TOEGEPAST ELEKTROMAGNETISME
Applied Electromagnetics

Daniël De Zutter
Department of Information Technology (INTEC)
Ghent University
Sint-Pietersnieuwstraat 41 9000 Gent Belgium
daniel.dezutter@intec.ugent.be

Chapter 1: Introduction

Outline

- **Electromagnetics**
- **Notations**
- **Frequency domain, sinusoidal regime, phasors**
- **Polarisation**
  - **linear polarisation**
  - **circular polarisation**

Electromagnetics

The basic equations of Electromagnetics are Maxwell's equations first put forward by James Clark Maxwell in 1864. They describe the coupling between electric and magnetic fields and lead to the concept of electromagnetic waves. Maxwell's equations constitute the first "modern" physics theory, soon to be followed by the Special Theory of Relativity and by Quantum Physics. Maxwell's equations predict the speed of light for wave propagation in vacuum and remain invariant under the Lorentz transformation, i.e. they are compatible with the Special Theory of Relativity.

Electromagnetics

In this course, Maxwell's equations form the starting point to study the propagation and scattering of waves, the principles of antennas and antenna communication, transmission line theory, signal propagation along multiconductor lines and waveguides but also to study electrostatics, magnetostatics and the so-called quasi-static behaviour of circuits and their description in terms of lumped elements. Advantage is taken of the extensive mathematical background of the third year's bachelor student to provide a mathematically rigourous treatment of the various topics.

Some modern applications

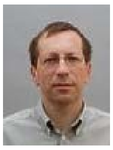
- **wireless communication: GSM, UMTS, Hyperlan, Bluetooth, …**
- **electronic circuits and wireline applications for multiGbit/s applications**
- **on-chip interconnect including harmonics up to 10GHz**
- **nanoscale opto-electronics to "mold the flow of light"**
- **remote sensing and inverse scattering for non-destructive testing**
- **medical imaging and imaging for security purposes**
- **design of artificial materials**
- **…………..**

The Electromagnetics Group

The Electromagnetics Group (5 staff members, 10 Ph.D. students) is one of the research groups of the Department of Information Technology.

During the past 15 years the Electromagnetics Group has been on the forefront of electromagnetic simulation techniques with applications in areas such as microwave and RF circuits, (inverse) scattering, waveguides, packaging for digital systems, EMC, indoor propagation and CAD-tool development. Research results were reported in more than 160 international journal papers and resulted in 25 Ph.D's .

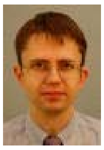
STAFF



D. De Zutter

A. Franchois

L. Knockaert



F. Olyslager

H. Rogier

Notations

a or α stand for scalar quantities

$\mathbf{a}$ and $\boldsymbol{\alpha}$ are used for vectors

unit vector $\mathbf{u}$ **e.g.** $\mathbf{u}_x$, $\mathbf{u}_y$ and $\mathbf{u}_z$ **or** $\mathbf{u}_r$, $\mathbf{u}_\phi$ and $\mathbf{u}_\theta$ **or** $\mathbf{u}_n$

place vector: $\mathbf{r} = x\mathbf{u}_x + y\mathbf{u}_y + z\mathbf{u}_z = r\mathbf{u}_r$

matrix $$\overline{\overline{A}} = \begin{pmatrix} A_{xx} & A_{xy} & A_{xz} \\ A_{yx} & A_{yy} & A_{yz} \\ A_{zx} & A_{zy} & A_{zz} \end{pmatrix}$$

$$\begin{aligned} \overline{\overline{A}} \cdot \mathbf{a} \quad = \quad & (A_{xx}a_x + A_{xy}a_y + A_{xz}a_z)\mathbf{u}_x + (A_{yx}a_x + A_{yy}a_y + A_{yz}a_z)\mathbf{u}_y \\ & +(A_{zx}a_x + A_{zy}a_y + A_{zz}a_z)\mathbf{u}_z. \end{aligned}$$

Frequency domain - sinusoidal regime

general time domain signal f(t) and its frequency domain counterpart f(ω)

$$f(\omega) = \int_{-\infty}^{+\infty} f(t)e^{-j\omega t}\mathrm{d}t \qquad\qquad f(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} f(\omega)e^{j\omega t}\mathrm{d}\omega$$

pure sinusoidal signal with angular frequency ω_0

$$f(t) = |f|\cos(\omega_0 t + \phi) = \Re[|f|e^{j\phi}e^{j\omega_0 t}] = \Re[f e^{j\omega_0 t}]$$

phasor representation: $\boxed{f = |f|e^{j\phi}}$

Fourier transform sinusoidal signal: $f(\omega) = \pi[f\delta(\omega - \omega_0) + f^*\delta(\omega + \omega_0)]$

Frequency domain - sinusoidal regime - cont.

Product of two sinusoidal signals (same frequency)

$$f(t) = |f|\cos(\omega t + \phi) = \Re[f e^{j\omega t}] \qquad g(t) = |g|\cos(\omega t + \psi) = \Re[g e^{j\omega t}]$$

$$\Downarrow$$

$$f(t)g(t) = \frac{1}{2}|f||g|[\cos(2\omega t + \phi + \psi) + \cos(\phi - \psi)]$$

Time average value over a single period

$$\overline{f(t)g(t)} = \frac{1}{2}|f||g|\cos(\phi - \psi) = \frac{1}{2}\Re[f g^*]$$

Polarisation

general property of a vector with sinusoidal time dependence!

$$\begin{aligned}\mathbf{a}(t) &= |a_x|\cos(\omega t + \phi_x)\mathbf{u}_x + |a_y|\cos(\omega t + \phi_y)\mathbf{u}_y + |a_z|\cos(\omega t + \phi_z)\mathbf{u}_z \\ &= \Re[(a_x\mathbf{u}_x + a_y\mathbf{u}_y + a_z\mathbf{u}_z)e^{j\omega t}]. \\ &= \Re[\mathbf{a}e^{j\omega t}]\end{aligned}$$

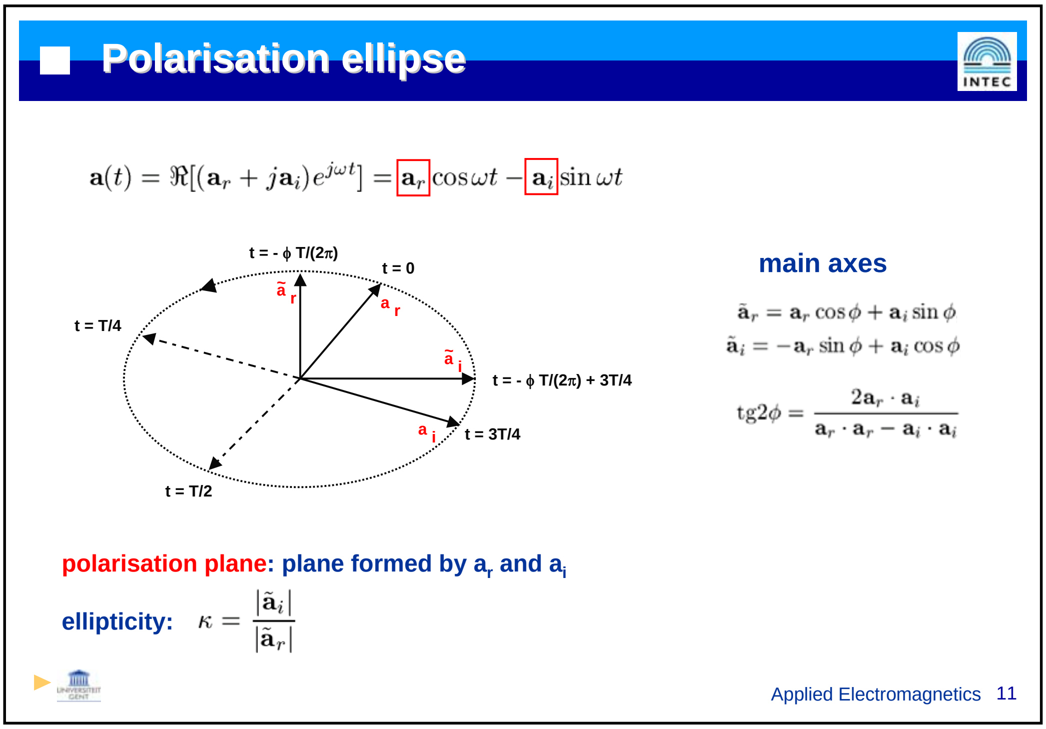
with $\mathbf{a} = \mathbf{a}_r + j\mathbf{a}_i$

the corresponding time domain signal becomes

$$\mathbf{a}(t) = \Re[(\mathbf{a}_r + j\mathbf{a}_i)e^{j\omega t}] = \mathbf{a}_r\cos\omega t - \mathbf{a}_i\sin\omega t$$

$$\Downarrow$$

**with ωt = τ as a parameter this is
the representation of an ellipse**



Polarisation ellipse

$$\mathbf{a}(t) = \Re[(\mathbf{a}_r + j\mathbf{a}_i)e^{j\omega t}] = \boxed{\mathbf{a}_r}\cos\omega t - \boxed{\mathbf{a}_i}\sin\omega t$$

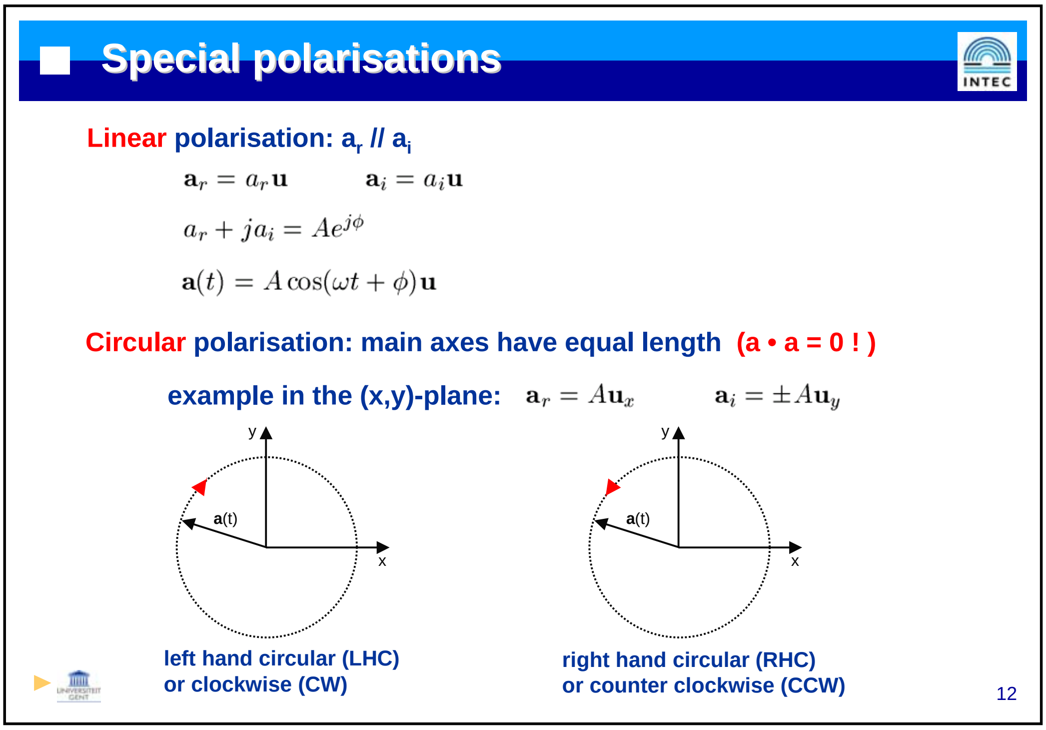
main axes

$$\tilde{\mathbf{a}}_r = \mathbf{a}_r \cos\phi + \mathbf{a}_i \sin\phi$$
$$\tilde{\mathbf{a}}_i = -\mathbf{a}_r \sin\phi + \mathbf{a}_i \cos\phi$$

$$\mathrm{tg}2\phi = \frac{2\mathbf{a}_r \cdot \mathbf{a}_i}{\mathbf{a}_r \cdot \mathbf{a}_r - \mathbf{a}_i \cdot \mathbf{a}_i}$$

polarisation plane: plane formed by a_r and a_i

ellipticity: $\kappa = \dfrac{|\tilde{\mathbf{a}}_i|}{|\tilde{\mathbf{a}}_r|}$



Special polarisations

Linear polarisation: a_r // a_i

$$\mathbf{a}_r = a_r\mathbf{u} \qquad \mathbf{a}_i = a_i\mathbf{u}$$

$$a_r + ja_i = Ae^{j\phi}$$

$$\mathbf{a}(t) = A\cos(\omega t + \phi)\mathbf{u}$$

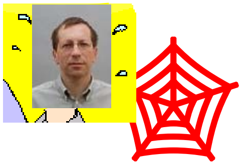
Circular polarisation: main axes have equal length (a • a = 0 !)

example in the (x,y)-plane: $\mathbf{a}_r = A\mathbf{u}_x \qquad \mathbf{a}_i = \pm A\mathbf{u}_y$

left hand circular (LHC)
or clockwise (CW)

right hand circular (RHC)
or counter clockwise (CCW)

Questions?



Chapter 2: Maxwell's Equations

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Outline

- Differential and integral formulation
- Constitutive equations
- Conservation of energy - Poynting's vector
- Boundary conditions
- Elementary dipole sources
- Potentials and Green's functions
- Wave equations
- Image theory and image sources

Maxwell's equations in the time domain

curl equations

$$\nabla \times \mathbf{e}(\mathbf{r},t) = -\frac{\partial}{\partial t}\mathbf{b}(\mathbf{r},t)$$

$$\nabla \times \mathbf{h}(\mathbf{r},t) = \frac{\partial}{\partial t}\mathbf{d}(\mathbf{r},t) + \mathbf{j}(\mathbf{r},t)$$

divergence equations

$$\nabla \cdot \mathbf{d}(\mathbf{r},t) = \rho(\mathbf{r},t)$$

$$\nabla \cdot \mathbf{b}(\mathbf{r},t) = 0$$

law of conservation of charge

$$\nabla \cdot \mathbf{j}(\mathbf{r},t) = -\frac{\partial}{\partial t}\rho(\mathbf{r},t)$$

Maxwell's equations in the frequency domain

curl equations

$$\nabla \times \mathbf{e}(\mathbf{r}) = -j\omega\mathbf{b}(\mathbf{r})$$

$$\nabla \times \mathbf{h}(\mathbf{r}) = j\omega\mathbf{d}(\mathbf{r}) + \mathbf{j}(\mathbf{r})$$

divergence equations

$$\nabla \cdot \mathbf{d}(\mathbf{r}) = \rho(\mathbf{r})$$

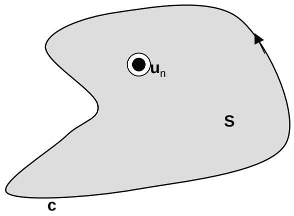
$$\nabla \cdot \mathbf{b}(\mathbf{r}) = 0$$

law of conservation of charge

$$\nabla \cdot \mathbf{j}(\mathbf{r}) = -j\omega\rho(\mathbf{r})$$

Integral form of Maxwell's equations

Integrating the curl equations over a surface S yields



Faraday's law

$$\oint_c \mathbf{e}(\mathbf{r},t)\cdot\mathbf{dc} = -\frac{\partial}{\partial t}\int_S \mathbf{b}(\mathbf{r},t)\cdot\mathbf{u}_n dS = -\frac{\partial}{\partial t}\Phi_b$$

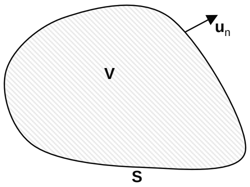
Ampère's law

$$\oint_c \mathbf{h}(\mathbf{r},t)\cdot\mathbf{dc} = \int_S \mathbf{j}(\mathbf{r},t)\cdot\mathbf{u}_n dS + \frac{\partial}{\partial t}\int_S \mathbf{d}(\mathbf{r},t)\cdot\mathbf{u}_n dS = i(\mathbf{r},t) + \frac{\partial}{\partial t}\Phi_d$$

$$\oint_c \mathbf{h}(\mathbf{r},t)\cdot\mathbf{dc} = \int_S \mathbf{j}(\mathbf{r},t)\cdot\mathbf{u}_n dS = i(\mathbf{r},t)$$ **(neglecting Φ_d)**

Integral form of Maxwell's equations

Integrating the divergence equations over a volume V yields



Gauss's law

$$\int_S \mathbf{d}(\mathbf{r},t)\cdot\mathbf{u}_n dS = \int_V \rho(\mathbf{r},t)dV = q_V(t)$$

= total charge in V

Gauss's law for the magnetic induction

$$\int_S \mathbf{b}(\mathbf{r},t)\cdot\mathbf{u}_n dS = 0$$ ⇨ **no magnetic charges**

Integrating the charge conservation law over a volume V yields

$$\int_S \mathbf{j}(\mathbf{r},t)\cdot\mathbf{u}_n dS = -\frac{\partial}{\partial t}q_V(t)$$

Constitutive equations in free space

To solve Maxwell we need relations between e, b, d, h and j

Vacuum (free space)

$$\mathbf{d}(\mathbf{r}) = \epsilon_0 \mathbf{e}(\mathbf{r})$$
$$\mathbf{b}(\mathbf{r}) = \mu_0 \mathbf{h}(\mathbf{r})$$

μ_0 = permeability of free space = $4\pi 10^{-7} H/m$

$\epsilon_0 = \frac{1}{c^2 \mu_0} = 8.854187818\ 10^{-12} F/m$ = permittivity of free space

$c = \frac{1}{\sqrt{\epsilon_0 \mu_0}}$ = velocity of light in vacuum = 2.99792458 10^8 m/s

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Constitutive equations in material media

typically formulated in the frequency domain

macroscopic

$$\mathbf{d}(\mathbf{r},\omega) = \epsilon(\mathbf{r},\omega)\mathbf{e}(\mathbf{r},\omega)$$
$$\mathbf{b}(\mathbf{r},\omega) = \mu(\mathbf{r},\omega)\mathbf{h}(\mathbf{r},\omega)$$

linear
isotropic
dispersive
anisotropic
homogeneous
inhomogeneous
??

$\epsilon_r = \epsilon/\epsilon_0,\qquad \mu_r = \mu/\mu_0$ relative permittivity and permeability

$\epsilon = \epsilon_R + j\epsilon_I \qquad \mu = \mu_R + j\mu_I$ complex valued !!

$n = \sqrt{\epsilon_r \mu_r}$ refractive index

$\mathrm{tg}\delta = -\frac{\epsilon_I}{\epsilon_R}$ loss tangent

Conduction current and source current

The current **j** in Maxwell's equations consists of 2 contributions

- a **known** externally enforced **source current** density $\mathbf{j}_e$ (A/m^2)
- charge displacement **induced** by the fields: **conduction current** $\mathbf{j}_i$

⇩

$$\mathbf{j}(\mathbf{r}) = \mathbf{j}_i(\mathbf{r}) + \mathbf{j}_e(\mathbf{r})$$

Constitutive equation for the conduction current

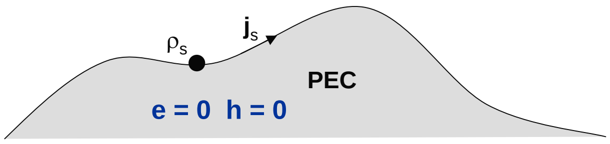
$$\mathbf{j}_i(\mathbf{r}) = \sigma(\mathbf{r})\mathbf{e}(\mathbf{r})$$

"Complex" permittivity

$$\left.\begin{aligned}\nabla \times \mathbf{h}(\mathbf{r}) &= j\omega\epsilon(\mathbf{r})\mathbf{e}(\mathbf{r}) + \sigma(\mathbf{r})\mathbf{e}(\mathbf{r}) + \mathbf{j}_e(\mathbf{r})\\ \nabla \times \mathbf{h}(\mathbf{r}) &= j\omega\tilde{\epsilon}(\mathbf{r})\mathbf{e}(\mathbf{r}) + \mathbf{j}_e(\mathbf{r})\end{aligned}\right\} \tilde{\epsilon}(\mathbf{r}) = \epsilon(\mathbf{r}) + \frac{\sigma(\mathbf{r})}{j\omega}$$

Perfect conductor

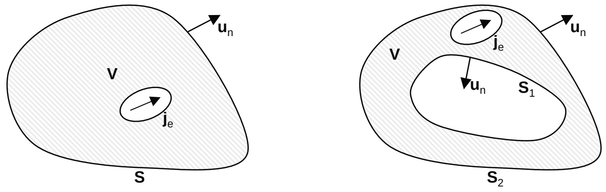
$\sigma = \infty$ ⇨ **PEC: Perfect Electric Conductor**



zero fields inside the PEC

supports surface charges and surface currents

Conservation of energy - Poynting's vector



S: S_1 U S_2

$$\int_S (\mathbf{e}\times\mathbf{h}^*)\cdot\mathbf{u}_n dS = \int_V (-j\omega\mathbf{h}^*\cdot\mathbf{b} - \mathbf{e}\cdot(\mathbf{j}_e^* + \mathbf{j}_i^*) + j\omega\mathbf{e}\cdot\mathbf{d}^*)dV$$

Poynting's vector: $\mathbf{p}(\mathbf{r}) = \frac{1}{2}\mathbf{e}(\mathbf{r})\times\mathbf{h}^*(\mathbf{r})$

⇩

$$-\frac{1}{2}\int_V \mathbf{e}\cdot\mathbf{j}_e^* dV = \int_S \mathbf{p}\cdot\mathbf{u}_n dS + \frac{1}{2}\int_V \mathbf{e}.\mathbf{j}_i^* dV + \frac{1}{2}\int_V (j\omega\mathbf{h}^*\cdot\mathbf{b} - j\omega\mathbf{e}\cdot\mathbf{d}^*)dV$$

Poynting's vector in free space

$$-\frac{1}{2}\int_V \mathbf{e}\cdot\mathbf{j}_e^* dV = \int_S \mathbf{p}\cdot\mathbf{u}_n dS + \frac{1}{2}\int_V \mathbf{e}.\mathbf{j}_i^* dV + \frac{1}{2}\int_V (j\omega\mathbf{h}^*\cdot\mathbf{b} - j\omega\mathbf{e}\cdot\mathbf{d}^*)dV$$

real power generated by source

$$\mathbf{d}(\mathbf{r}) = \epsilon_0\mathbf{e}(\mathbf{r})$$
$$\mathbf{b}(\mathbf{r}) = \mu_0\mathbf{h}(\mathbf{r})$$

j_i = 0

$$u_m(t) = \frac{1}{2}\int_V \mu_0[\mathbf{h}(t)]^2 dV$$
$$u_e(t) = \frac{1}{2}\int_V \epsilon_0[\mathbf{e}(t)]^2 dV$$

⇩

imaginary power

$$-\frac{1}{2}\Re\int_V \mathbf{e}\cdot\mathbf{j}_e^* dV = \Re\int_S \mathbf{p}\cdot\mathbf{u}_n dS$$

radiated power through S

$$-\frac{1}{2}\Im\int_V \mathbf{e}\cdot\mathbf{j}_e^* dV = \Im\int_S \mathbf{p}\cdot\mathbf{u}_n dS + \frac{1}{2}\omega\mu_0\int_V |\mathbf{h}|^2 dV - \frac{1}{2}\omega\epsilon_0\int_V |\mathbf{e}|^2 dV$$

II

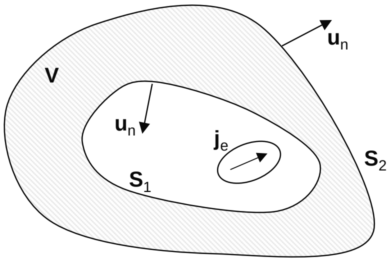
$$2\omega(\overline{u_m(t)} - \overline{u_e(t)})$$

radiated imaginary power

difference of average electric and magnetic energy stored in V

Sourceless volume V

power entering V through S_1 must leave through S_2



$$\Re \int_{S_1 \cup S_2} \mathbf{p} \cdot \mathbf{u}_n dS = 0$$

$$\Re \int_{S_1} \mathbf{p} \cdot (-\mathbf{u}_n) dS = \Re \int_{S_2} \mathbf{p} \cdot \mathbf{u}_n dS$$

and similarly for the imaginary part!

