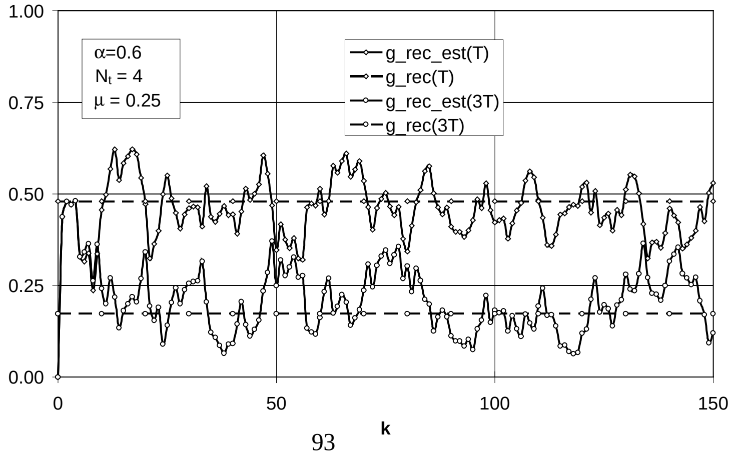
# Adaptive equalization /channel estimation (26)

**Example (cont.)** Evolution of filter taps (cont.)

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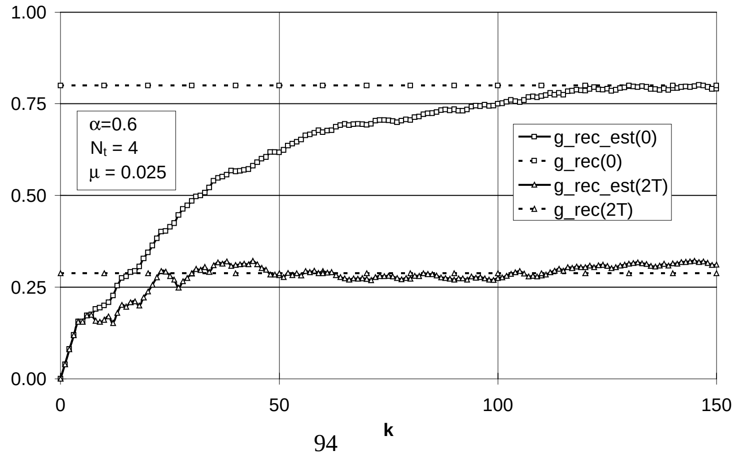
# Adaptive equalization /channel estimation (27)

**Example (cont.)** Evolution of filter taps (cont.)

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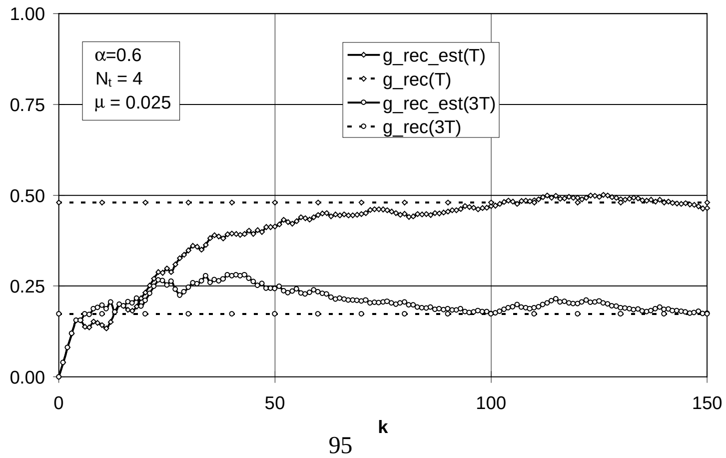
# Adaptive equalization /channel estimation (28)

**Example (cont.)** Evolution of filter taps (cont.)

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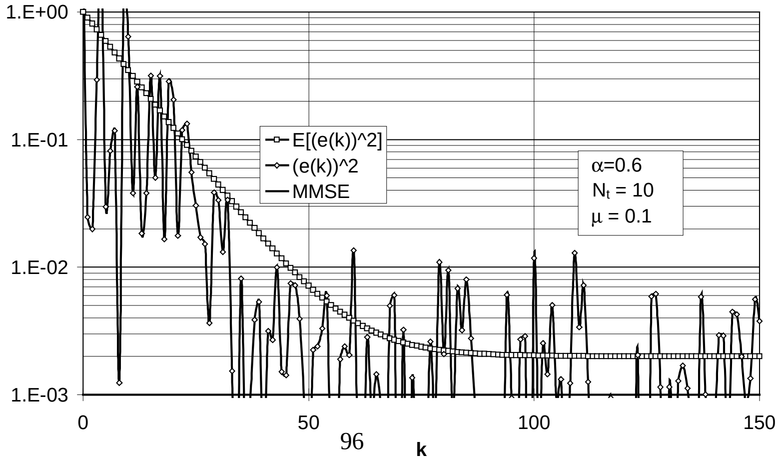
## Adaptive equalization /channel estimation (29)

**Example (cont.)** $N_t \uparrow$ , $\mu = 1/N_t$

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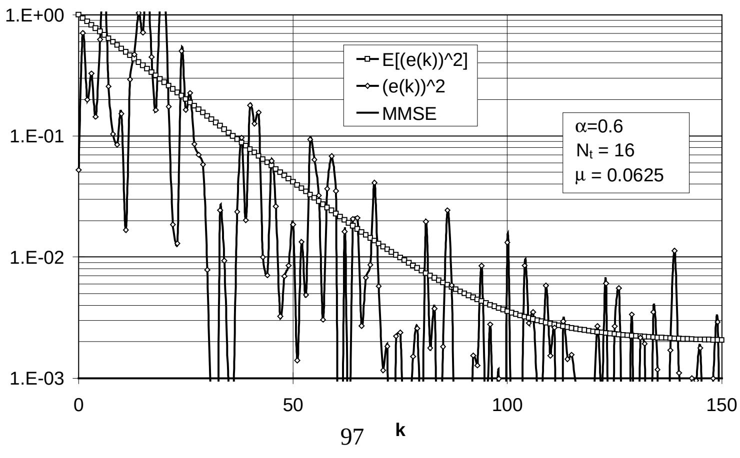
## Adaptive equalization /channel estimation (30)

**Example (cont.)** $N_t$ too large , $\mu = 1/N_t$

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## Adaptive equalization /channel estimation (31)

**Other adaptation algorithms**

LMS algorithm requires few computations, but convergence is rather slow.

Faster convergence can be obtained at the expense of a higher computational load (Kalman algorithm, ...)

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## Adaptive equalization /channel estimation (32)

**Conclusions**

| | |
|---|---|
| $\mu N_t \sqrt{E_s} > 2$ | instable |
| $\mu N_t \sqrt{E_s} = 1$ | fastest convergence (as $(1 - 1/N_t)^k$),<br>steady-state MSE = 2MMSE |
| $\mu N_t \sqrt{E_s} < 1$ | slow convergence,<br>steady-state MSE $\in$ (MMSE, 2MMSE) |

Keep $N_t$ to a reasonable minimum !

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