Isotropic lossy medium

$$-\frac{1}{2}\Re \int_V \mathbf{e} \cdot \mathbf{j}_e^* dV = \Re \int_S \mathbf{p} \cdot \mathbf{u}_n dS + \frac{1}{2} \int_V \sigma |\mathbf{e}|^2 dV$$

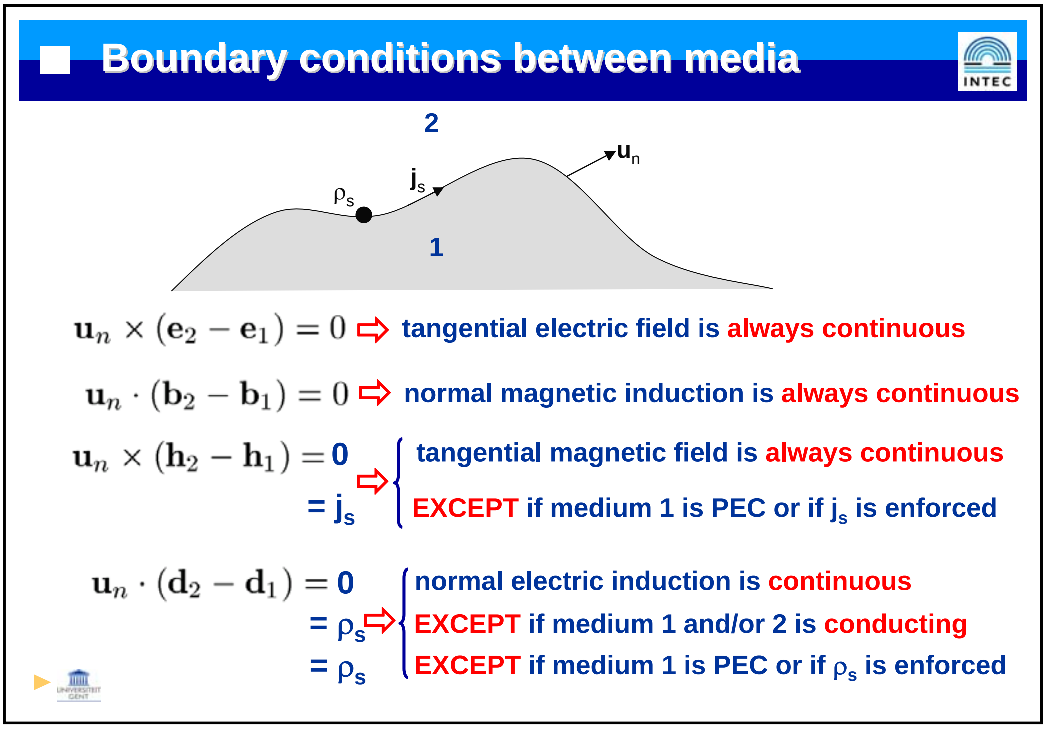
$$-\frac{1}{2}\omega \int_V \mu_I |\mathbf{h}|^2 dV + \frac{1}{2}\omega \int_V \epsilon_I |\mathbf{e}|^2 dV,$$

Joule losses (time average)

magnetic losses

electric losses

$$-\frac{1}{2}\Im \int_V \mathbf{e} \cdot \mathbf{j}_e^* dV = \Im \int_S \mathbf{p} \cdot \mathbf{u}_n dS + \frac{1}{2}\omega \int_V \mu_R |\mathbf{h}|^2 dV - \frac{1}{2}\omega \int_V \epsilon_R |\mathbf{e}|^2 dV.$$



Boundary conditions between media

$\mathbf{u}_n \times (\mathbf{e}_2 - \mathbf{e}_1) = 0$ ⇨ tangential electric field is always continuous

$\mathbf{u}_n \cdot (\mathbf{b}_2 - \mathbf{b}_1) = 0$ ⇨ normal magnetic induction is always continuous

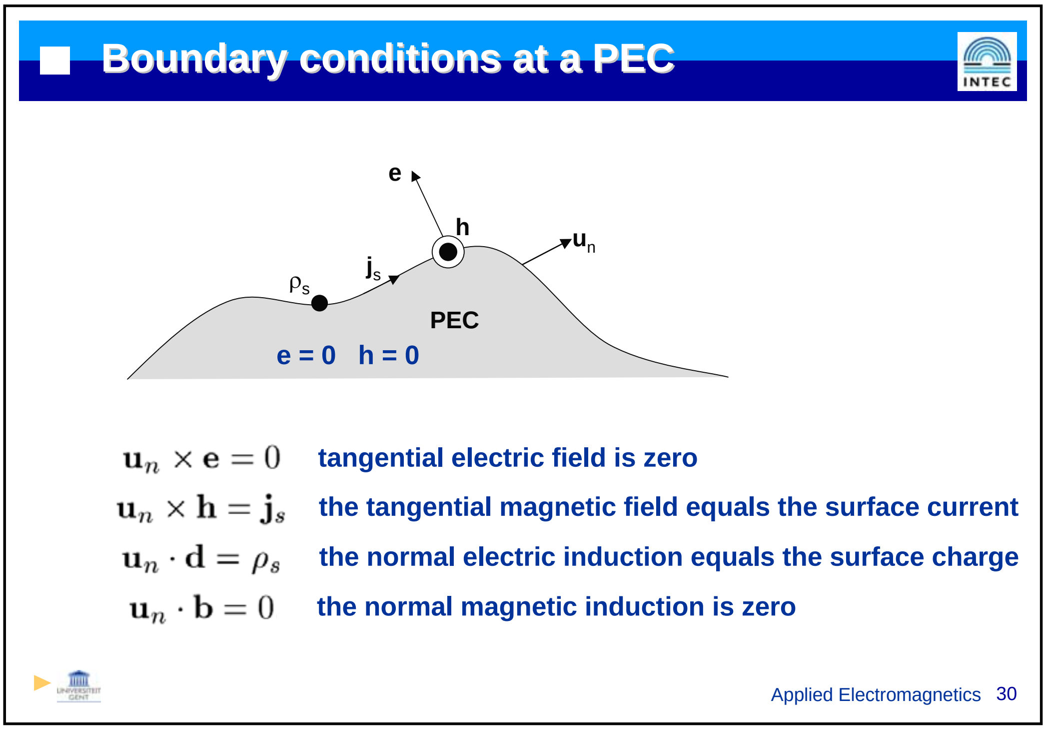
$\mathbf{u}_n \times (\mathbf{h}_2 - \mathbf{h}_1) = 0$ ⇨ tangential magnetic field is always continuous

$= \mathbf{j}_s$ ⇨ EXCEPT if medium 1 is PEC or if j_s is enforced

$\mathbf{u}_n \cdot (\mathbf{d}_2 - \mathbf{d}_1) = 0$ ⇨ normal electric induction is continuous

$= \rho_s$ ⇨ EXCEPT if medium 1 and/or 2 is conducting

$= \rho_s$ ⇨ EXCEPT if medium 1 is PEC or if ρ_s is enforced



Boundary conditions at a PEC

$\mathbf{u}_n \times \mathbf{e} = 0$ tangential electric field is zero

$\mathbf{u}_n \times \mathbf{h} = \mathbf{j}_s$ the tangential magnetic field equals the surface current

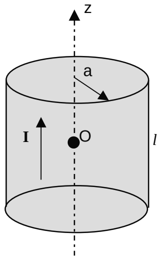
$\mathbf{u}_n \cdot \mathbf{d} = \rho_s$ the normal electric induction equals the surface charge

$\mathbf{u}_n \cdot \mathbf{b} = 0$ the normal magnetic induction is zero

Elementary dipole sources

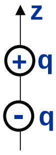
z-directed electric (Hertz) dipole

p_e : electric dipole moment



$$\mathbf{j}(\mathbf{r}) = Il\delta(\mathbf{r})\mathbf{u}_z$$
$$= j\omega\mathbf{p}_e\delta(\mathbf{r})$$

⇨

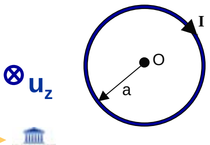


$q = I / j\omega$

$q = - I / j\omega$

$p_e = I\,l / j\omega$

elementary loop current or magnetic (Hertz) dipole



$$\mathbf{j}(\mathbf{r}) = \nabla \times \mathbf{p}_m\delta(\mathbf{r})$$
$$\mathbf{p}_m = p_m\mathbf{u}_z$$

p_m : magnetic dipole moment

$p_m = I\,\pi\, a^2\, u_z$

Potentials in a homogeneous medium

- from Electrostatics we are familiar with the potential ϕ with $e = -\nabla \phi$
- can we still use potentials to solve Maxwell?
- from calculus we know that

$\nabla \times f = 0$	$f = -\nabla \phi$	ϕ is a scalar potential
$\nabla \cdot f = 0$	$f = \nabla \times a$	a is a vector potential

$$\nabla \cdot \mathbf{b}(\mathbf{r}) = 0 \quad ⇨ \quad \boxed{\mathbf{b}(\mathbf{r}) = \nabla \times \mathbf{a}(\mathbf{r})}$$

$$\nabla \times \mathbf{e}(\mathbf{r}) = -j\omega\mathbf{b}(\mathbf{r})$$

$$⇨ \ \nabla \times [\mathbf{e}(\mathbf{r}) + j\omega\mathbf{a}(\mathbf{r})] = 0$$

⇩

$$\boxed{\mathbf{e}(\mathbf{r}) = -j\omega\mathbf{a}(\mathbf{r}) - \nabla\phi(\mathbf{r})}$$

a and ϕ ?? → use remaining Maxwell equations

Homogeneous mediun with “complex” ε and μ

Insertion in the remaing Maxwell equations gives

$$\nabla \cdot \mathbf{d}(\mathbf{r}) = \rho_e(\mathbf{r}) \quad \Rightarrow \quad \nabla^2 \phi(\mathbf{r}) + j\omega \nabla \cdot \mathbf{a}(\mathbf{r}) = -\frac{\rho_e(\mathbf{r})}{\epsilon}$$

$$\nabla \times \mathbf{h}(\mathbf{r}) = j\omega \mathbf{d}(\mathbf{r}) + \mathbf{j}(\mathbf{r})$$

$$\Rightarrow \quad \nabla^2 \mathbf{a}(\mathbf{r}) + k^2 \mathbf{a}(\mathbf{r}) = \nabla[\nabla \cdot \mathbf{a}(\mathbf{r}) + j\omega\epsilon\mu\phi(\mathbf{r})] - \mu \mathbf{j}_e(\mathbf{r})$$

Now use the Lorentz gauge (Lorentz ijk of ijkvoorwaarde)

$$\nabla \cdot \mathbf{a}(\mathbf{r}) + j\omega\epsilon\mu\phi(\mathbf{r}) = 0$$

$$\Downarrow$$

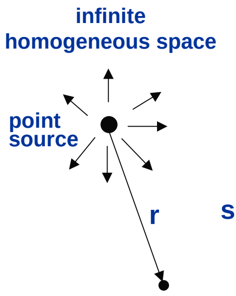
$$\boxed{\begin{aligned} \nabla^2 \mathbf{a}(\mathbf{r}) + k^2 \mathbf{a}(\mathbf{r}) &= -\mu \mathbf{j}_e(\mathbf{r}) \\ \nabla^2 \phi(\mathbf{r}) + k^2 \phi(\mathbf{r}) &= -\frac{\rho_e(\mathbf{r})}{\epsilon} \end{aligned}}$$

a and ϕ : Lorentz potentials

k : wavenumber = $\omega \sqrt{\varepsilon\mu}$

Helmholtz equation

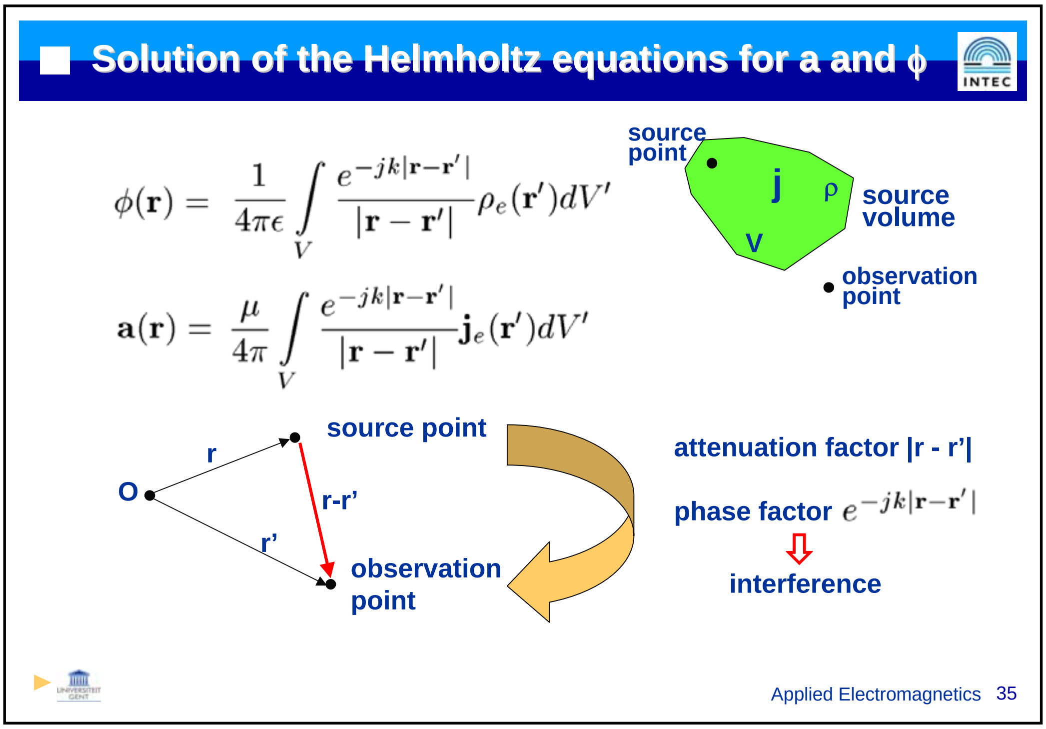
scalar Helmholtz equation : $\nabla^2 f(\mathbf{r}) + k^2 f(\mathbf{r}) = source(\mathbf{r})$



$$\nabla^2 G(\mathbf{r}) + k^2 G(\mathbf{r}) = -\delta(\mathbf{r})$$

solution or Green’s function: $G(r) = \frac{e^{-jkr}}{4\pi r}$

This Green’s function not only satisfies the differential equation
but also the radiation condition at infinity !



Solution of the Helmholtz equations for a and ϕ

$$\phi(\mathbf{r}) = \frac{1}{4\pi\epsilon}\int_V \frac{e^{-jk|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}\rho_e(\mathbf{r}')dV'$$

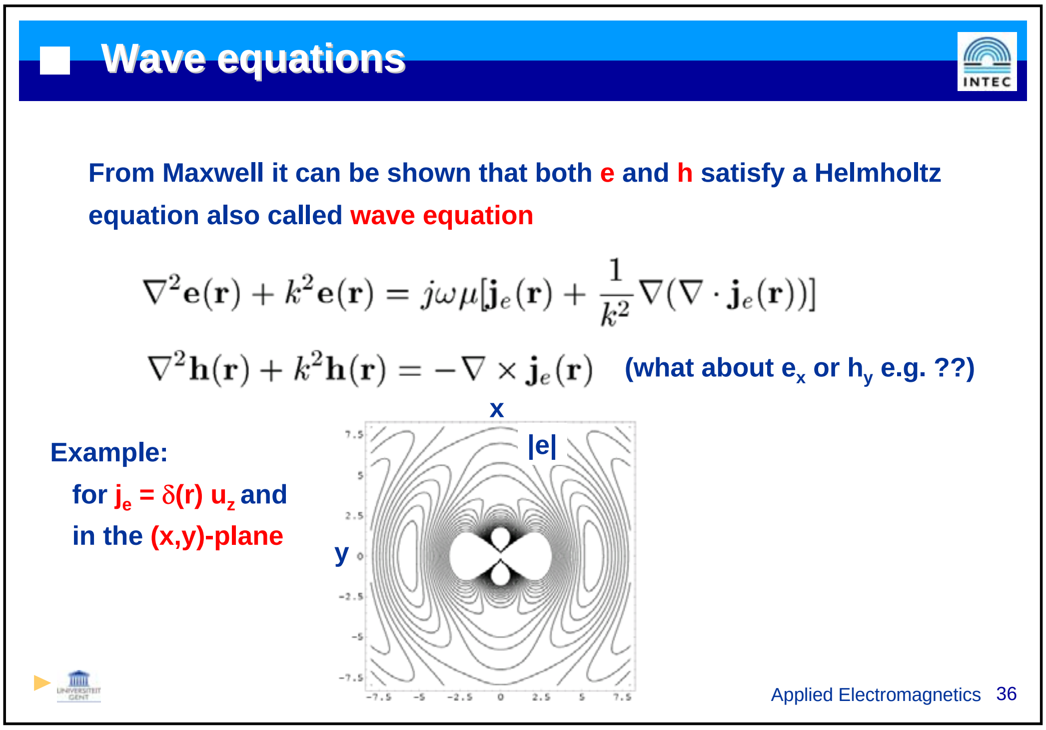
$$\mathbf{a}(\mathbf{r}) = \frac{\mu}{4\pi}\int_V \frac{e^{-jk|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}\mathbf{j}_e(\mathbf{r}')dV'$$

attenuation factor |r - r'|

phase factor $e^{-jk|\mathbf{r}-\mathbf{r}'|}$

⇩

interference



Wave equations

From Maxwell it can be shown that both e and h satisfy a Helmholtz equation also called wave equation

$$\nabla^2\mathbf{e}(\mathbf{r}) + k^2\mathbf{e}(\mathbf{r}) = j\omega\mu[\mathbf{j}_e(\mathbf{r}) + \frac{1}{k^2}\nabla(\nabla\cdot\mathbf{j}_e(\mathbf{r}))]$$

$$\nabla^2\mathbf{h}(\mathbf{r}) + k^2\mathbf{h}(\mathbf{r}) = -\nabla\times\mathbf{j}_e(\mathbf{r})$$ (what about e_x or h_y e.g. ??)

Example:

for $j_e = \delta(r)\, u_z$ and in the (x,y)-plane

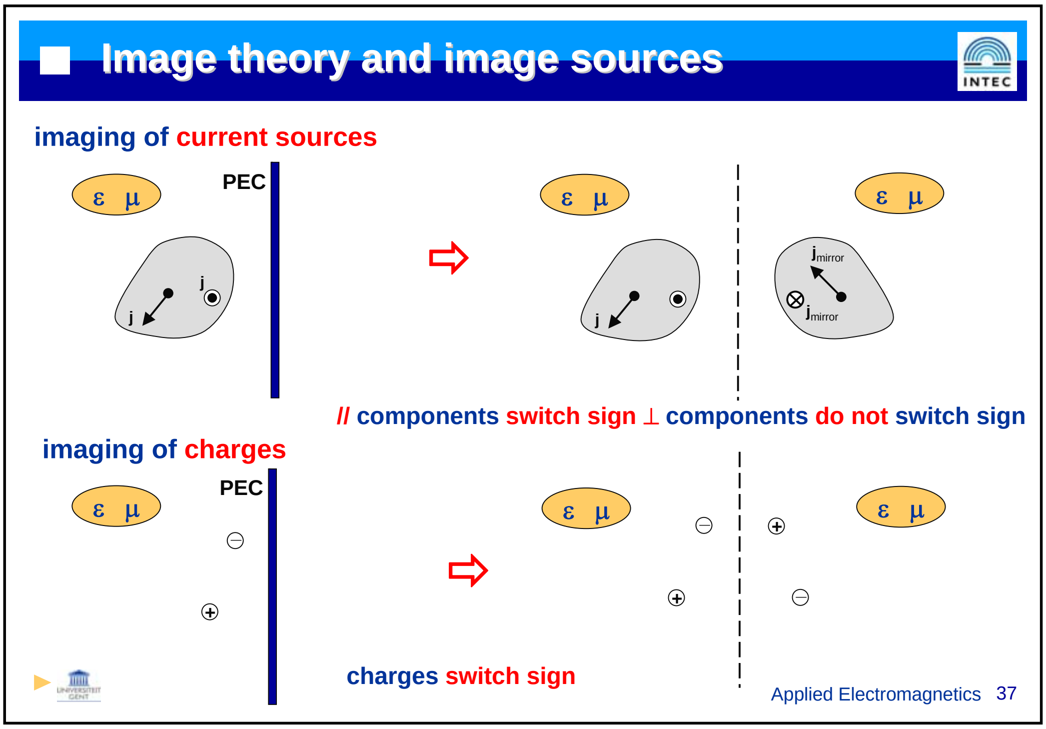


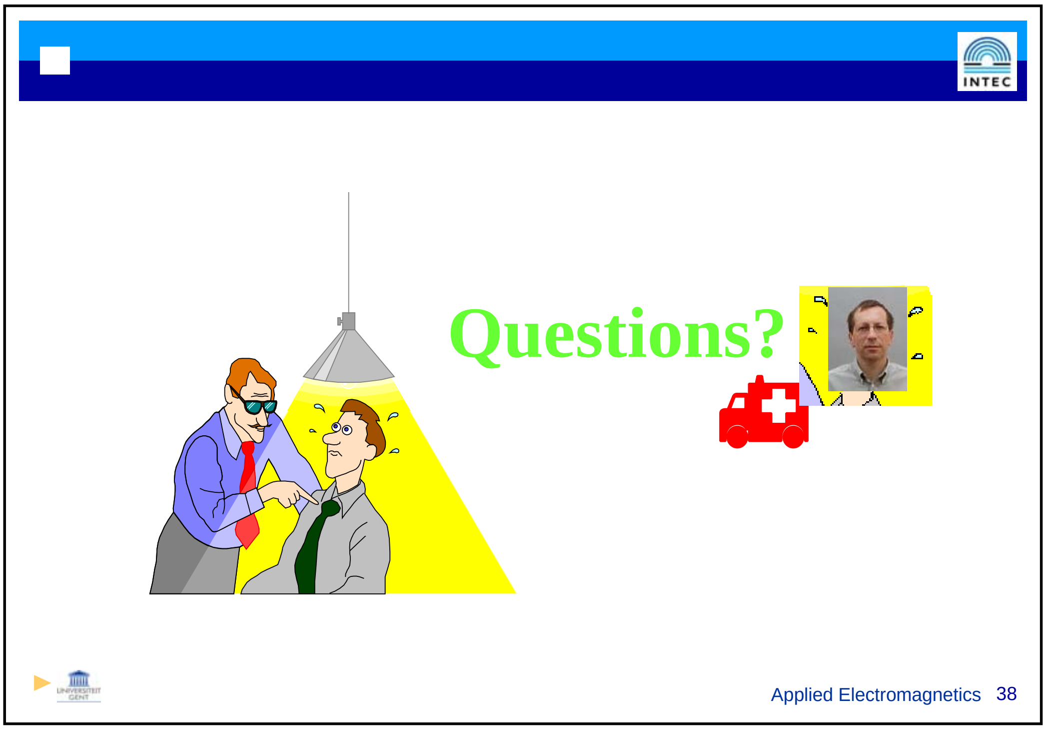
Image theory and image sources

imaging of current sources

// components switch sign ⊥ components do not switch sign

imaging of charges

charges switch sign



Questions?

Chapter 3: Electrostatics

Outline

- Maxwell's equations in the static case
- Coulomb force and electric field
- The electric potential
- Dielectric - Electric dipole - Polarisation
- Boundary conditions
- Conductors - Resistance - Joule's law
- Capacitance - Capacitance matrix
- Electrostatic energy
- Solution of Laplace's equation

Maxwell's equations in the static case

Static = independent of time ⇨ ∂ / ∂ t = 0

$$\nabla \times \mathbf{e}(\mathbf{r},\cancel{t}) = -\cancel{\frac{\partial}{\partial t}\mathbf{b}(\mathbf{r},t)}$$

$$\nabla \times \mathbf{h}(\mathbf{r},\cancel{t}) = \cancel{\frac{\partial}{\partial t}\mathbf{d}(\mathbf{r},t)} + \mathbf{j}(\mathbf{r},\cancel{t})$$

$$\nabla \cdot \mathbf{d}(\mathbf{r},\cancel{t}) = \rho(\mathbf{r},\cancel{t})$$

$$\nabla \cdot \mathbf{b}(\mathbf{r},\cancel{t}) = 0$$

⇨

Electrostatics

$$\nabla \times \mathbf{e}(\mathbf{r}) = 0,$$

$$\nabla \cdot \mathbf{d}(\mathbf{r}) = \rho(\mathbf{r})$$

Magnetostatics

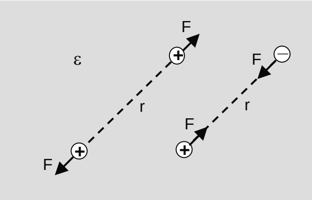
$$\nabla \times \mathbf{h}(\mathbf{r}) = \mathbf{j}(\mathbf{r}) \quad \text{(*)}$$

$$\nabla \cdot \mathbf{b}(\mathbf{r}) = 0.$$

(*) both problems can be coupled for a conduction current j = σ e

---

Coulumb force and the electric field

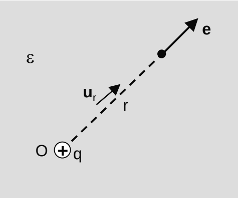


Coulomb force

$$F = \frac{1}{4\pi\epsilon}\frac{q_1 q_2}{r^2}$$

⇩

Electric field



$$\mathbf{e}(\mathbf{r}) = \mathbf{e}(r) = \frac{1}{4\pi\epsilon}\frac{q}{r^2}\mathbf{u}_r \quad \text{(V/m)}$$

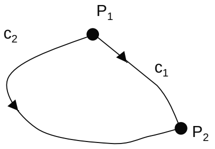
Electric potential ϕ

The static electric field is conservative or irrotational, i.e. :

$$\nabla \times \mathbf{e}(\mathbf{r}) = 0 \quad \Rightarrow \quad \mathbf{e}(\mathbf{r}) = -\nabla\phi(\mathbf{r}) \qquad \mathbf{e}(\mathbf{r}) = -j\omega\cancel{\mathbf{a}}(\mathbf{r}) - \nabla\phi(\mathbf{r})$$

Only the potential difference is meaningful and unique (why?)

$$\phi_{12} = \phi_1 - \phi_2 = -\int_{P_1}^{P_2} \nabla\phi \cdot \mathbf{dc} = \int_{P_1}^{P_2} \mathbf{e} \cdot \mathbf{dc}$$



The potential can be defined by choosing a reference point for which ϕ = 0

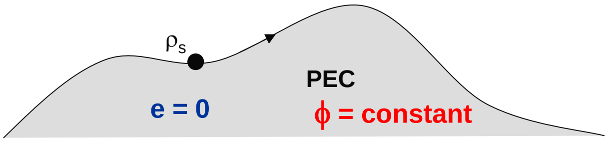
$$\Rightarrow \quad \phi(P) = -\int_{P_{ref}}^{P} \nabla\phi \cdot \mathbf{dc}$$

ref: very often at infinity

☹

Potential of a Perfect Electric Conductor

Electrostatic behaviour of a PEC



Why ?

The equations of Poisson and Laplace

The potential satisfies Poisson's equation:

$$\nabla \cdot [\epsilon(\mathbf{r})\nabla\phi(\mathbf{r})] = -\rho(\mathbf{r})$$

⇩ **homogeneous region**

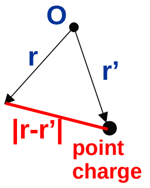
$$\nabla^2\phi(\mathbf{r}) = -\frac{\rho(\mathbf{r})}{\epsilon}$$

⇩ **charge free region**

$$\nabla^2\phi(\mathbf{r}) = 0$$ **i.e. Laplace's equation**

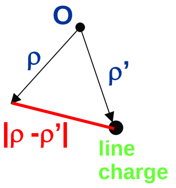
Potential of a point charge and a line charge

infinite
homogeneous 3D space



$$\phi(\mathbf{r}) = \frac{1}{4\pi\epsilon}\int_V \frac{1}{|\mathbf{r}-\mathbf{r}'|}\rho(\mathbf{r}')dV'$$

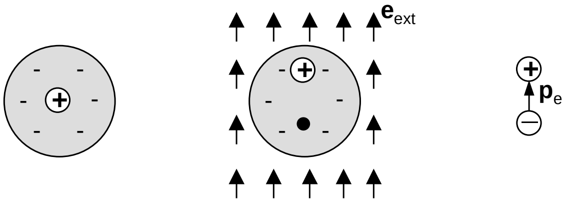
infinite
homogeneous 2D space



$$\phi(\boldsymbol{\rho}) = -\frac{\rho_l}{2\pi\epsilon}\ln\frac{|\boldsymbol{\rho}-\boldsymbol{\rho}'|}{|\boldsymbol{\rho}_{ref}-\boldsymbol{\rho}'|}$$

Dielectrics - dipole moment - polarisation

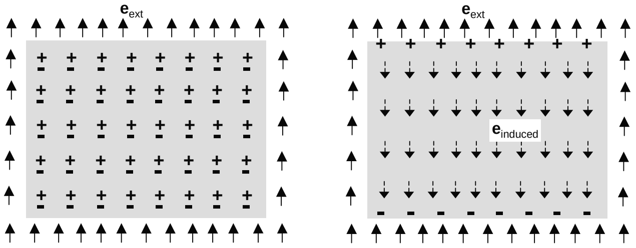
How to model a nonpolar material using a simple model of the atom?



neutral situation

effect of external field

modelling by a dipole



polarisation

Dielectric displacement vector

$$\mathbf{d} = \epsilon_0 \mathbf{e} + \mathbf{p}_e$$

dielectric displacement

electric polarisation field

electric susceptibility

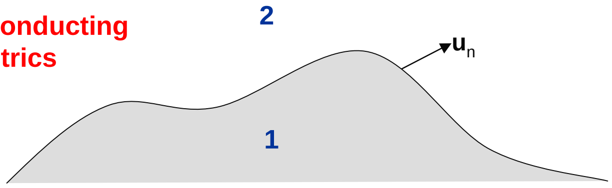
linear and isotropic material: $\mathbf{p}_e(\mathbf{r}) = \epsilon_0 \chi_e \mathbf{e}(\mathbf{r})$

$$\mathbf{d}(\mathbf{r}) = \epsilon_0 (1 + \chi_e) \mathbf{e}(\mathbf{r}) = \epsilon_0 \epsilon_r \mathbf{e}(\mathbf{r}) = \epsilon \mathbf{e}(\mathbf{r})$$

relative pemittivity

Boundary conditions for electrostatics

1,2 : non-conducting dielectrics



$\mathbf{u}_n \times (\mathbf{e}_2 - \mathbf{e}_1) = 0$ ⇨ tangential electric field is always continuous

⇨ $\phi_1 = \phi_2$

$\mathbf{u}_n \cdot (\mathbf{d}_2 - \mathbf{d}_1) = 0$ ⇨ normal electric induction is continuous

$$\Rightarrow \quad \epsilon_1 \left(\frac{\partial \phi}{\partial n}\right)_1 = \epsilon_2 \left(\frac{\partial \phi}{\partial n}\right)_2$$

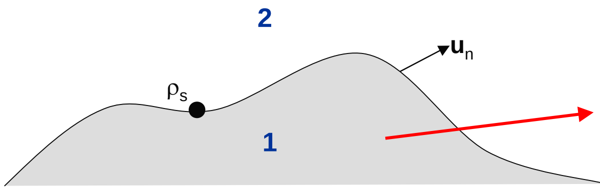
Conductors

Constitutive law : conductivity σ

$$\begin{aligned} \mathbf{j} &= \sigma \mathbf{e} \\ &= e N_e \mu_e \mathbf{e} \quad \text{for a conductor,} \\ &= e(N_e \mu_e + N_h \mu_h)\mathbf{e} \quad \text{for a semiconductor} \end{aligned}$$

|electron charge|

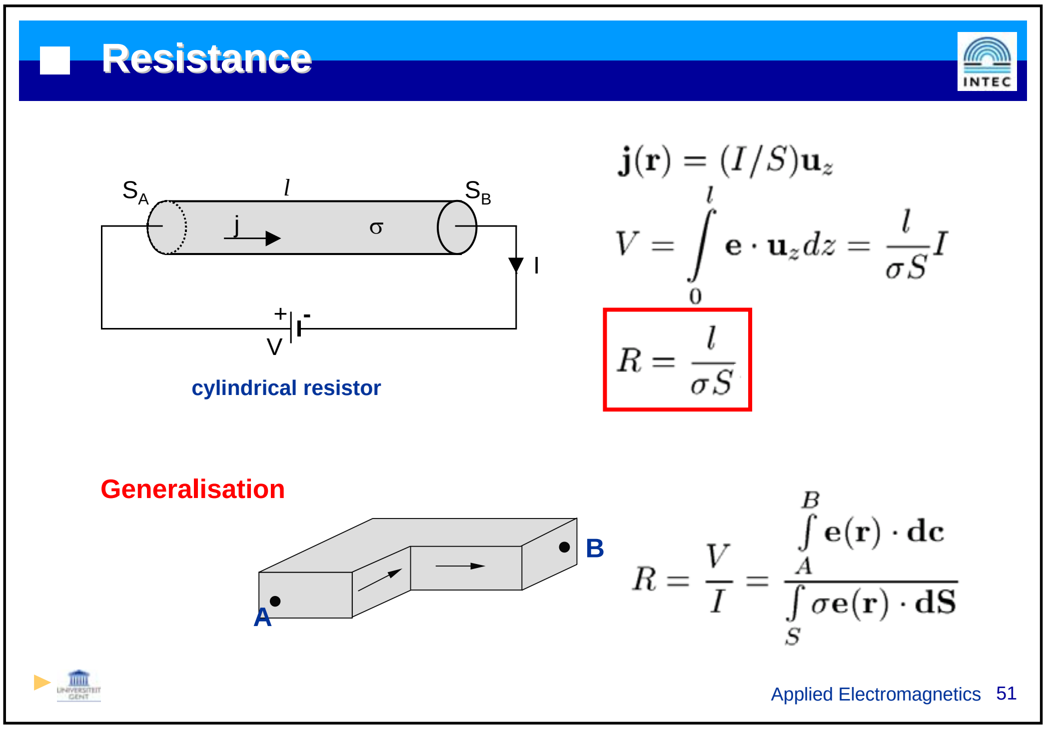
N: number per m^3
μ : mobility in m^2 / Vs
subscript e: electrons
h: holes



all free volume charges rapidly diffuse to the surface !!

relaxation time for copper e.g. is about 1.5 10^{-9} s

good conductor σ >> ωε



Resistance

cylindrical resistor

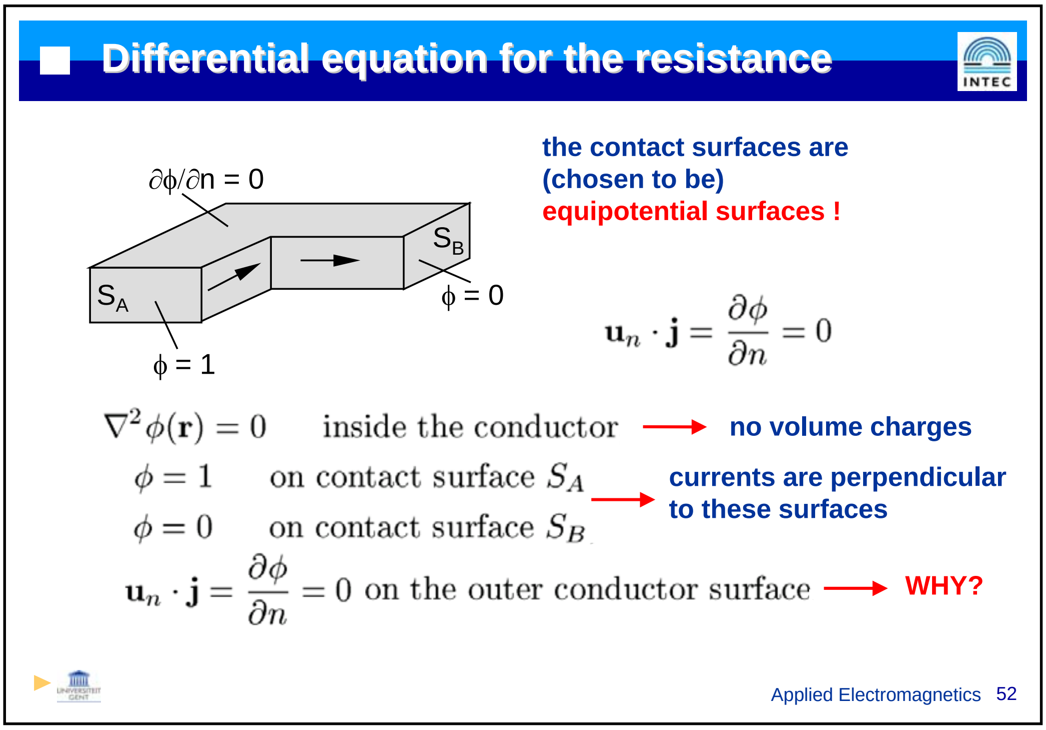
$$\mathbf{j}(\mathbf{r}) = (I/S)\mathbf{u}_z$$

$$V = \int_0^l \mathbf{e}\cdot\mathbf{u}_z dz = \frac{l}{\sigma S} I$$

$$\boxed{R = \frac{l}{\sigma S}}$$

Generalisation

$$R = \frac{V}{I} = \frac{\int_A^B \mathbf{e}(\mathbf{r})\cdot d\mathbf{c}}{\int_S \sigma\mathbf{e}(\mathbf{r})\cdot d\mathbf{S}}$$



Differential equation for the resistance

the contact surfaces are (chosen to be) equipotential surfaces !

$$\mathbf{u}_n\cdot\mathbf{j} = \frac{\partial\phi}{\partial n} = 0$$

$\nabla^2\phi(\mathbf{r}) = 0$ inside the conductor → **no volume charges**

$\phi = 1$ on contact surface S_A

$\phi = 0$ on contact surface S_B

→ **currents are perpendicular to these surfaces**

$\mathbf{u}_n\cdot\mathbf{j} = \frac{\partial\phi}{\partial n} = 0$ on the outer conductor surface → **WHY?**

Dissipated power

Joule losses

$$P_{dis} = \int_V \sigma |\mathbf{e}(\mathbf{r})|^2 dV = \int_V \sigma(\nabla\phi \cdot \nabla\phi) dV$$

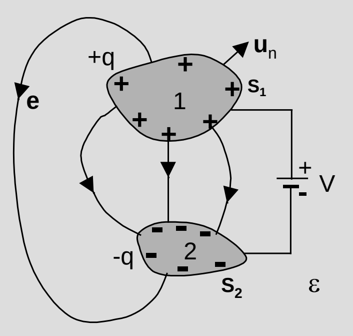
⇩

$$\boxed{P_{dis} = RI^2}$$

Proof ? Use one of Green's theorems

$$\int_V (f\nabla^2 g + \nabla f \cdot \nabla g) dV = \int_S f \frac{\partial g}{\partial n} dS$$

Capacitance



static (ω = 0): $\mathbf{e}(\mathbf{r}) = -\nabla\phi(\mathbf{r})$

potential difference:

$$\phi_{12} = \phi_1 - \phi_2 = -\int_{P_1}^{P_2} \nabla\phi \cdot \mathbf{dc} = \int_{P_1}^{P_2} \mathbf{e} \cdot \mathbf{dc}.$$

path independent!

The potential ϕ satisfies

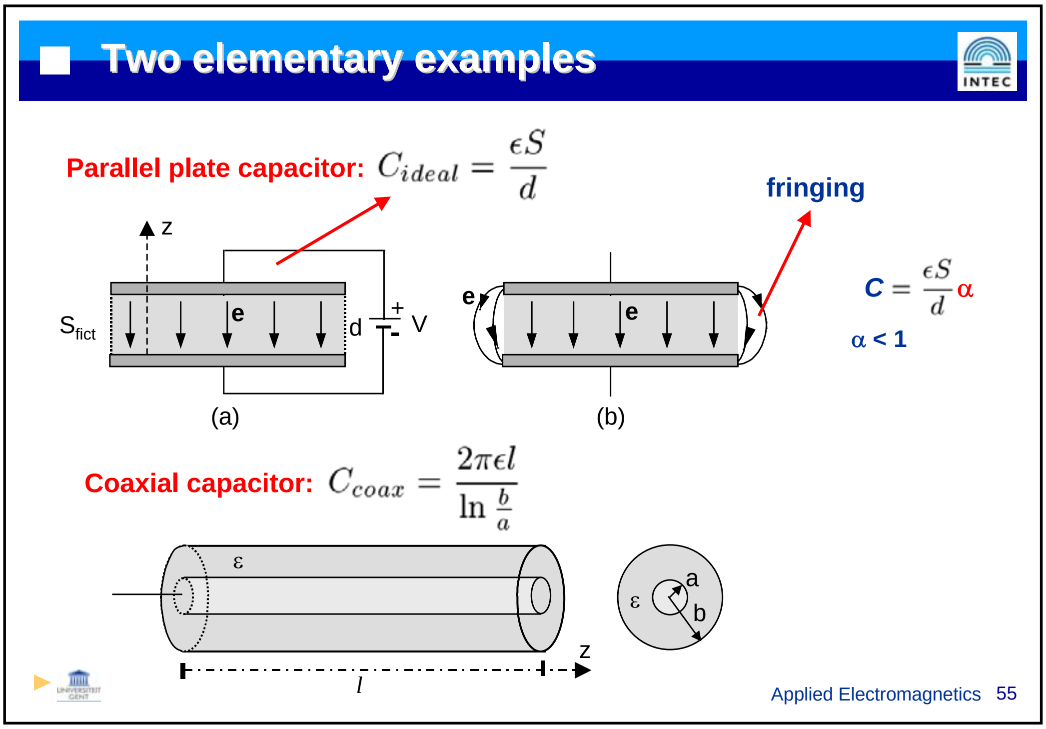
$\nabla^2\phi(\mathbf{r}) = 0$ **outside the conductors**

$\phi = V$ **on S_1**

$\phi = 0$ **on S_2**

q = C V **with**

$$q = \int_{S_1} \rho_{s1}(\mathbf{r}) dS = \int_{S_1} \epsilon \mathbf{u}_n \cdot \mathbf{e}(\mathbf{r}) dS = -\int_{S_1} \epsilon \frac{\partial \phi}{\partial n} dS$$



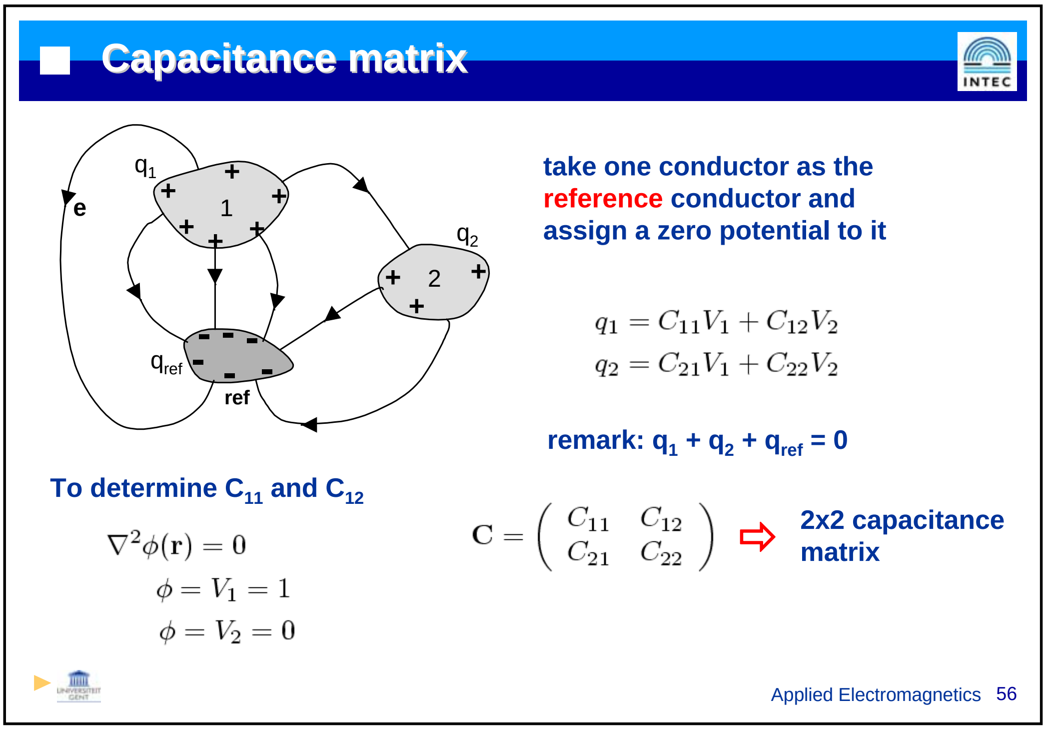
Two elementary examples

Parallel plate capacitor: $C_{ideal} = \frac{\epsilon S}{d}$

$C = \frac{\epsilon S}{d}\alpha$

$\alpha < 1$

Coaxial capacitor: $C_{coax} = \frac{2\pi\epsilon l}{\ln\frac{b}{a}}$



Capacitance matrix

take one conductor as the reference conductor and assign a zero potential to it

$$q_1 = C_{11}V_1 + C_{12}V_2$$
$$q_2 = C_{21}V_1 + C_{22}V_2$$

remark: $q_1 + q_2 + q_{ref} = 0$

To determine C_{11} and C_{12}

$$\nabla^2\phi(\mathbf{r}) = 0$$
$$\phi = V_1 = 1$$
$$\phi = V_2 = 0$$

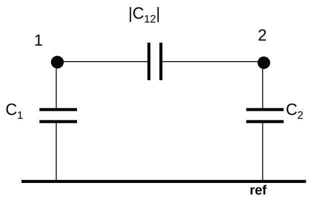
$$\mathbf{C} = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}$$ ⇨ **2x2 capacitance matrix**

Capacitance matrix - cont.

M + 1 conductor case

$$\mathcal{Q} = \mathbf{C}\,\mathcal{V}$$

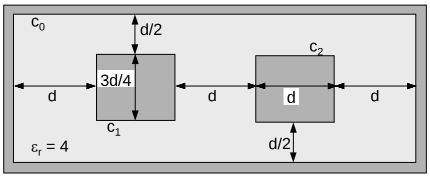
Q : M x 1 column vector of charges
C : M x M capacitance matrix
V : M x 1 column vector of potentials w.r.t. the reference conductor

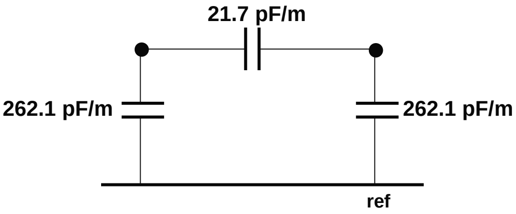


$$C_i = C_{ii} - \sum_{j(j \neq i)} |C_{ij}|$$

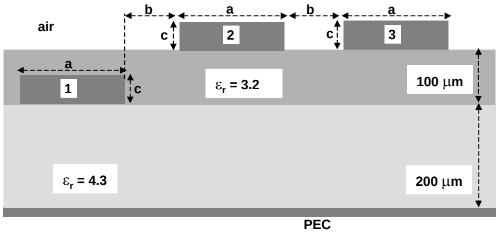
- the capacitance matrix is symmetric, $C_{ij} = C_{ji}$;
- all the non-diagonal elements are strictly negative, $C_{ij} < 0$;
- all the diagonal elements are strictly positive, $C_{ii} > 0$;
- the matrix is strictly diagonally dominant, i.e. $C_{ii} > \sum_{j(j \neq i)} |C_{ij}|$.

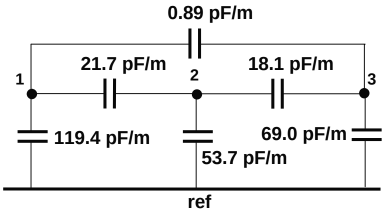
A set of signal lines: examples





$$\mathbf{C} = \begin{pmatrix} 283.8 & -21.7 \\ -21.7 & 283.8 \end{pmatrix} pF/m$$

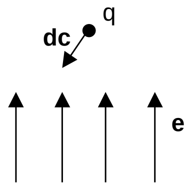




a = 350µm; b = 150µm; c = 70µm

Electrostatic energy

Energy needed to move a charge against an electric field



$$du_e = \mathbf{F}_{ext} \cdot \mathbf{dc} = -q\mathbf{e} \cdot \mathbf{dc}$$

To charge a capacitance we have to transfer charge from one conductor to the other conductor. The total energy needed for this is

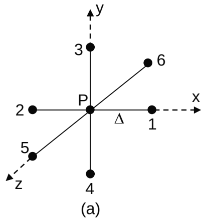
$$u_e = \frac{1}{2}CV^2 \overset{\text{proof !}}{=} \int_V \frac{1}{2}\mathbf{e} \cdot \mathbf{d}\, dV$$ ⇨ **see Poynting's theorem**

Extension to the multiconductor case: $u_e = \frac{1}{2}\mathcal{V}^T \mathbf{C}\, \mathcal{V}$

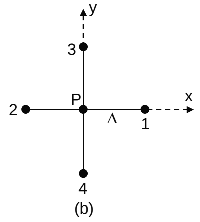
Solution of Laplace's equation

- **allows to determine the resitance or the capacitance (matrix)**
- **important topic in Computational Electromagnetics**
- **research topic in the Electromagnetics Group**
- **most important techniques: Finite Differerences, Finite Elements, Integral equations (Method of Moments)**

Elementary Finite Difference Technique



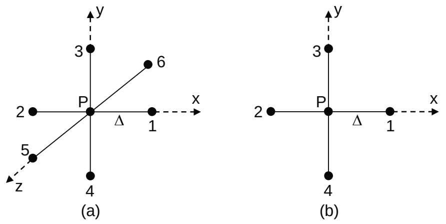
(a)



(b)

$$\phi_1 = \phi_P + \left(\frac{\partial}{\partial x}\phi\right)_P \Delta + \left(\frac{\partial^2}{\partial x^2}\phi\right)_P \frac{\Delta^2}{2} + \cdots$$

Finite difference technique



$$\left(\nabla^2\phi\right)_P \approx \frac{1}{\Delta^2}(\phi_1+\phi_2+\phi_3+\phi_4+\phi_5+\phi_6-6\,\phi_P)$$

⇩

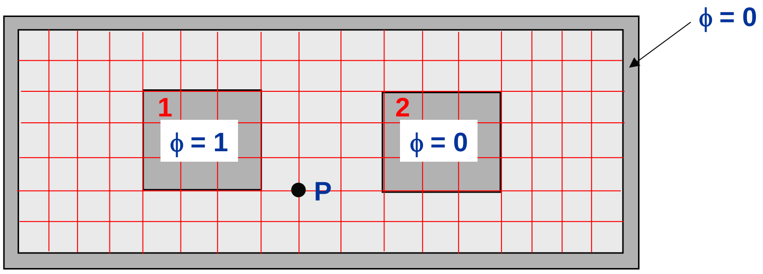
Laplace's equation is satisfied provided

$$\phi_P = \frac{\phi_1+\phi_2+\phi_3+\phi_4+\phi_5+\phi_6}{6}$$ **3D**

$$\phi_P = \frac{\phi_1+\phi_2+\phi_3+\phi_4}{4}$$ **2D**

Finite difference technique

Iterative solution



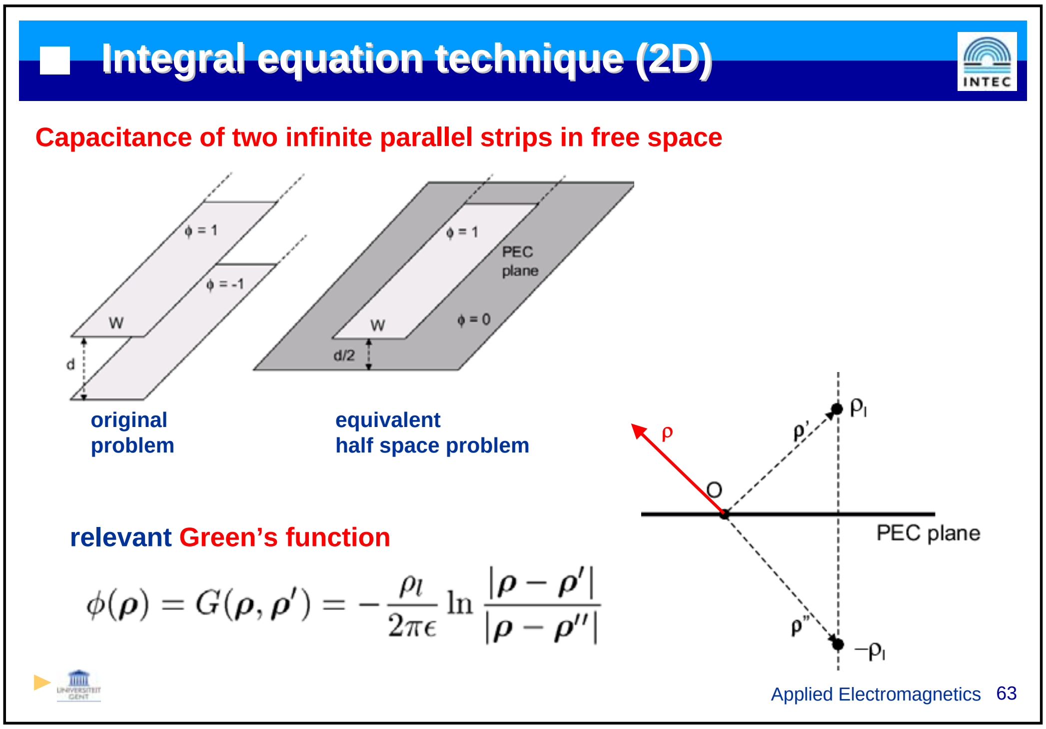
(a) ϕ is zero everywhere except on conductor 1

(b) apply $\phi_P = \frac{\phi_1+\phi_2+\phi_3+\phi_4}{4}$ to each point P

(c) repeat (b) until convergence is obtained (note: convergence is poor)

(d) calculate total charge q_1 on conductor 1 ⇨ $q_1 = C_{11}$

(e) calculate total charge q_2 on conductor 2 ⇨ $q_2 = C_{12}$



Integral equation technique (2D)

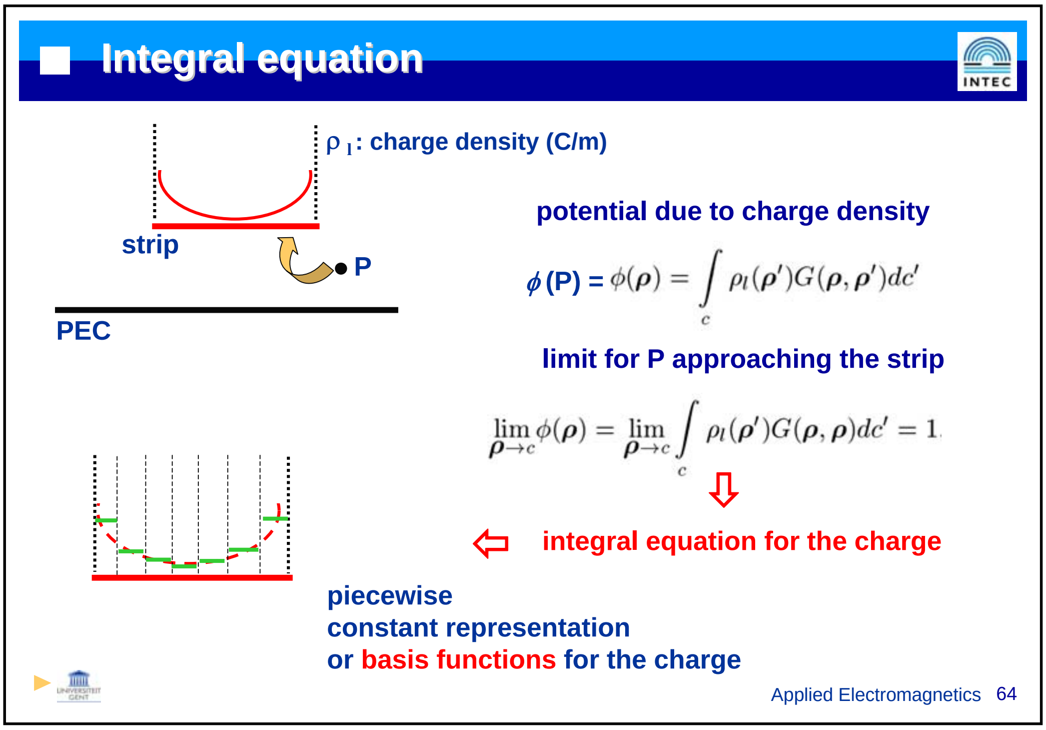
Capacitance of two infinite parallel strips in free space

original problem

equivalent half space problem

relevant Green's function

$$\phi(\boldsymbol{\rho}) = G(\boldsymbol{\rho},\boldsymbol{\rho}') = -\frac{\rho_l}{2\pi\epsilon}\ln\frac{|\boldsymbol{\rho}-\boldsymbol{\rho}'|}{|\boldsymbol{\rho}-\boldsymbol{\rho}''|}$$



Integral equation

ρ_l: charge density (C/m)

potential due to charge density

$$\phi\,(P) = \phi(\boldsymbol{\rho}) = \int_c \rho_l(\boldsymbol{\rho}')G(\boldsymbol{\rho},\boldsymbol{\rho}')dc'$$

limit for P approaching the strip

$$\lim_{\boldsymbol{\rho}\to c}\phi(\boldsymbol{\rho}) = \lim_{\boldsymbol{\rho}\to c}\int_c \rho_l(\boldsymbol{\rho}')G(\boldsymbol{\rho},\boldsymbol{\rho})dc' = 1$$

⇩

⇦ **integral equation for the charge**

piecewise
constant representation
or basis functions for the charge

Integral equation - cont.

$$\lim_{\boldsymbol{\rho}\to c}\phi(\boldsymbol{\rho}) = \lim_{\boldsymbol{\rho}\to c}\int_c \rho_l(\boldsymbol{\rho}')G(\boldsymbol{\rho},\boldsymbol{\rho})dc' = 1$$

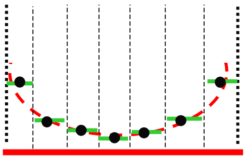
⇩ discretisation (basis functions)

$$\lim_{\boldsymbol{\rho}\to c}\sum_{i=1}^{N}\rho_{li}\int_{c_i} G(\boldsymbol{\rho},\boldsymbol{\rho}')dc_i' = 1$$

⇩ enforcing this in the midpoints (point matching)

$$\lim_{\boldsymbol{\rho}\to\boldsymbol{\rho}_j}\sum_{i=1}^{N}\rho_{li}\int_{c_i} G(\boldsymbol{\rho},\boldsymbol{\rho}')dS_i' = 1 \quad j = 1,2,...,N$$

N linear equations in N unknowns



Matrix representation - numerical results

$$\mathcal{I}_{\mathcal{N}} = \mathbf{C}^{-1}\,\mathcal{Q}$$

→ N x 1 column matrix with charge densities

N x 1 unity column matrix

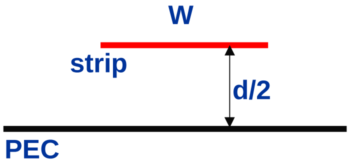
N x N system matrix

$$\mathbf{C}_{\mathbf{ji}}^{-1} = \lim_{\boldsymbol{\rho}\to\boldsymbol{\rho}_j}\int_{c_i} G(\boldsymbol{\rho}_j,\boldsymbol{\rho}')dS_i'$$

i ≠ j

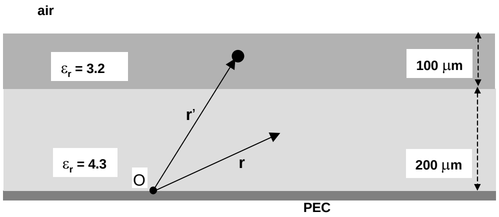
$$\mathbf{C}_{\mathbf{jj}}^{-1} \approx -\frac{1}{2\pi\epsilon}\tau + \frac{\Delta}{2\pi\epsilon}\ln|\boldsymbol{\rho}_j - \boldsymbol{\rho}_{j,mir}|$$

$$\tau = 2\Delta(\ln\frac{\Delta}{2} - 1)$$

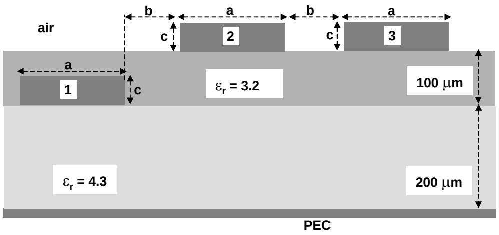


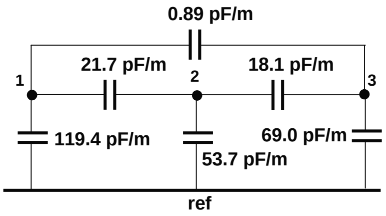
d/W	C (pF/m)	C_{ideal} (pF/m)
10	7.49 pF/m	0.89 pF/m
5	9.20 pF/m	1.77 pF/m
1	18.47 pF/m	8.85 pF/m
0.5	28.34 pF/m	17.71 pF/m
0.1	98.06 pF/m	88.54pF/m
0.05	176.65pF/m	177.08 pF/m

Signal lines embedded in a dielectric

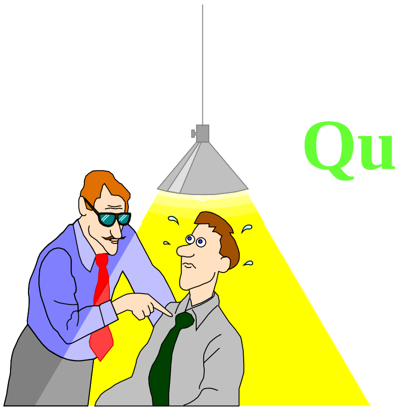


Green's function problem





a = 350µm; b = 150µm; c = 70µm



Questions?

Chapter 4: Magnetostatics

Outline

- Introduction
- Lorentz force and the magnetic induction
- The magnetic field and Biot-Savart’s law
- The vector potential
- Magnetic dipole - Magnetisation
- Boundary conditions
- Inductance - Inductance matrix
- Magnetostatic energy
- Two-dimensional signal lines

Maxwell's equations in the static case

Static = independent of time ⇨ ∂ / ∂ t = 0

$$\nabla \times \mathbf{e}(\mathbf{r},\cancel{t}) = -\cancel{\frac{\partial}{\partial t}\mathbf{b}(\mathbf{r},t)}$$

$$\nabla \times \mathbf{h}(\mathbf{r},\cancel{t}) = \cancel{\frac{\partial}{\partial t}\mathbf{d}(\mathbf{r},t)} + \mathbf{j}(\mathbf{r},\cancel{t})$$

$$\nabla \cdot \mathbf{d}(\mathbf{r},\cancel{t}) = \rho(\mathbf{r},\cancel{t})$$

$$\nabla \cdot \mathbf{b}(\mathbf{r},\cancel{t}) = 0$$

⇨

Electrostatics

$$\nabla \times \mathbf{e}(\mathbf{r}) = 0,$$

$$\nabla \cdot \mathbf{d}(\mathbf{r}) = \rho(\mathbf{r})$$

Magnetostatics

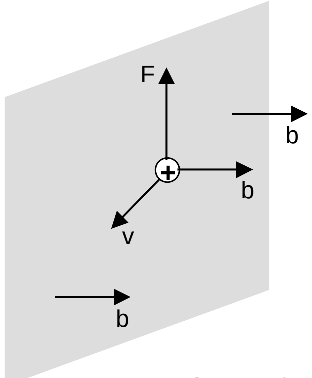
$$\nabla \times \mathbf{h}(\mathbf{r}) = \mathbf{j}(\mathbf{r}) \quad (*)$$

$$\nabla \cdot \mathbf{b}(\mathbf{r}) = 0.$$

(*) both problems become coupled for a conduction current j = σ e

---

Lorentz force and the magnetic induction



Lorentz force

$$\mathbf{F} = q(\mathbf{v} \times \mathbf{b})$$

$$\mathbf{F} = q(\mathbf{e} + \mathbf{v} \times \mathbf{b})$$

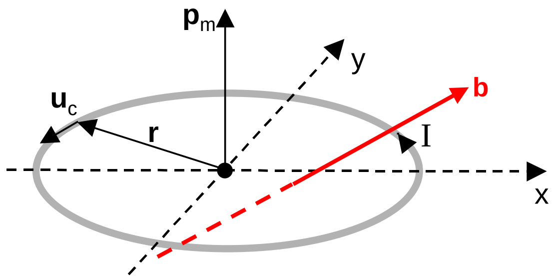
⇩

can be used to define the magnetic induction b (Wb/m²)

note: take a reference frame in which q does NOT move

⇨ $\mathbf{F}' = q\mathbf{e}'$ ⇨ $\mathbf{F} = q(\mathbf{e} + \mathbf{v} \times \mathbf{b})$ using the principles put forward in Einstein's "Zur Elektrodynamic Bewegten Körper"

Torque



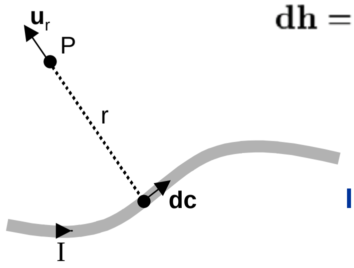
total torque: $\mathbf{t} = I \int_c \mathbf{r} \times (\mathbf{dc} \times \mathbf{b})$

magnetic dipole moment of the loop
⇧

for a constant b-field this becomes: $\mathbf{t} = SI\mathbf{u}_z \times \mathbf{b} = \mathbf{p}_m \times \mathbf{b}$

Biot-Savart's (Oersted) law

Magnetic field due to an elementary current filament



$$\mathbf{dh} = \frac{1}{4\pi}\frac{I}{r^2}\mathbf{dc} \times \mathbf{u}_r$$

Generalisation

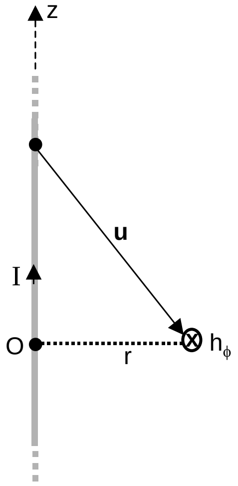
line current I: $\mathbf{h}(\mathbf{r}) = \frac{I}{4\pi}\int_c \frac{\mathbf{dc}' \times \mathbf{u}}{|\mathbf{r}-\mathbf{r}'|^2}\, dc'$

surface current j_s: $\mathbf{h}(\mathbf{r}) = \frac{1}{4\pi}\int_S \frac{\mathbf{j}_s(\mathbf{r}') \times \mathbf{u}}{|\mathbf{r}-\mathbf{r}'|^2}\, dS'$

volume current j: $\mathbf{h}(\mathbf{r}) = \frac{1}{4\pi}\int_V \frac{\mathbf{j}(\mathbf{r}') \times \mathbf{u}}{|\mathbf{r}-\mathbf{r}'|^2}\, dV'$

Magnetic field of a linear conductor

Proof the following formula's for a linear conductor with radius a and carrying a total current I



$$\mathbf{h}(r) = \frac{I}{2\pi r}\mathbf{u}_\phi \qquad r > a$$
$$= \frac{Ir}{2\pi a^2}\mathbf{u}_\phi \qquad r < a$$

Use either Biot-Savart or Ampère's law

Vector potential

$$\nabla \cdot \mathbf{b}(\mathbf{r}) = 0 \quad \Rightarrow \quad \mathbf{b}(\mathbf{r}) = \nabla \times \mathbf{a}(\mathbf{r})$$

In the static case the vector potential a satisfies the vectorial Poisson equation

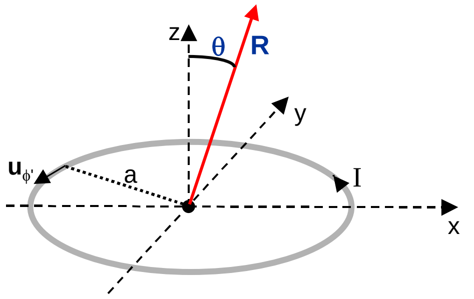
$$\nabla^2 \mathbf{a}(\mathbf{r}) = -\mu \mathbf{j}(\mathbf{r})$$
$$\Downarrow$$
$$\mathbf{a}(\mathbf{r}) = \frac{\mu}{4\pi}\int\limits_V \frac{\mathbf{j}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}dV'$$
$$\Downarrow$$

Biot-Savart

$$\Uparrow$$

$$\mathbf{h}(\mathbf{r}) = \nabla \times \frac{1}{4\pi}\int\limits_V \frac{\mathbf{j}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}dV' \overset{?}{\Rightarrow} \mathbf{h}(\mathbf{r}) = \frac{1}{4\pi}\int\limits_V \nabla\frac{1}{|\mathbf{r}-\mathbf{r}'|} \times \mathbf{j}(\mathbf{r}')dV'$$

Magnetic dipole



circular current-carying loop

$$\mathbf{j}(\mathbf{r}) = I\delta(r-a)\delta(z)\mathbf{u}_\phi$$

⇩

$$\mathbf{a}(r,\phi,z) =$$

$$\frac{\mu}{4\pi} I \int_V \frac{\delta(r'-a)\delta(z')\mathbf{u}_{\phi'}}{|\mathbf{r}-\mathbf{r}'|} r'dr'd\phi'dz'$$

⇩ a << R !

$$\mathbf{a}(r,\phi,z) = \frac{\mu}{4\pi R^2}(\mathbf{p}_m \times \mathbf{u}_R)$$

⇩

$$\mathbf{h}(\mathbf{r}) = \frac{p_m}{4\pi R^2}(2\cos\theta\mathbf{u}_R + \sin\theta\mathbf{u}_\theta)$$

DEFINE the magnetic dipole moment

$$\boxed{\mathbf{p}_m = \pi a^2 I \mathbf{u}_z}$$

Magnetisation

Magnetisation of a material results from the magnetic moments caused by:

1. current loops generated by the motion of the electrons
2. intrinsic magnetic moment of spinning electrons
3. current loops generated by the motion of protons in the nucleus (negligible)

⇩

$$\mathbf{b} = \mu_0(\mathbf{h} + \mathbf{m})$$

magnetic induction

magnetisation vector

Magnetisation - cont.

DIAMAGNETISM (copper, silver, gold, lead, …)

- distortion of electron orbits due to incident magnetic field

⇨ $\mathbf{m} = \chi_m \mathbf{h}$ $\quad$ $\mathbf{b} = \mu_0(\mathbf{h}+\mathbf{m}) = \mu_0(1+\chi_m)\mathbf{h} = \mu\mathbf{h}$

$$\mu = \mu_0(1+\chi_m) = \mu_0\mu_r$$

magnetic susceptibility
of the order of -10^{-5}

permeability

relative permeability

PARAMAGNETISM (aluminium, magnesium, chromium, …)

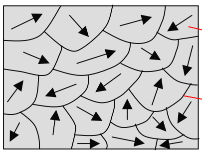
- materials already exhibit a permanent but small magnetisation

⇨ $\mathbf{m} = \chi_m \mathbf{h}$

magnetic susceptibility
of the order of $+10^{-5}$

Ferromagnetism and hysteresis

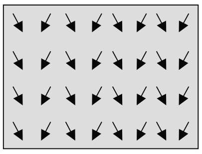
FERROMAGNETISM (iron, nickel, cobalt, …)



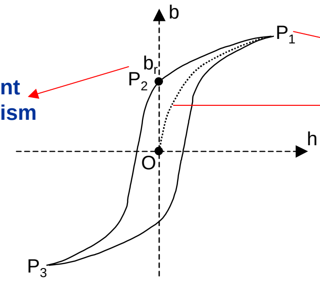
about 10^{-10} m^3

domain wall

unmagnetised domains



magnetised domains



saturation point

remanent
magnetism

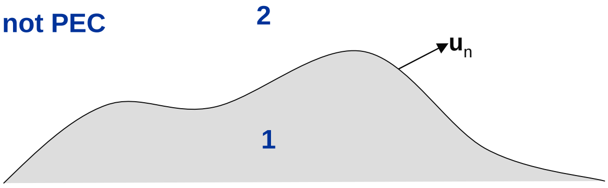
magnetisation curve

hysteresis curve

ferromagnetic material:
- non - linear
- large μ_r values (> 4000 for iron)
- very large μ_r values for mumetals (> 10^5)
- effect disappears above Curie temperature

Boundary conditions for magnetostatics

with 1 and 2 not PEC

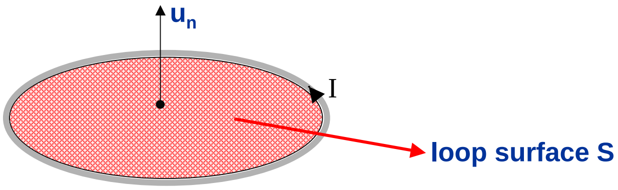


$\mathbf{u}_n \times (\mathbf{h}_2 - \mathbf{h}_1) = 0$ ⇨ tangential magnetic field is always continuous

$\mathbf{u}_n \cdot (\mathbf{b}_2 - \mathbf{b}_1) = 0$ ⇨ normal magnetic induction is continuous

Identical to the dynamic case !

Self-inductance



current carrying loop

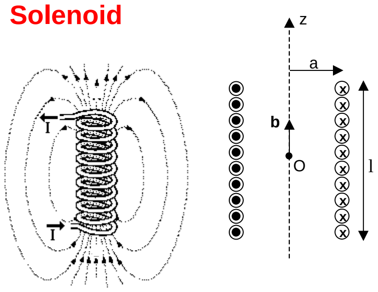
The current creates a magnetic field and with this field a flux ψ can be associated

$$\psi = \int_S \mathbf{b} \cdot \mathbf{u}_n \, dS = LI$$

self - inductance (Henry)

Self-inductance: examples

Solenoid

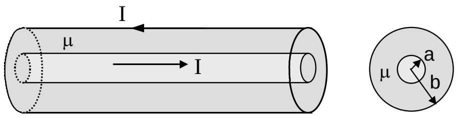


$$L_{ideal} = \frac{\mu\pi a^2 N^2}{l}$$

(a << *l*)

N: number of turns

Coaxial inductor

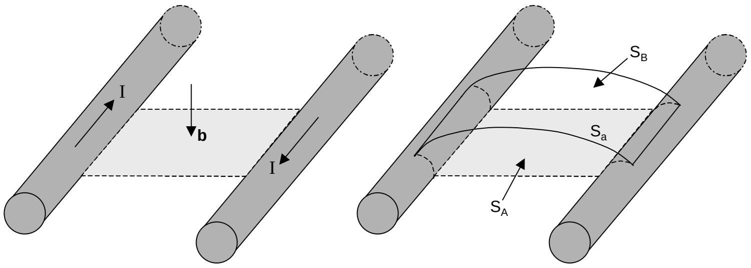


$$L_{coax} = \frac{\mu}{2\pi}\ln\frac{b}{a}$$

(H/m)

Self-inductance: examples - cont.

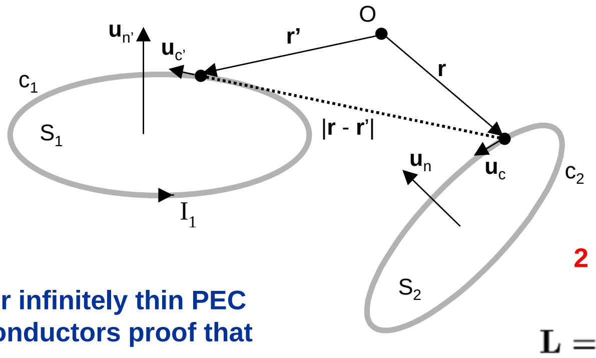
Pair of parallel PEC wires



The flux is independent of the surface !
Why ?

$$L_{wirepair} = \frac{\mu}{\pi}\ln\frac{d-a}{a}$$

Mutual inductance - Inductance matrix



$$\psi_1 = L_{11}I_1 + L_{12}I_2$$
$$\psi_2 = L_{21}I_1 + L_{22}I_2$$

2 x 2 inductance matrix

$$\mathbf{L} = \begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix}$$

with $L_{12} = L_{21}$

for infinitely thin PEC conductors proof that

$$L_{21} = \frac{\mu}{4\pi}\int_{c_1}\int_{c_2}\frac{\mathbf{dc}'\cdot\mathbf{dc}}{|\mathbf{r}-\mathbf{r}'|}$$

Inductance matrix - cont.

M + 1 conductor case

$$\mathcal{F} = \mathbf{L}\,\mathcal{I}$$

F **: M x 1 column vector of fluxes**
L : M x M capacitance matrix
I **: M x 1 column vector of currents**

In the two-dimensional (!!!!) case and only in that case it can be proven that

- the inductance matrix is symmetric, $L_{ij} = L_{ji}$;
- all the non-diagonal elements are strictly positive, $L_{ij} > 0$;
- all the diagonal elements are strictly positive, $L_{ii} > 0$;
- the matrix $\mathbf{L}$ is positive definite and non-singular.

Magnetostatic energy

Energy needed to build up the flux through a loop

elementary energy u_m when increasing the current

$$du_m = v di = L di \frac{di}{dt} = \frac{L}{2}\frac{di^2}{dt} \qquad \text{and} \qquad v(t) = L\frac{di(t)}{dt}$$

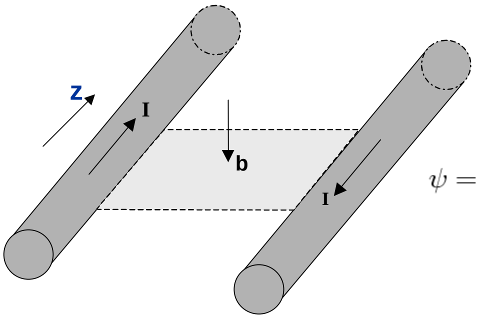
Hence

$$u_m = \frac{1}{2}LI^2 \overset{\text{proof !}}{=} \int_V \frac{1}{2}\mathbf{h}\cdot\mathbf{b}\,dV \;\Rightarrow\; \text{see Poynting's theorem}$$

Extension to the multiconductor case:

$$u_m = \frac{1}{2}\mathcal{I}^T\mathbf{L}\,\mathcal{I}$$

Two-dimensional situation



For the scalar potential ψ we can prove that

$$\nabla_t^2\psi = 0, \quad \text{outside conductors}$$

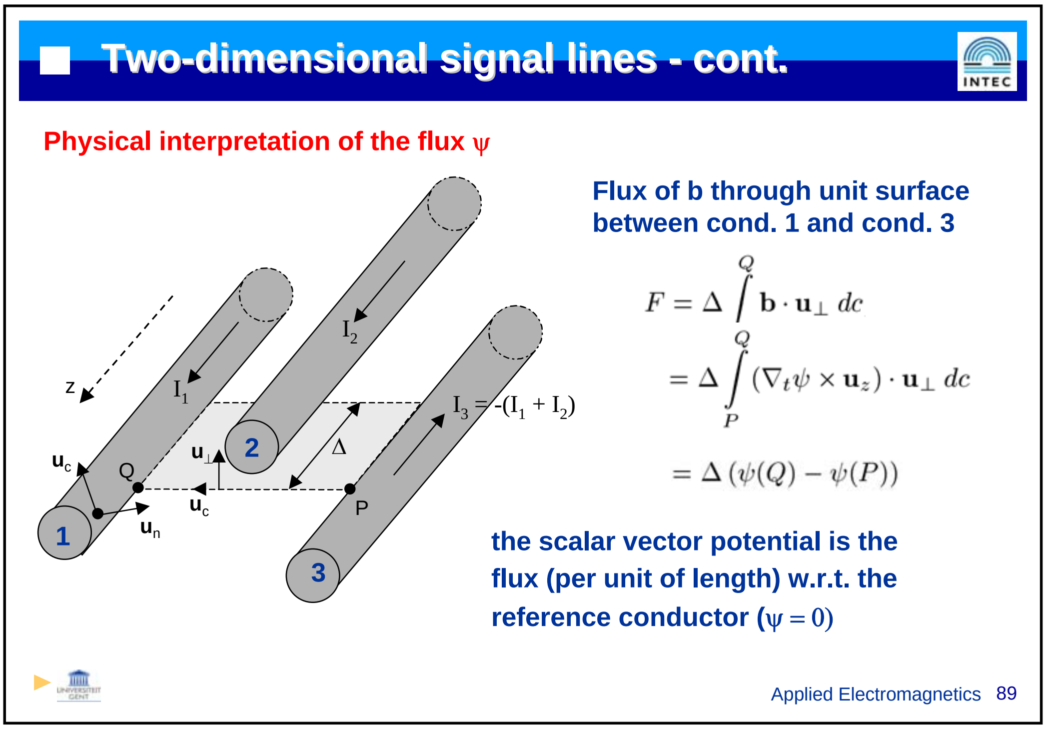
$$\psi = \text{constant on each conductor surface.}$$

Important note:

the scalar vector potential is the flux (per unit of length) w.r.t. the reference conductor ($\psi = 0$)

- **only z-directed current**
- **the vector potential is also z-directed, hence**

$$\mathbf{h}_t(\boldsymbol{\rho}) = \frac{1}{\mu}\nabla\times(\psi(\boldsymbol{\rho})\mathbf{u}_z) = \frac{1}{\mu}\nabla_t\psi(\boldsymbol{\rho})\times\mathbf{u}_z$$



Two-dimensional signal lines - cont.

Physical interpretation of the flux ψ

Flux of b through unit surface between cond. 1 and cond. 3

$$F = \Delta \int^{Q} \mathbf{b} \cdot \mathbf{u}_\perp \, dc$$

$$= \Delta \int_{P}^{Q} (\nabla_t \psi \times \mathbf{u}_z) \cdot \mathbf{u}_\perp \, dc$$

$$= \Delta \left(\psi(Q) - \psi(P) \right)$$

the scalar vector potential is the flux (per unit of length) w.r.t. the reference conductor ($\psi = 0$)

Determining *L* from an equivalent *C*

One can prove that in the 2D (signal lines) case the inductance matrix L is the inverse of a capacitance matric C_m

$$\mathbf{L} = \mathbf{C_m}^{-1}$$

The capacitance problem for C_m is obtained by replacing ε by $1/\mu$

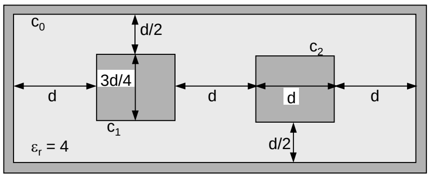
For a completely homogeneous medium we have: $\mathbf{L} \cdot \mathbf{C} = \mathbf{C} \cdot \mathbf{L} = \epsilon\mu\mathbf{1}$

Example: coaxial line

$$C_{coax} = \frac{2\pi\epsilon}{\ln \frac{b}{a}}$$

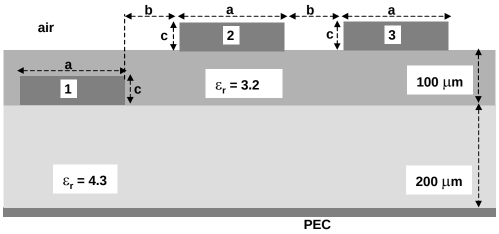
$$L_{coax} = \frac{\mu}{2\pi} \ln \frac{b}{a}$$

A set of signal lines: examples



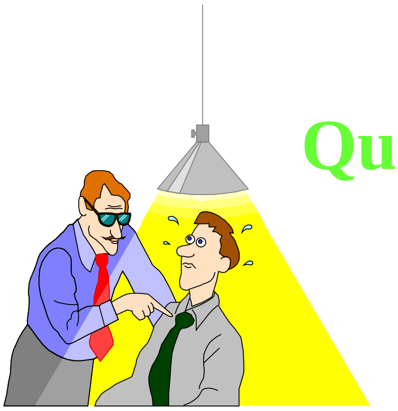
$$\mathbf{L} = \begin{pmatrix} 157.74 & 12.06 \\ 12.06 & 157.74 \end{pmatrix} nH/m$$

L•C = C •L = 4 ε_0 μ_0



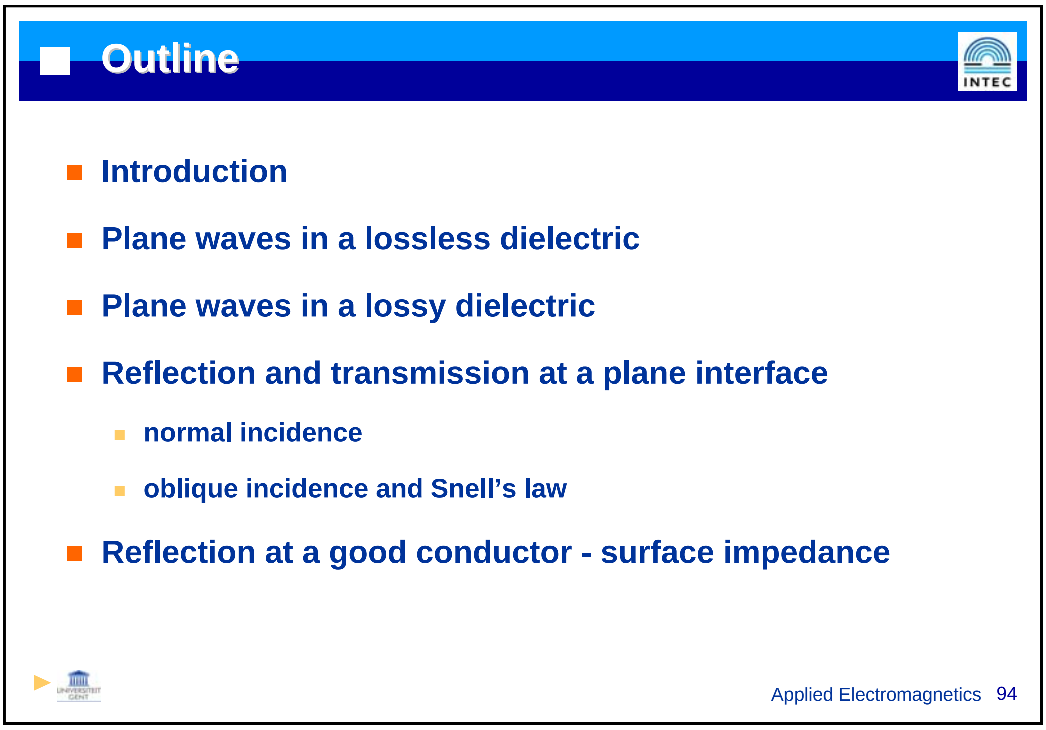
$$\mathbf{L} = \begin{pmatrix} 277.73 & 87.76 & 36.77 \\ 87.76 & 328.60 & 115.77 \\ 36.77 & 115.77 & 337.98 \end{pmatrix} nH/m$$

a = 350µm; b = 150µm; c = 70µm



Questions?

Chapter 5: Plane Waves



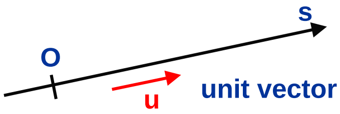
Outline

- Introduction
- Plane waves in a lossless dielectric
- Plane waves in a lossy dielectric
- Reflection and transmission at a plane interface
  - normal incidence
  - oblique incidence and Snell’s law
- Reflection at a good conductor - surface impedance

Introduction

In this chapter we look for the most simple solutions of Maxwell's equations

- homogenous, isotropic, infinitely extending medium
- sourceless situation (eigenmode)
- sinusoidal regime at frequency ω
- solution only depending upon a single coordinate along a propagation direction u



unit vector along the propagation direction

Solution ?? E(s,ω) and H(s,ω) ⇨ plane wave solution

Plane waves in a lossless dielectric

Medium characterised by real-valued and constant ε_r and μ_r with $\sigma = 0$

Suppose that our prefered direction is the z-direction, i.e. s = z

⇨ Maxwell's curl equations become

$$\frac{\mathrm{d}}{\mathrm{d}z}\mathbf{u}_z \times \mathbf{e}(z) = -j\omega\mu\mathbf{h}(z)$$

$$\frac{\mathrm{d}}{\mathrm{d}z}\mathbf{u}_z \times \mathbf{h}(z) = j\omega\epsilon\mathbf{e}(z)$$

Scalar multiplication with u_z shows that

$e_z = h_z = 0$ ⇨ the plane wave has no longitudinal components

Plane waves in a lossless dielectric - cont.

Elimination of e of h leads to the following wave equations

$$\frac{\mathrm{d}^2}{\mathrm{d}z^2}\mathbf{e}(z)+k^2\mathbf{e}(z)=0$$

+ first order ! ⇨

$$\mathbf{e}(z)=\mathbf{E}_t^+e^{-jkz}+\mathbf{E}_t^-e^{jkz}$$

$$\frac{\mathrm{d}^2}{\mathrm{d}z^2}\mathbf{h}(z)+k^2\mathbf{h}(z)=0$$

$$\mathbf{h}(z)=\frac{1}{Z_c}\mathbf{u}_z\times\mathbf{E}_t^+e^{-jkz}-\frac{1}{Z_c}\mathbf{u}_z\times\mathbf{E}_t^-e^{jkz}$$

transversal
NO z- component

real wavenumber in 1/m: k = $\omega\sqrt{\varepsilon\mu} = \omega / c = 2\pi / \lambda$

real characteristic impedance: $Z_c=\sqrt{\mu/\epsilon}$ in Ω

characteristic impedance
of free space: $Z_{c0}=\sqrt{\mu_0/\epsilon_0}=376.7303135\ \Omega\approx 377\ \Omega\approx 120\pi\ \Omega$

Plane waves in a lossless dielectric - cont.

Conclusions

- the electric field is perpendicular to the propagation direction
- the magnetic field is perpendicular to the propagation direction
- electric and magnetic field are perpendicular
- the amplitude of the magnetic and electric field differ by a factor called the characteristic impedance of the medium

Time-domain expression

$$\mathbf{e}(z,t)=\Re[\mathbf{E}_t^+e^{-jkz+j\omega t}]+\Re[\mathbf{E}_t^-e^{jkz+j\omega t}]$$

why?

wave propagating in the
positive z-direction

wave propagating in the
negative z-direction

Plane waves in a lossless dielectric - cont.

Poynting vector for the wave propagating in the positive z-direction

$$\mathbf{p}(z) = \frac{|\mathbf{E}_t^+|^2}{2Z_c^*}\mathbf{u}_z = \frac{|\mathbf{E}_t^+|^2}{2Z_c}\mathbf{u}_z$$

⇨ energy density in Watt / m^2 propagated by the plane wave

Role of polarisation

example: left hand circularly polarised electric field

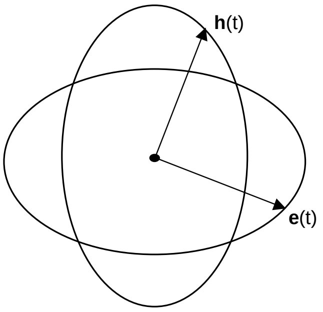
$$\mathbf{E}_t^+ = E^+(\mathbf{u}_x + j\mathbf{u}_y)$$

$$\mathbf{e}(z,t) = E^+\cos(\omega t - kz)\mathbf{u}_x - E^+\sin(\omega t - kz)\mathbf{u}_y$$

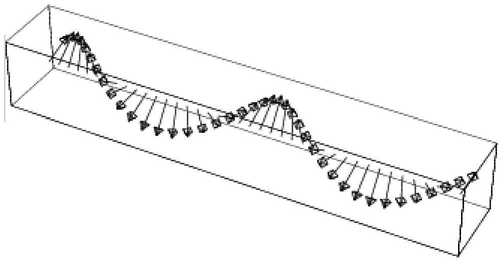
$$Z_c\mathbf{h}(z,t) = E^+\cos(\omega t - kz)\mathbf{u}_y + E^+\sin(\omega t - kz)\mathbf{u}_x$$

Plane waves in a lossless dielectric - cont.

Polarisation ellipses



e(z, t = 0)



Plane wave propagation in the direction u

Previous results can be generalised to

$$\mathbf{e}(\mathbf{r}) = \mathbf{A}e^{-jk\mathbf{u}\cdot\mathbf{r}}$$

$$\mathbf{h}(\mathbf{r}) = \frac{1}{Z_c}\mathbf{u} \times \mathbf{A}e^{-jk\mathbf{u}\cdot\mathbf{r}} = \frac{1}{Z_c}\mathbf{u} \times \mathbf{e}(\mathbf{r})$$

with

$$\mathbf{A} \cdot \mathbf{u} = 0.$$

Conclusions

- the electric field is perpendicular to the propagation direction
- the magnetic field is perpendicular to the propagation direction
- electric and magnetic field are perpendicular
- the amplitude of the magnetic and electric field differ by a factor called the characteristic impedance of the medium

remain valid !

Plane waves in a lossy dielectric

complex wavenumber: k = $\omega \sqrt{\varepsilon \mu}$ $\Re k \geq 0$ and $\Im k \leq 0$

complex characteristic impedance: $Z_c = \frac{\omega\mu}{k} = \frac{k}{\omega\epsilon}$

write k as k = $\beta - j\,\alpha$

propagation constant ⇩ propagation speed
v = ω / β

attenuation constant ⇩

$$\mathbf{p}(\mathbf{r}) = \frac{|\mathbf{A}|^2}{2Z_c^*}e^{-2\alpha\,\mathbf{u}\cdot\mathbf{r}}\,\mathbf{u}$$

⇨ Poynting's vector decreases exponentially

Plane waves in a lossy dielectric - cont.

Plane wave expression remains identical

$$\mathbf{e}(\mathbf{r}) = \mathbf{A}e^{-jk\mathbf{u}\cdot\mathbf{r}}$$

$$\mathbf{h}(\mathbf{r}) = \frac{1}{Z_c}\mathbf{u}\times\mathbf{A}e^{-jk\mathbf{u}\cdot\mathbf{r}} = \frac{1}{Z_c}\mathbf{u}\times\mathbf{e}(\mathbf{r}) \qquad \mathbf{A}\cdot\mathbf{u} = 0.$$

example: electric field for propagation along the z-direction

$$\mathbf{e}(z,t) = \Re[\mathbf{E}_t^+ e^{-j\beta z+j\omega t}]e^{-\alpha z} + \Re[\mathbf{E}_t^- e^{j\beta z+j\omega t}]e^{\alpha z}$$

power loss L in dB/m per metre: $L = 8.686\alpha$

Conclusions

- the electric field is perpendicular to the propagation direction ☑
- the magnetic field is perpendicular to the propagation direction ☑
- ~~electric and magnetic field are perpendicular~~
- ~~the amplitude of the magnetic and electric field differ by a factor called the characteristic impedance of the medium~~

Isolators and good conductors

Special case:

non-magnetic material with conductivity σ

$$\varepsilon \Rightarrow \varepsilon + \sigma / j\omega \Rightarrow \left\{ \begin{array}{l} k = \omega\sqrt{\epsilon\mu}\sqrt{1+\dfrac{\sigma}{j\omega\epsilon}} \\ Z_c = \sqrt{\dfrac{\mu}{\epsilon}}\dfrac{1}{\sqrt{1+\frac{\sigma}{j\omega\epsilon}}} \end{array} \right.$$

(a) low-loss dielectric (isolator) $\boxed{\sigma \ll \omega\varepsilon}$

$$k = \beta - j\alpha = \omega\sqrt{\epsilon\mu} - j\frac{\sigma}{2}\sqrt{\frac{\mu}{\epsilon}} \qquad Z_c = \sqrt{\frac{\mu}{\epsilon}}$$

→ **attenuation only**

polyethylene: $\varepsilon_r = 2.3$ $\quad \sigma/\omega\varepsilon = 2\ 10^{-4}$ **to** $3\ 10^{-4}$
between 50 Hz and 1 GHz

Isolators and good conductors - cont.

(b) good conductor $\boxed{\sigma >> \omega\, \varepsilon}$

$$k = \beta - j\alpha = \frac{(1-j)}{\delta} \qquad Z_c = \frac{(1+j)}{\sigma\delta}$$

skin-depth δ

$$\delta = \sqrt{\frac{2}{\omega\mu\sigma}}$$

Plane wave in a good conductor

$$\mathbf{e}(\mathbf{r}) = \mathbf{A}e^{-\frac{\mathbf{u}\cdot\mathbf{r}}{\delta}}e^{-j\frac{\mathbf{u}\cdot\mathbf{r}}{\delta}}$$

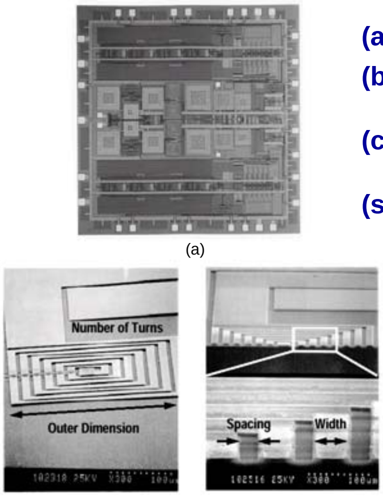
$$\mathbf{h}(\mathbf{r}) = (1-j)\frac{\sigma\delta}{2}(\mathbf{u}\times\mathbf{A})e^{-\frac{\mathbf{u}\cdot\mathbf{r}}{\delta}}e^{-j\frac{\mathbf{u}\cdot\mathbf{r}}{\delta}}$$

attenuation phase shift

copper: $\varepsilon_r = 1$ $\sigma = 5.9\ 10^7$

freq.	δ
50 Hz	9mm
1 MHz	0.06mm
1 GHz	0.02mm
10 GHz	0.66μm

From DC to skin-effect



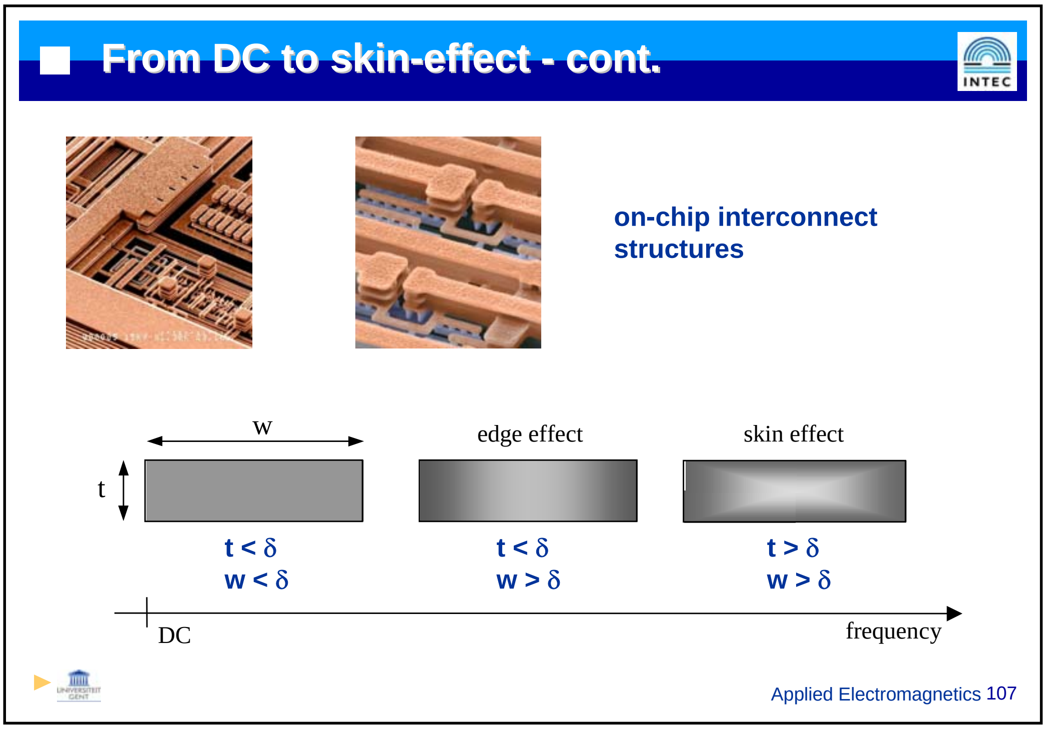
(a)

(b) (c)

(a) Layout of a single-chip CMOS GPS receiver
(b) On-chip spiral inductor component

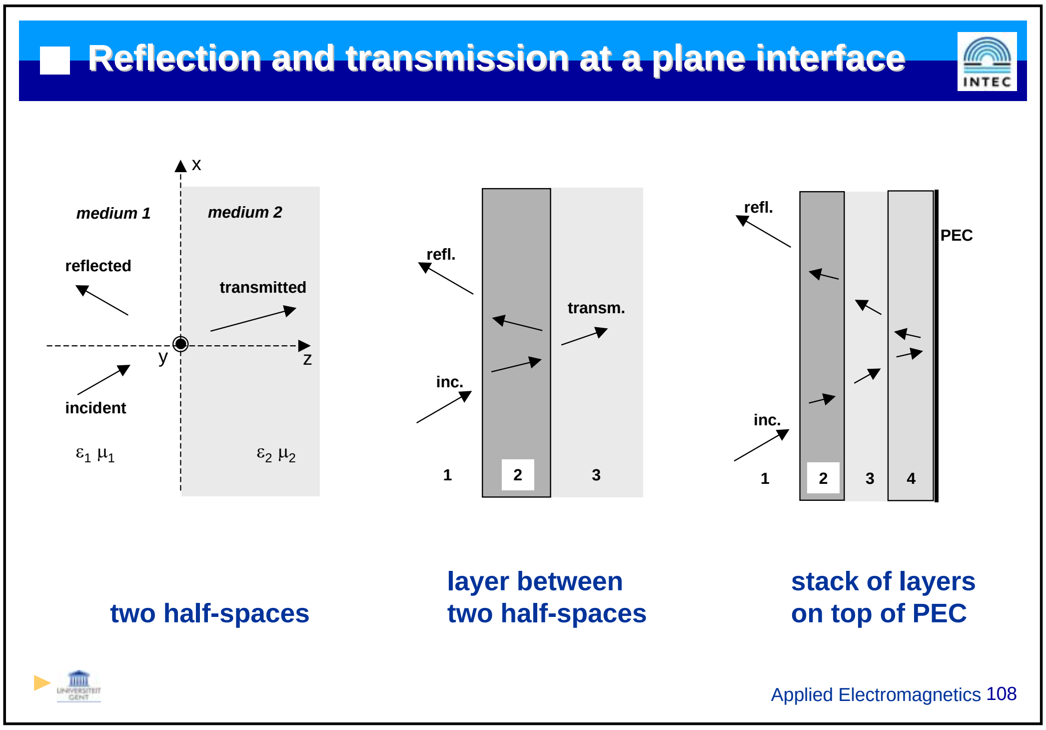
(c) Cross-section of the spiral inductor

(source: Stanford University, CA)



From DC to skin-effect - cont.

on-chip interconnect structures



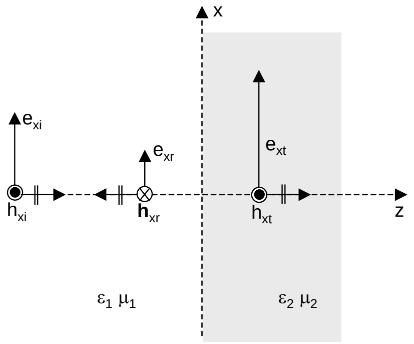
Reflection and transmission at a plane interface

two half-spaces

layer between two half-spaces

stack of layers on top of PEC

Normal incidence



incident linearly polarised plane wave

$$e_{xi}(z) = e^{-jk_1 z}$$
$$Z_1 h_{yi}(z) = e^{-jk_1 z}$$

reflected wave (why?)

$$e_{xr}(z) = R_n e^{+jk_1 z}$$
$$Z_1 h_{yr}(z) = -R_n e^{+jk_1 z}$$

reflection coeff.

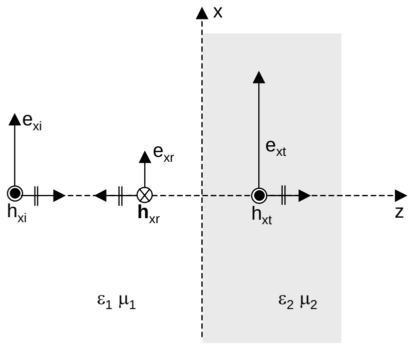
transmitted wave (why?)

$$e_{xt}(z) = T_n e^{-jk_2 z}$$
$$Z_2 h_{yt}(z) = T_n e^{-jk_2 z}$$

transmission coeff.

Is the polarisation of the incident wave relevant ?

Normal incidence - cont.



Application of boundary conditions

(a) continuity of tangential electric field

⇨ $1 + R_n = T_n$

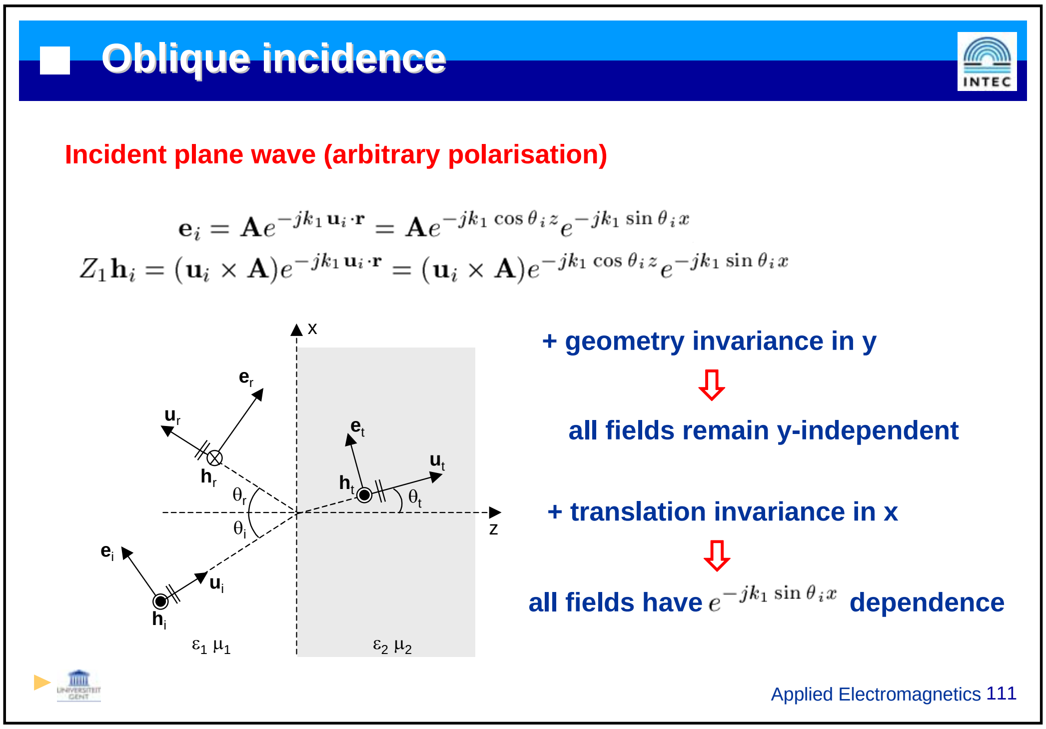
(b) continuity of tangential magnetic field

⇨ $\dfrac{1 - R_n}{Z_1} = \dfrac{T_n}{Z_2}$

Final result

$$T_n = \frac{2}{1 + \frac{Z_1}{Z_2}} \qquad R_n = \frac{1 - \frac{Z_1}{Z_2}}{1 + \frac{Z_1}{Z_2}}$$

(what happens if medium 2 is PEC ?)



Oblique incidence

Incident plane wave (arbitrary polarisation)

$$\mathbf{e}_i = \mathbf{A}e^{-jk_1\,\mathbf{u}_i\cdot\mathbf{r}} = \mathbf{A}e^{-jk_1\cos\theta_i z}e^{-jk_1\sin\theta_i x}$$

$$Z_1\mathbf{h}_i = (\mathbf{u}_i\times\mathbf{A})e^{-jk_1\,\mathbf{u}_i\cdot\mathbf{r}} = (\mathbf{u}_i\times\mathbf{A})e^{-jk_1\cos\theta_i z}e^{-jk_1\sin\theta_i x}$$

+ geometry invariance in y

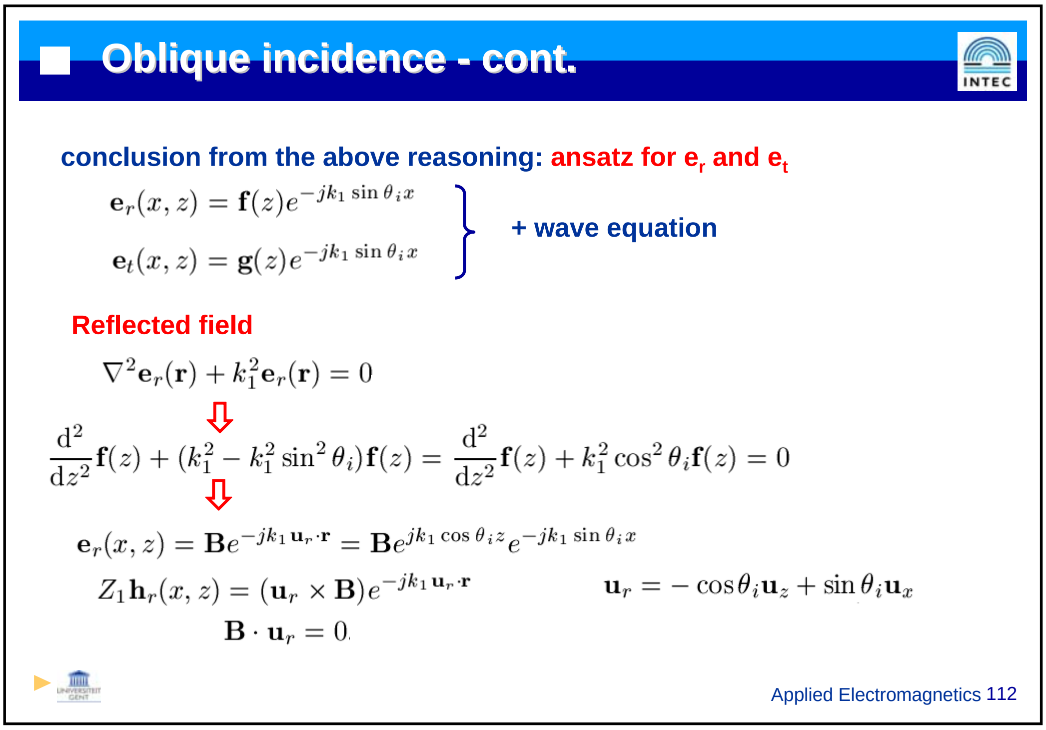
⇩

all fields remain y-independent

+ translation invariance in x

⇩

all fields have $e^{-jk_1\sin\theta_i x}$ dependence



Oblique incidence - cont.

conclusion from the above reasoning: ansatz for $\mathbf{e}_r$ and $\mathbf{e}_t$

$$\left.\begin{aligned}\mathbf{e}_r(x,z) &= \mathbf{f}(z)e^{-jk_1\sin\theta_i x}\\ \mathbf{e}_t(x,z) &= \mathbf{g}(z)e^{-jk_1\sin\theta_i x}\end{aligned}\right\}$$

+ wave equation

Reflected field

$$\nabla^2\mathbf{e}_r(\mathbf{r}) + k_1^2\mathbf{e}_r(\mathbf{r}) = 0$$

⇩

$$\frac{\mathrm{d}^2}{\mathrm{d}z^2}\mathbf{f}(z) + (k_1^2 - k_1^2\sin^2\theta_i)\mathbf{f}(z) = \frac{\mathrm{d}^2}{\mathrm{d}z^2}\mathbf{f}(z) + k_1^2\cos^2\theta_i\mathbf{f}(z) = 0$$

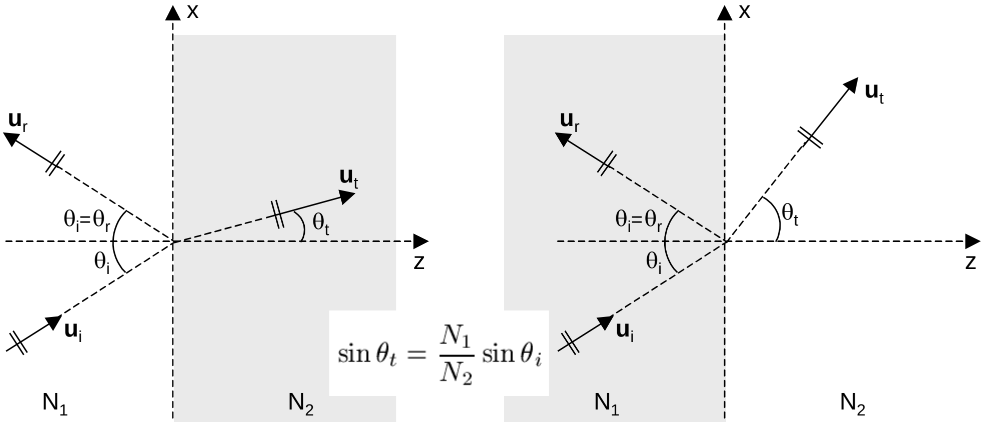
⇩

$$\mathbf{e}_r(x,z) = \mathbf{B}e^{-jk_1\,\mathbf{u}_r\cdot\mathbf{r}} = \mathbf{B}e^{jk_1\cos\theta_i z}e^{-jk_1\sin\theta_i x}$$

$$Z_1\mathbf{h}_r(x,z) = (\mathbf{u}_r\times\mathbf{B})e^{-jk_1\,\mathbf{u}_r\cdot\mathbf{r}} \qquad \mathbf{u}_r = -\cos\theta_i\mathbf{u}_z + \sin\theta_i\mathbf{u}_x$$

$$\mathbf{B}\cdot\mathbf{u}_r = 0$$

Oblique incidence - Snell’s law



$$\sin\theta_t = \frac{N_1}{N_2}\sin\theta_i$$

$\theta_r = \theta_i$
N1 < N2
$\theta_t < \theta_i$

$\theta_r = \theta_i$
N1 > N2
$\theta_t > \theta_i$

Oblique incidence - cont.

Transmitted field

$$\nabla^2\mathbf{e_t}(\mathbf{r}) + k_1^2\mathbf{e_t}(\mathbf{r}) = 0$$

⇩

$$\frac{\mathrm{d}^2}{\mathrm{d}z^2}\mathbf{g}(z) + (k_2^2 - k_1^2\sin^2\theta_i)\mathbf{g}(z) = 0$$

⇩

$$\begin{aligned}\mathbf{e}_t(x,z) &= \mathbf{C}e^{-jk_2\mathbf{u}_t\cdot\mathbf{r}}\\ &= \mathbf{C}e^{-jk_2\cos\theta_t z}e^{-jk_1\sin\theta_i x} = \mathbf{C}e^{-jk_2\cos\theta_t z}e^{-jk_2\sin\theta_t x}\end{aligned}$$

$$Z_2\mathbf{h}_t(x,z) = (\mathbf{u}_t\times\mathbf{C})e^{-jk_1\mathbf{u}_t\cdot\mathbf{r}}$$

$$\mathbf{C}\cdot\mathbf{u}_t = 0$$

$$k_2\sin\theta_t = k_1\sin\theta_i$$

$$\mathbf{u}_t = \cos\theta_t\mathbf{u}_z + \sin\theta_t\mathbf{u}_x$$

$$\Re(k_2\cos\theta_t) \geq 0 \text{ and } \Im(k_2\cos\theta_t) \leq 0$$

Transmission vs. total internal reflection

$$\begin{cases} \dfrac{\mathrm{d}^2}{\mathrm{d}z^2}\mathbf{g}(z) + (k_2^2 - k_1^2 \sin^2\theta_i)\mathbf{g}(z) = 0 \\ k_2 \sin\theta_t = k_1 \sin\theta_i \end{cases}$$

+ lossless materials

1. medium 2 is more dense than medium 1 ($k_2 > k_1$ or $N_2 > N_1$)

$$\mathbf{e}_t(x,z) = \mathbf{C}e^{-j\sqrt{k_2^2-k_1^2\sin^2\theta_i}\,z}e^{-jk_2\sin\theta_t x}$$

$$= \mathbf{C}e^{-jk_2\cos\theta_t z}e^{-jk_2\sin\theta_t x}$$

+

$$\sin\theta_t = \frac{N_1}{N_2}\sin\theta_i$$

Snell's law

$$\boxed{\theta_t < \theta_i}$$

example: from air ($N_1 \approx 1$) to water ($N_2 \approx 8.4$)

Transmission vs. total internal reflection - cont.

2. medium 2 is less dense than medium 1 ($k_2 < k_1$ or $N_2 < N_1$)

(a) $(k_1^2 - k_1^2\sin^2\theta_i)$ **> 0** ⇨ **Snell's law still applies BUT** $\boxed{\theta_t > \theta_i}$

(b) $(k_1^2 - k_1^2\sin^2\theta_i)$ **< 0** ⇨ **total internal reflection (≈ mirror)**

$$\mathbf{e}_t(x,z) = \mathbf{C}e^{-\sqrt{k_1^2\sin^2\theta_i - k_2^2}\,z}e^{-jk_2\sin\theta_t x}$$

(inhomogeneous plane wave)

exponential damping

Transition between (a) and (b): defined by critical angle $\sin\theta_i < \dfrac{N_2}{N_1} = \sin\theta_c$

example: from water ($N_1 \approx 8.4$) to air ($N_2 \approx 1$) $\theta_c = 6.8°$

TE and TM polarisation

$$\left.\begin{aligned}
&\mathbf{e}_i = \mathbf{A}e^{-jk_1\mathbf{u}_i\cdot\mathbf{r}}\\
&Z_1\mathbf{h}_i = (\mathbf{u}_i \times \mathbf{A})e^{-jk_1\mathbf{u}_i\cdot\mathbf{r}}\\
&\mathbf{A}\cdot\mathbf{u}_i = 0\\
&\mathbf{e}_r(x,z) = \mathbf{B}e^{-jk_1\mathbf{u}_r\cdot\mathbf{r}}\\
&Z_1\mathbf{h}_r(x,z) = (\mathbf{u}_r \times \mathbf{B})e^{-jk_1\mathbf{u}_r\cdot\mathbf{r}}\\
&\mathbf{B}\cdot\mathbf{u}_r = 0\\
&\mathbf{e}_t(x,z) \;=\; \mathbf{C}e^{-jk_2\mathbf{u}_t\cdot\mathbf{r}}\\
&Z_2\mathbf{h}_t(x,z) = (\mathbf{u}_t \times \mathbf{C})e^{-jk_1\mathbf{u}_t\cdot\mathbf{r}}\\
&\mathbf{C}\cdot\mathbf{u}_t = 0.
\end{aligned}\right\}$$

4 scalar unknowns +
2 scalar source terms (why?)

+

continuity of the tangential electric and magnetic field, i.e. 4 scalar conditions

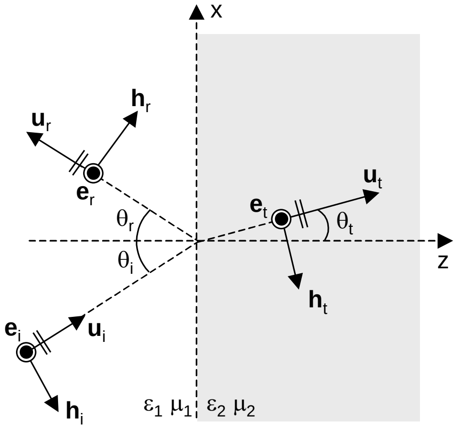
BUT decoupling into two independent sets of problems or polarisations is possible

---

TE polarisation

TE-polarisation or Transverse Electric polarisation

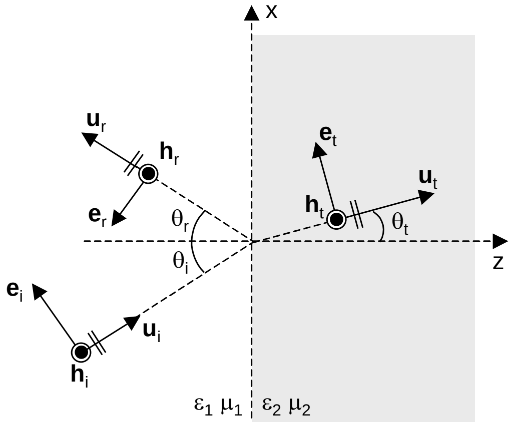
⇨ **electric field linearly polarised and perpendicular to plane of incidence**



TM polarisation

TM-polarisation or Transverse Magnetic polarisation

⇨ magnetic field linearly polarised and perpendicular to plane of incidence



TE-polarisation: results

$$\mathbf{e}_i(\mathbf{r}) = Ae^{-jk_1\mathbf{u}_i\cdot\mathbf{r}}\mathbf{u}_y$$
$$Z_1\mathbf{h}_i(\mathbf{r}) = A(\mathbf{u}_i\times\mathbf{u}_y)e^{-jk_1\mathbf{u}_i\cdot\mathbf{r}}$$

$$\mathbf{e}_r(\mathbf{r}) = R_{TE}Ae^{-jk_1\mathbf{u}_r\cdot\mathbf{r}}\mathbf{u}_y$$
$$Z_1\mathbf{h}_r(\mathbf{r}) = R_{TE}A(\mathbf{u}_r\times\mathbf{u}_y)e^{-jk_1\mathbf{u}_r\cdot\mathbf{r}}$$

$$\mathbf{e}_t(\mathbf{r}) = T_{TE}Ae^{-jk_2\mathbf{u}_t\cdot\mathbf{r}}\mathbf{u}_y$$
$$Z_1\mathbf{h}_i(\mathbf{r}) = T_{TE}A(\mathbf{u}_t\times\mathbf{u}_y)e^{-jk_2\mathbf{u}_t\cdot\mathbf{r}}$$

$$R_{TE} = \frac{Z_2\cos\theta_i - Z_1\cos\theta_t}{Z_2\cos\theta_i + Z_1\cos\theta_t} = \frac{\sin(\theta_t-\theta_i)}{\sin(\theta_t+\theta_i)}$$

$$\downarrow$$

always different from 0 !

$$T_{TE} = \frac{2Z_2\cos\theta_i}{Z_2\cos\theta_i + Z_1\cos\theta_t}$$

TM-polarisation: results

$$Z_1\mathbf{h}_i(\mathbf{r}) = Ae^{-jk_1\mathbf{u}_i\cdot\mathbf{r}}\mathbf{u}_y$$
$$\mathbf{e}_i(\mathbf{r}) = -A(\mathbf{u}_i\times\mathbf{u}_y)e^{-jk_1\mathbf{u}_i\cdot\mathbf{r}}$$

$$Z_1\mathbf{h}_r(\mathbf{r}) = R_{TM}Ae^{-jk_1\mathbf{u}_r\cdot\mathbf{r}}\mathbf{u}_y$$
$$\mathbf{e}_r(\mathbf{r}) = -R_{TM}A(\mathbf{u}_r\times\mathbf{u}_y)e^{-jk_1\mathbf{u}_r\cdot\mathbf{r}}$$

$$Z_2\mathbf{h}_t(\mathbf{r}) = T_{TM}Ae^{-jk_2\mathbf{u}_t\cdot\mathbf{r}}\mathbf{u}_y$$
$$\mathbf{e}_t(\mathbf{r}) = -T_{TM}A(\mathbf{u}_t\times\mathbf{u}_y)e^{-jk_2\mathbf{u}_t\cdot\mathbf{r}}$$

$$R_{TM} = \frac{Z_1\cos\theta_i - Z_2\cos\theta_t}{Z_1\cos\theta_i + Z_2\cos\theta_t} = \frac{\tan(\theta_i-\theta_t)}{\tan(\theta_t+\theta_i)}$$

$\downarrow$

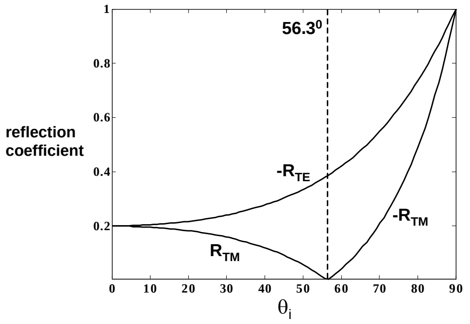
0 for $\tan\theta_B = \sqrt{\frac{\epsilon_2}{\epsilon_1}} = \frac{N_2}{N_1}$

i.e the Brewster angle

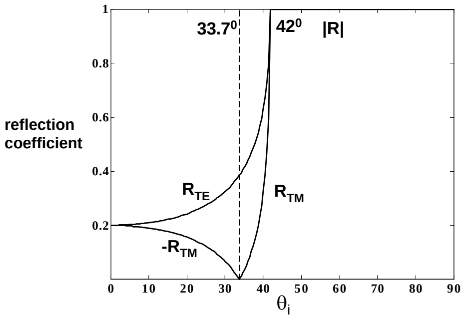
$$T_{TM} = \frac{2Z_2\cos\theta_i}{Z_1\cos\theta_i + Z_2\cos\theta_t}$$

(what happens for $\theta_i = 0$?)

Example: glass - air



air to glass



glass to air

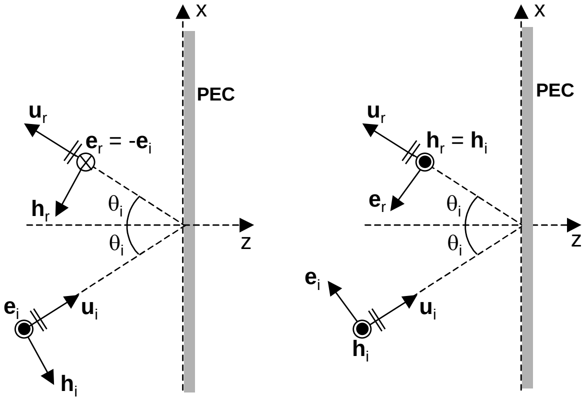
air ($N \approx 1$)
glass ($N \approx 1.5$)

Total reflection at a perfect conductor

perfect conductor: $\sigma \rightarrow \infty$ and $Z_2 \rightarrow 0$

⇩

$$R_{TE,PEC} = -1$$
$$T_{TE,PEC} = 0$$
$$R_{TM,PEC} = +1$$
$$T_{TM,PEC} = 0$$



TE polarisation

TM polarisation

Reflection by an impedance sheet

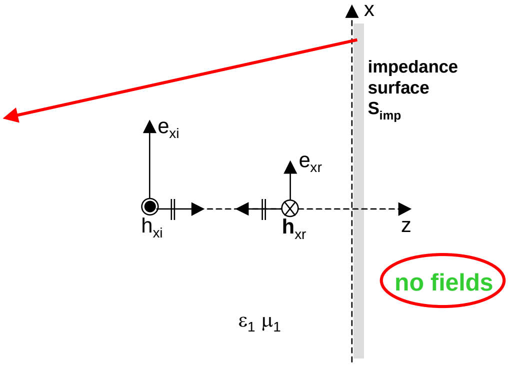
- impedance surface =

  infinitely thin current-carrying sheet
- at this sheet:

$$\mathbf{e}_{tan} = Z_s(\mathbf{u}_n \times \mathbf{h}_{tan})$$

⇩

$$\mathbf{e}_{tan} = Z_s \mathbf{j}_s$$



- reflection coefficient for

  perpendicular incidence:

$$R_n = \frac{1 - \frac{Z_1}{Z_s}}{1 + \frac{Z_1}{Z_s}}$$

Isolators and good conductors - cont.

(b) good conductor $\boxed{\sigma >> \omega\, \varepsilon}$

$$k = \beta - j\alpha = \frac{(1-j)}{\delta} \qquad Z_c = \frac{(1+j)}{\sigma\delta}$$

copper: $\varepsilon_r = 1$ $\sigma = 5.9\ 10^7$

freq.	δ
50 Hz	9mm
1 MHz	0.06mm
1 GHz	0.02mm
10 GHz	0.66μm

skin-depth δ

$$\delta = \sqrt{\frac{2}{\omega\mu\sigma}}$$

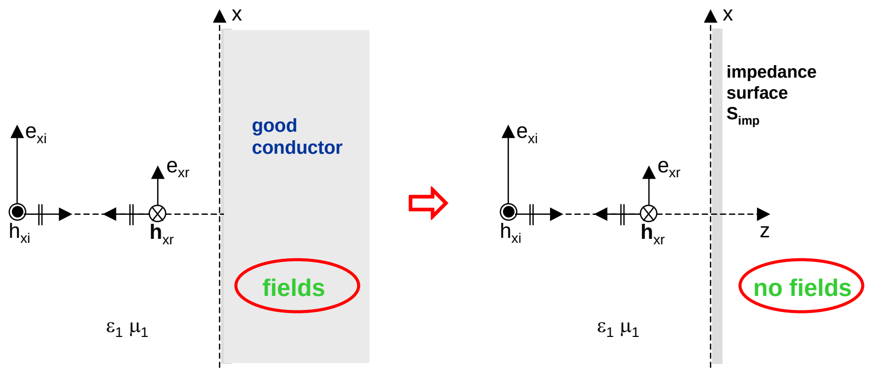
Plane wave in a good conductor

$$\mathbf{e}(\mathbf{r}) = \mathbf{A}e^{-\frac{\mathbf{u}\cdot\mathbf{r}}{\delta}}e^{-j\frac{\mathbf{u}\cdot\mathbf{r}}{\delta}}$$

$$\mathbf{h}(\mathbf{r}) = (1-j)\frac{\sigma\delta}{2}(\mathbf{u}\times\mathbf{A})e^{-\frac{\mathbf{u}\cdot\mathbf{r}}{\delta}}e^{-j\frac{\mathbf{u}\cdot\mathbf{r}}{\delta}}$$

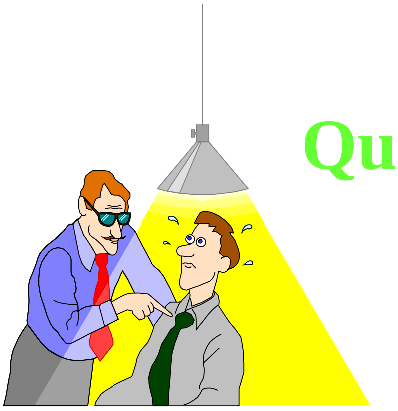
attenuation phase shift

Reflection by a good conductor



$$Z_s = \frac{(1+j)}{\sigma\delta} = Z_c$$ **+** the Joule losses are identical (proof?) $= \frac{1}{2}\Re(Z_s)|I|^2$

PEC: $\sigma \rightarrow \infty$ and $Z_s \rightarrow 0$ no losses !



Questions?

Chapter 6: Transmission Lines

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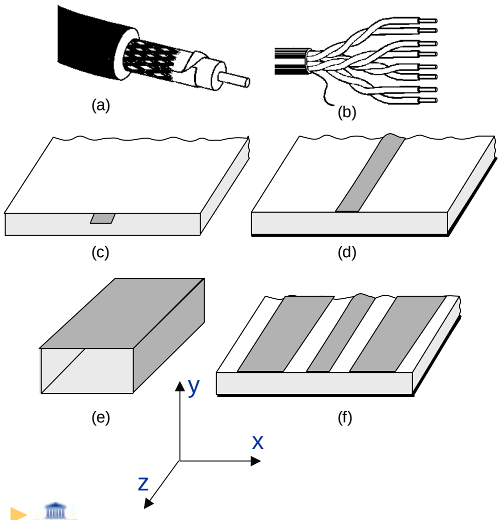
Outline

- Introduction
- Telegrapher’s equations
- Voltage reflection coefficient
- Input impedance
- Generalised reflection coefficient
- Power flow
- Standing waves and VSWR
- Smith chart
- Matching
- Transients

Introduction



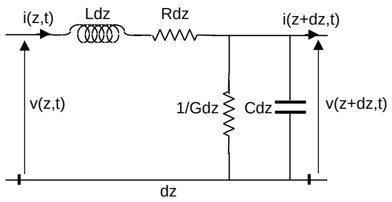
Purposes

- + describe signal propagation along z in terms of voltages and currents
- + extend lumped element description in circuit theory with a distributed element describing signal propagation and wave effects in the z-direction
- + avoid solving Maxwell?

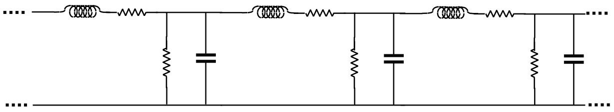
Solution

- + the transmission line concept
- + solve Maxwell in the cross-section

Telegrapher’s equations



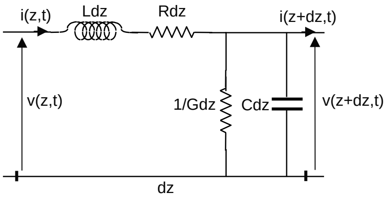
elementary lumped element section of a transmission line



lumped element representation of a transmission line

L - C - G - R : result from the solution of Maxwell’s equations !

Telegrapher's equations - time domain

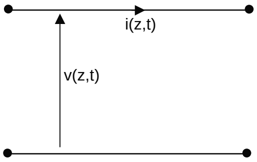


$$v(z+dz,t)-v(z,t)=-Rdz\,i(z,t)-Ldz\,\frac{\partial}{\partial t}i(z,t)$$

$$i(z+dz,t)-i(z,t)=-Gdz\,v(z+dz,t)-Cdz\,\frac{\partial}{\partial t}v(z+dz,t)$$

$$\frac{\partial v(z,t)}{\partial z}=-Ri(z,t)-L\frac{\partial i(z,t)}{\partial t}$$

$$\frac{\partial i(z,t)}{\partial z}=-Gv(z,t)-C\frac{\partial v(z,t)}{\partial t}$$



Telegrapher's equations - frequency domain

$$\frac{\mathrm{d}V(z)}{\mathrm{d}z}=-(R+j\omega L)I(z)=-Z(\omega)I(z),$$

$$\frac{\mathrm{d}I(z)}{\mathrm{d}z}=-(G+j\omega C)V(z)=-Y(\omega)V(z).$$

General Solution

$$V(z)=Ae^{-jkz}+Be^{jkz},\quad I(z)=Y_c(Ae^{-jkz}-Be^{jkz})=\frac{1}{Z_c}(Ae^{-jkz}-Be^{jkz})$$

$$k=\beta-j\alpha=\sqrt{-(R+j\omega L)(G+j\omega C)}.$$

$$Z_c=\frac{1}{Y_c}=\sqrt{\frac{R+j\omega L}{G+j\omega C}}$$

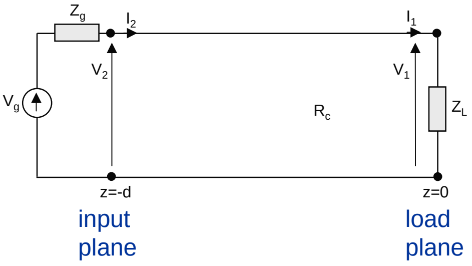
⇨ no losses

$$k=\omega\sqrt{LC}$$

$$Z_c=R_c=\sqrt{\frac{L}{C}}$$

$$c=\frac{1}{\sqrt{LC}}$$

Voltage reflection coefficient



$$\frac{B}{A} = \frac{b}{a} e^{j(\beta-\alpha)} = K_L = \frac{Z_L - R_c}{Z_L + R_c}$$

$$K_L = \frac{Z'_L - 1}{Z'_L + 1}$$

$Z_L(\Omega)$	K_L	VSWR
∞	1	∞
11 R_c	5/6	11
5 R_c	2/3	5
2 R_c	1/3	2
R_c	0	1
1/2 R_c	-1/3	2
1/5 R_c	-2/3	5
1/11 R_c	-5/6	11
0	-1	∞

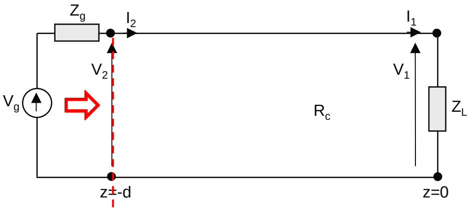
$$V(z) = Ae^{-jkz} + Be^{jkz}$$

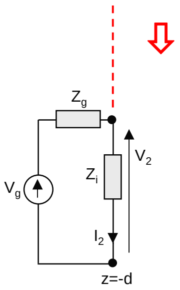
$$I(z) = \frac{1}{R_c}(Ae^{-jkz} - Be^{jkz})$$

$$A = ae^{j\alpha}$$

$$B = be^{j\beta}$$

Input impedance Z_i





$$Z_i = \frac{V_2}{I_2} = \frac{V(-d)}{I(-d)} = R_c \frac{A\ e^{jkd} + B\ e^{-jkd}}{A\ e^{jkd} - B\ e^{-jkd}}$$

$$Z'_i = \frac{Z_i}{R_c} = \frac{Z'_L + j\tan(kd)}{1 + jZ'_L\tan(kd)}$$

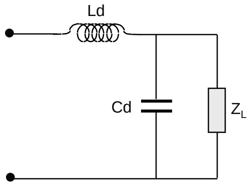
Input impedance Z_i: short line

Short line: kd = 2πd/λ << 1

$$Z_i' = \frac{Z_i}{R_c} = \frac{Z_L' + j\tan(kd)}{1 + jZ_L'\tan(kd)}$$

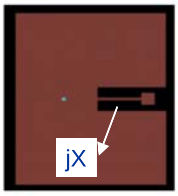
$$\Downarrow$$

$$Z_i = \frac{Z_L + j\omega Ld}{1 + jZ_L\omega Cd}$$



Input impedance Z_i: reactance synthesis

$$Z_i' = \frac{Z_i}{R_c} = \frac{Z_L' + j\tan(kd)}{1 + jZ_L'\tan(kd)}$$

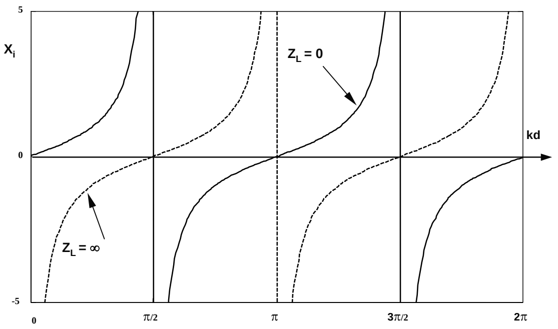


Open line

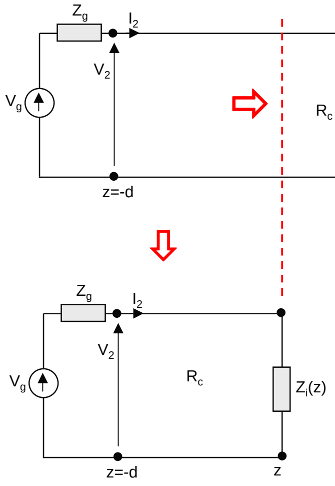
$$Z_i' = \frac{1}{j\tan kd} = jX$$

Short circuited line

$$Z_i' = j\tan kd = jX$$



Generalised reflection coefficient



$$K(z) = \frac{Be^{jkz}}{Ae^{-jkz}} = \frac{B}{A}e^{2jkz}$$

$$K(z) = K_L e^{2jkz} = K_L e^{-2jk|z|}$$

periodic with period λ/2 !

$$K(z) = \frac{Z_i'(z) - 1}{Z_i'(z) + 1}$$

$$Z_i'(z) = \frac{1 + K(z)}{1 - K(z)}$$

Power flow

$$P = \frac{1}{2}Re(I^* V) \quad ⇨ \quad P = \frac{1}{2}\frac{|A|^2}{R_c} - \frac{1}{2}\frac{|B|^2}{R_c}$$

no losses

power orthogonality

$$P = P_i - P_r = P_i(1 - |K_L|^2)$$

$$P_i = \frac{1}{2}\frac{|A|^2}{R_c}$$

$$P_r = |K_L|^2 P_i$$

$|K_L|$ = 1 results in total power reflection !

Standing waves and VSWR

frequency domain

$$V(z) = Ae^{-jkz} + Be^{jkz} \qquad A = ae^{j\alpha},$$

$$I(z) = \frac{1}{R_c}(Ae^{-jkz} - Be^{jkz}) \qquad B = be^{j\beta}$$

time domain

$$v(z,t) = a\cos(\omega t - kz + \alpha) + b\cos(\omega t + kz + \beta)$$

$$i(z,t) = \frac{1}{R_c}[a\cos(\omega t - kz + \alpha) - b\cos(\omega t + kz + \beta)]$$

constructive interference for the voltage = destructive interference for the current

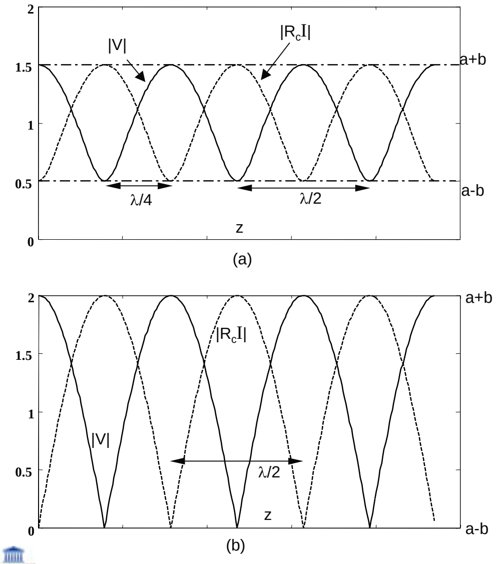
$$-kz_n + \alpha = +kz_n + \beta + n2\pi \Rightarrow$$

maximum voltage amplitude (a+b)

minimum current amplitude (a-b)/R_c

and VICE VERSA !

VSWR

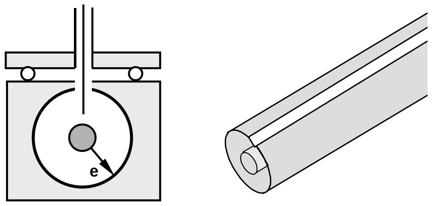


Voltage Standing Wave Ratio or VSWR

$$S = \frac{a+b}{a-b} = \frac{1+|K|}{1-|K|} = \frac{1+|K_L|}{1-|K_L|}$$

$Z_L(\Omega)$	K_L	VSWR
∞	1	∞
11 R_c	5/6	11
5 R_c	2/3	5
2 R_c	1/3	2
R_c	0	1
1/2 R_c	-1/3	2
1/5 R_c	-2/3	5
1/11 R_c	-5/6	11
0	-1	∞

VSWR detector



Standing wave detector

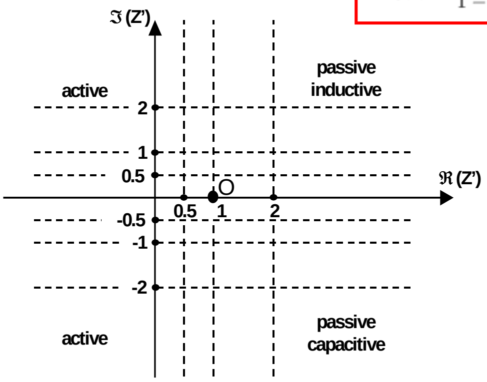
$$|K| = |K_L| = \frac{S-1}{S+1}$$

$$2kz_n = \alpha - \beta - n2\pi \qquad n = ..., -1, 0, 1, ...$$

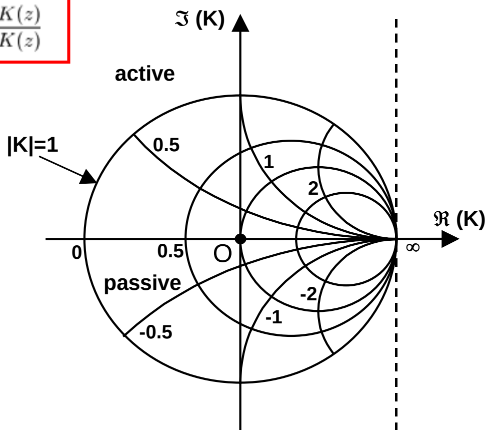
Smith chart

$$K(z) = \frac{Z_i'(z)-1}{Z_i'(z)+1}$$

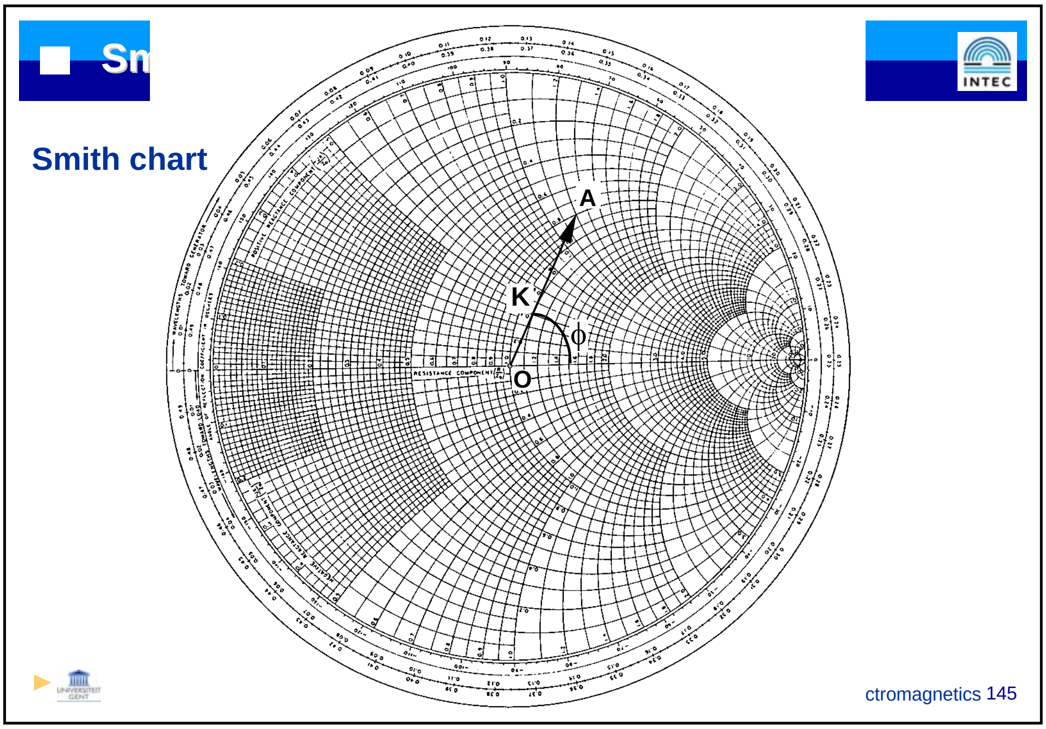
$$Z_i'(z) = \frac{1+K(z)}{1-K(z)}$$



impedance plane

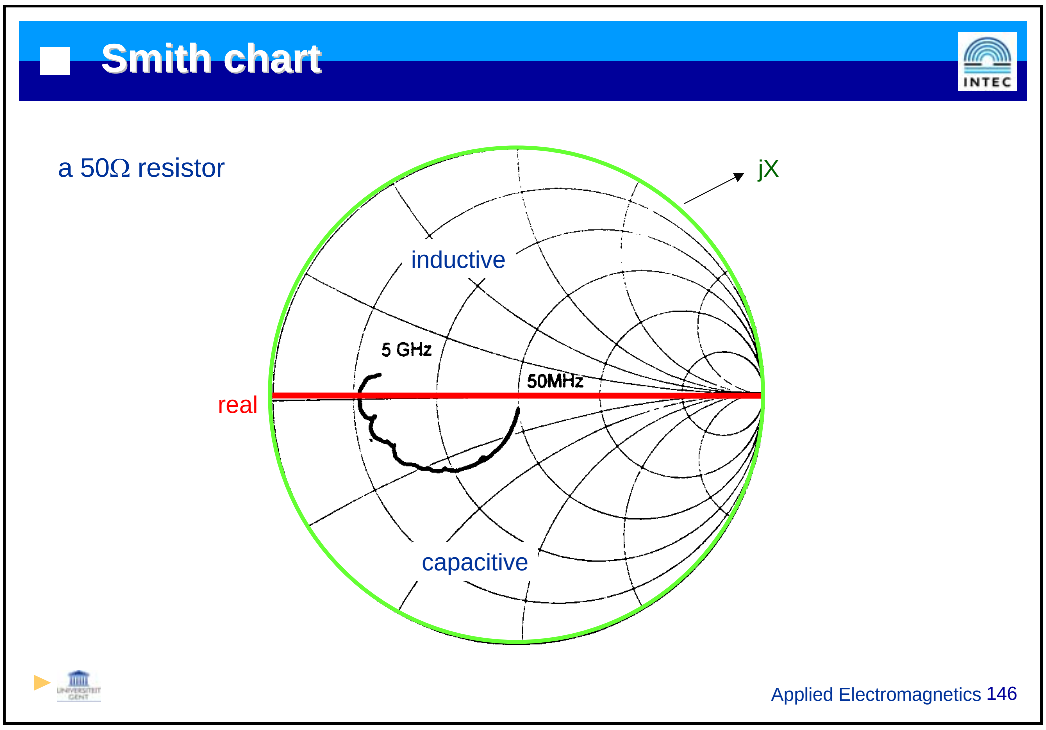


reflection coefficient plane



Sm

Smith chart

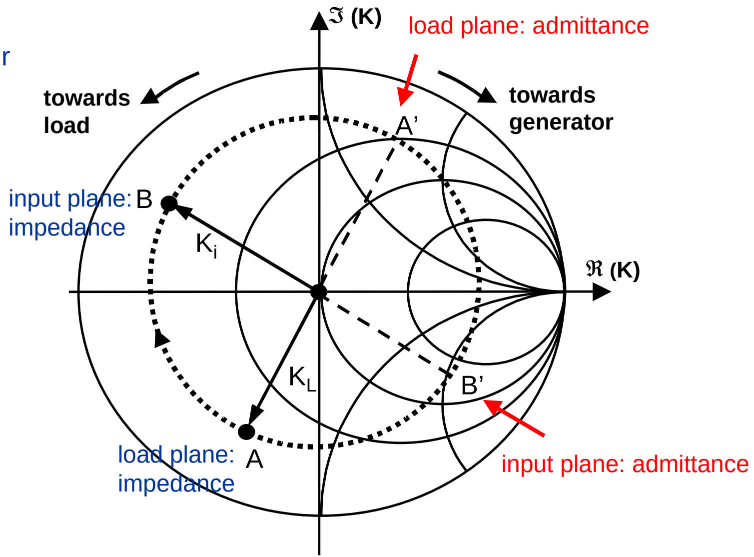


Smith chart

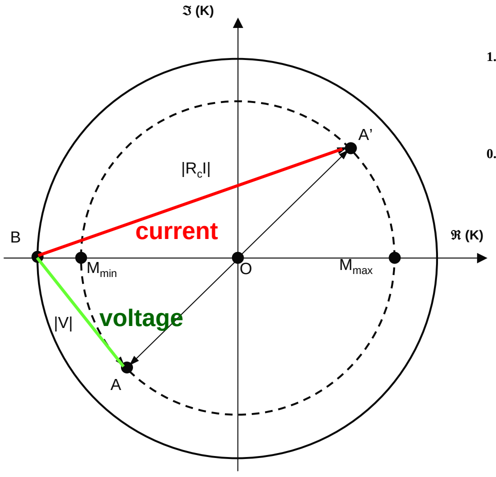
a 50Ω resistor

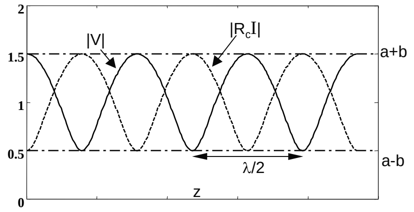
Smith chart: impedance transformation

180^0 = 1 tour



Smith chart: VSWR





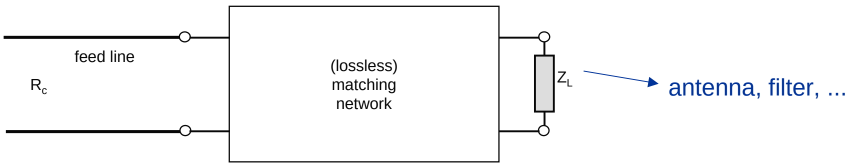
$$V(z) = Ae^{-jkz} + Be^{jkz}$$

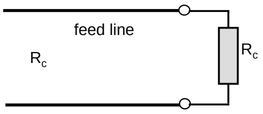
$$I(z) = \frac{1}{R_c}(Ae^{-jkz} - Be^{jkz})$$

$$|V(z)| = a|1 + K(z)|$$

$$R_c|I(z)| = a|1 - K(z)|$$

Matching





⇨

Mismatch

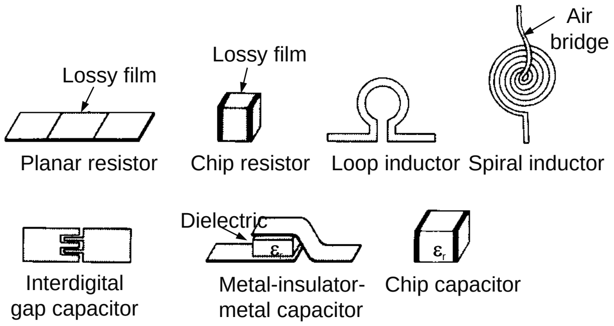
- reflection of power
- signal reflection and echoes
- standing waves (dielectric breakdown)

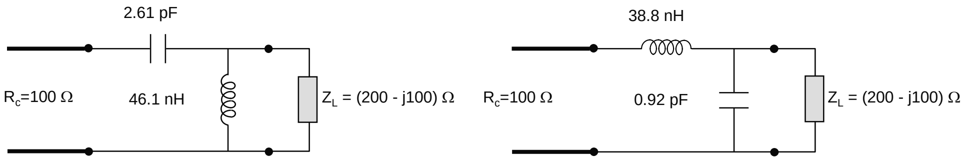
Matching techniques ⇨

- lumped element matching;
- single stub matching;
- double stub matching;
- the quarter-wavelength transformer;
- multisection matching;
- tapered lines.

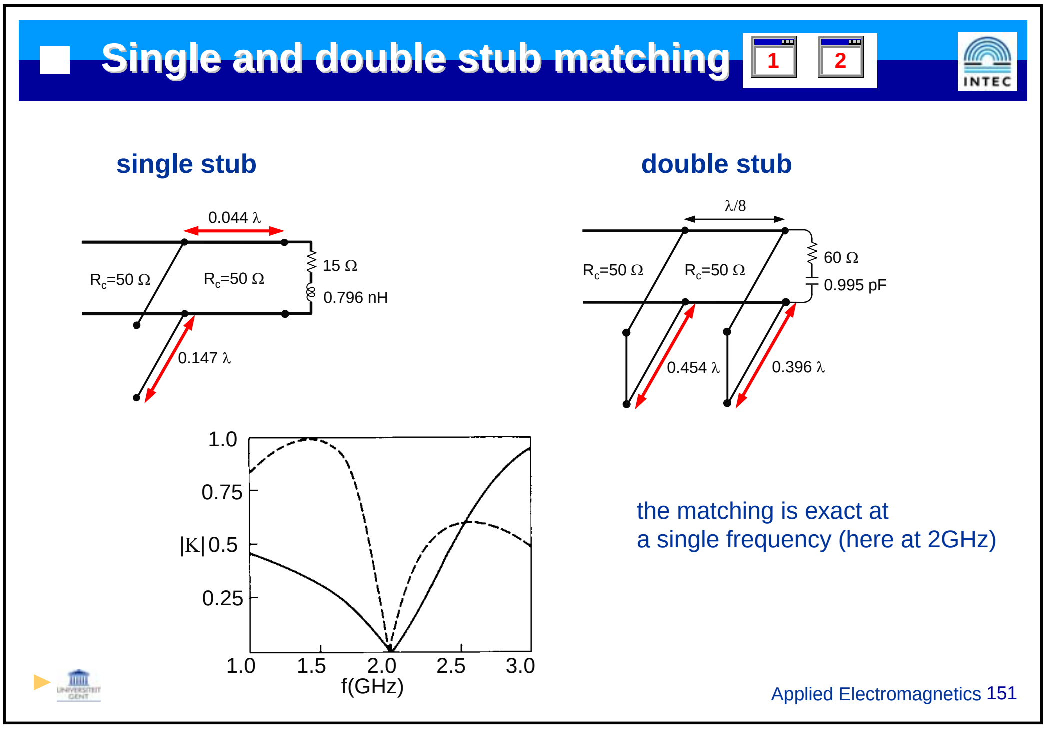
Lumped element matching

- dimensions < $\lambda/10$
- parasitics are very important





The matching is exact at a single frequency (here at 500 MHz)

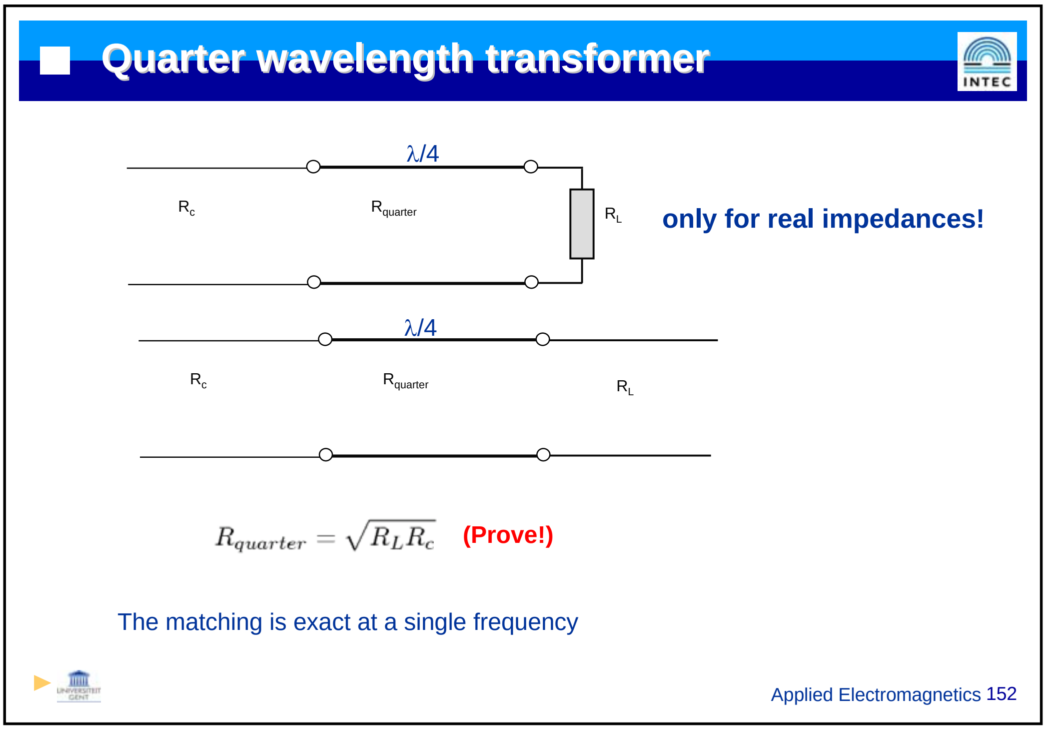


Single and double stub matching

single stub

double stub

the matching is exact at
a single frequency (here at 2GHz)



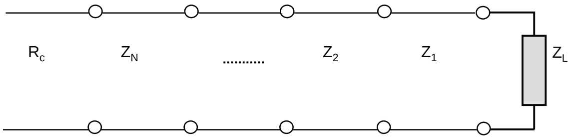
Quarter wavelength transformer

only for real impedances!

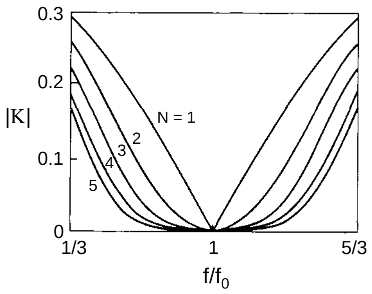
$$R_{quarter} = \sqrt{R_L R_c}$$ **(Prove!)**

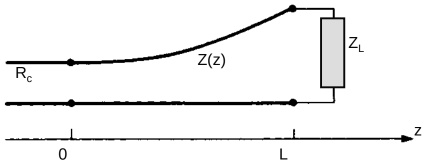
The matching is exact at a single frequency

Broadband matching



multisection matching

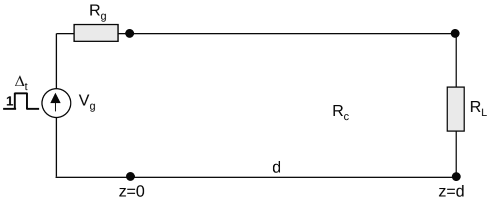




tapered matching

Transients

- no dispersion i.e. no frequency dependent velocity
- no losses



$$\frac{\partial v(z,t)}{\partial z} = -\cancel{Ri(z,t)} - L\frac{\partial i(z,t)}{\partial t}$$

$$\frac{\partial i(z,t)}{\partial z} = -\cancel{Gv(z,t)} - C\frac{\partial v(z,t)}{\partial t}$$

$$\frac{\partial^2}{\partial z^2}v(z,t) - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}v(z,t) = 0.$$

$$c = \frac{1}{\sqrt{LC}}$$

delay $\tau = d/c$

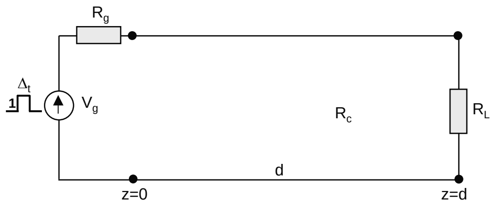
⇨

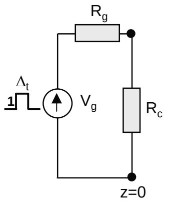
$$R_c = \sqrt{L/C}.$$

$$v(z,t) = v^+(z-ct) + v^-(z+ct)$$

$$i(z,t) = \frac{1}{R_c}\left(v^+(z-ct) - v^-(z+ct)\right)$$

Transients: equivalent input impedance

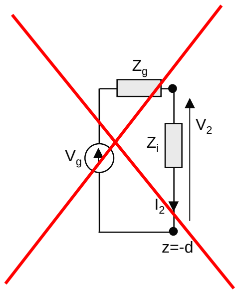




equivalent circuit at the beginning of the line

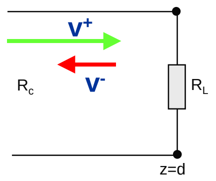
⇩

$$\kappa = \frac{R_c}{R_c + R_g}$$



Transients: reflection coefficients

load

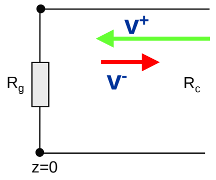


$$\frac{v^+ + v^-}{\frac{1}{Rc}(v^+ - v^-)} = \frac{v_{total}}{i_{total}} = R_L$$

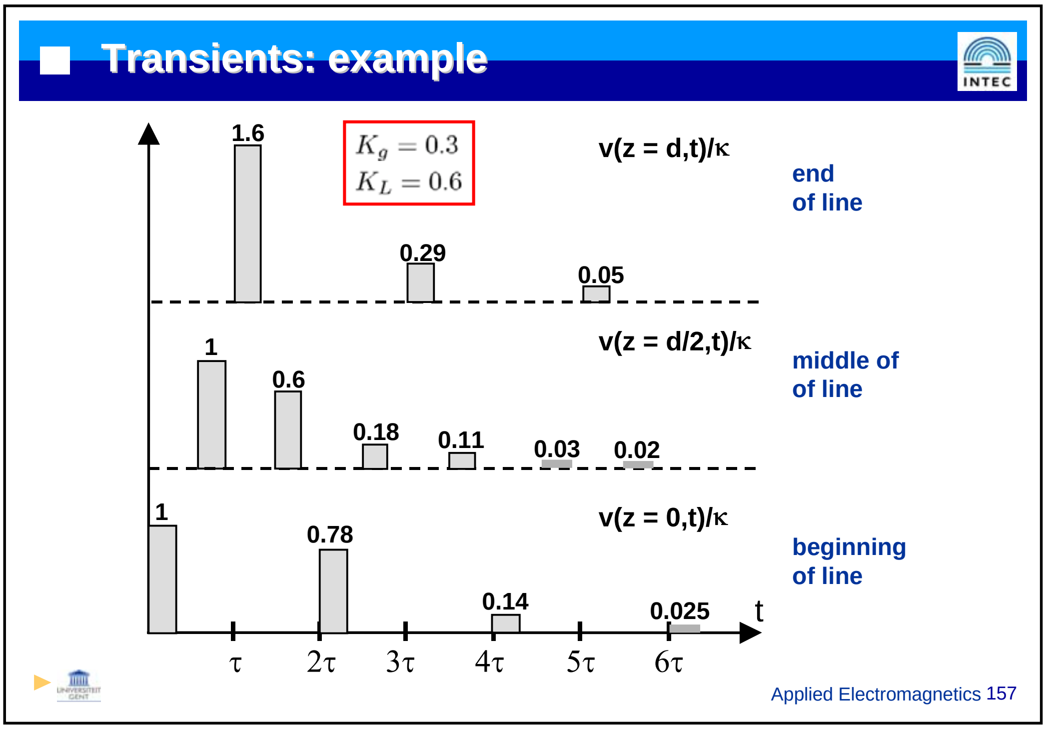
$$v^- = \frac{R_L - R_c}{R_L + R_c} v^+ = K_L v^+$$

$$K_L = \frac{Z_L - R_c}{Z_L + R_c}$$

generator



$$K_g = \frac{R_g - R_c}{R_g + R_c}$$



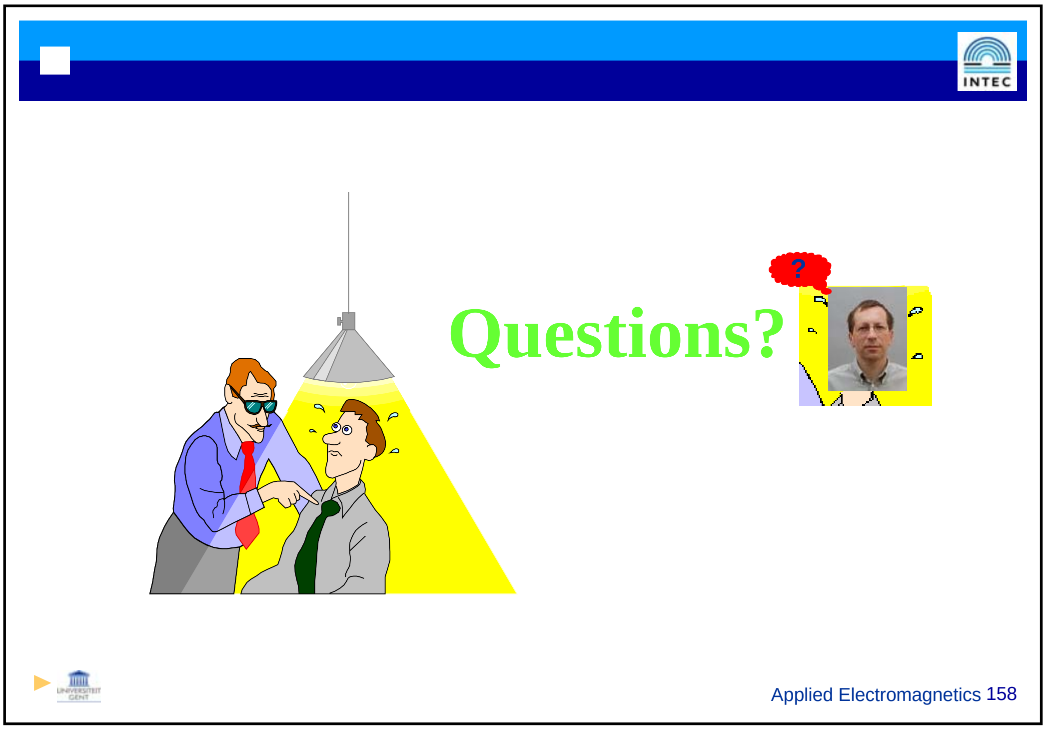
Transients: example

$K_g = 0.3$
$K_L = 0.6$

end
of line

middle of
of line

beginning
of line

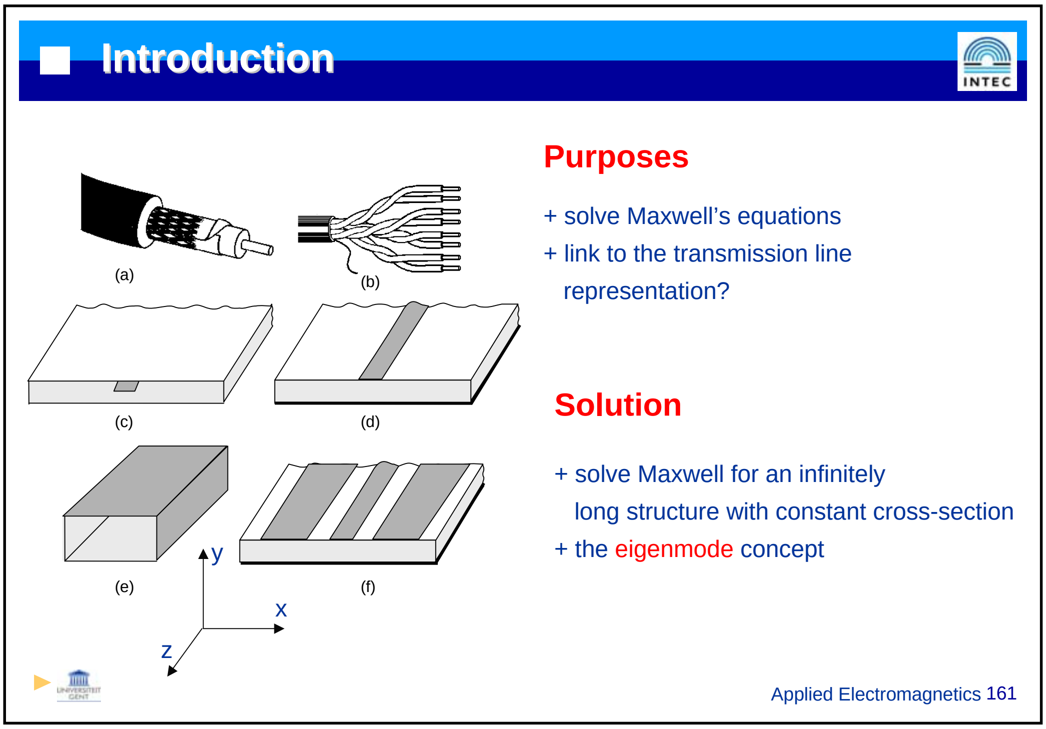


Questions?

Chapter 7: Multiconductor Transmission Lines and Waveguides

Outline

- Introduction
- General eigenmode equations
- Multiconductor transmission lines: quasi-TEM analysis
- TEM modes
- TE and TM modes
- Mode orthogonality
- Parallel-plate waveguide
  - losses
  - group velocity versus phase velocity - dispersion
- Rectangular waveguide
- Coaxial cable
- Microstrip and stripline



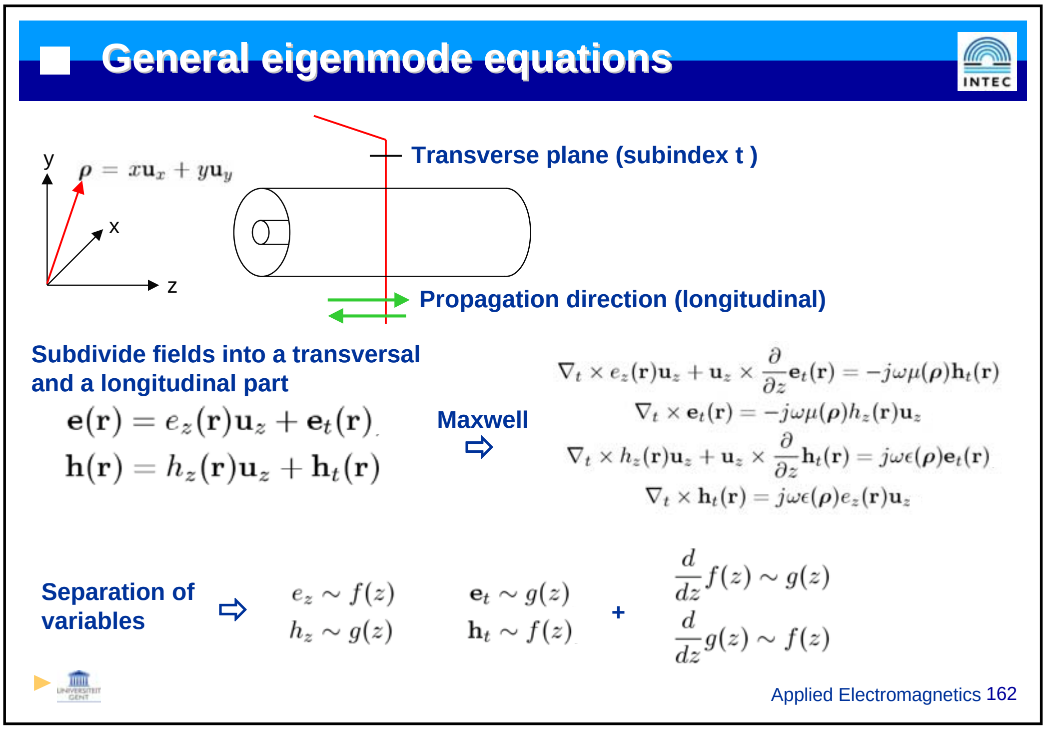
Introduction

Purposes

+ solve Maxwell's equations
+ link to the transmission line representation?

Solution

+ solve Maxwell for an infinitely long structure with constant cross-section
+ the eigenmode concept



General eigenmode equations

$\boldsymbol{\rho} = x\mathbf{u}_x + y\mathbf{u}_y$

Transverse plane (subindex t)

Propagation direction (longitudinal)

Subdivide fields into a transversal and a longitudinal part

$$\mathbf{e}(\mathbf{r}) = e_z(\mathbf{r})\mathbf{u}_z + \mathbf{e}_t(\mathbf{r})$$
$$\mathbf{h}(\mathbf{r}) = h_z(\mathbf{r})\mathbf{u}_z + \mathbf{h}_t(\mathbf{r})$$

Maxwell ⇨

$$\nabla_t \times e_z(\mathbf{r})\mathbf{u}_z + \mathbf{u}_z \times \frac{\partial}{\partial z}\mathbf{e}_t(\mathbf{r}) = -j\omega\mu(\boldsymbol{\rho})\mathbf{h}_t(\mathbf{r})$$
$$\nabla_t \times \mathbf{e}_t(\mathbf{r}) = -j\omega\mu(\boldsymbol{\rho})h_z(\mathbf{r})\mathbf{u}_z$$
$$\nabla_t \times h_z(\mathbf{r})\mathbf{u}_z + \mathbf{u}_z \times \frac{\partial}{\partial z}\mathbf{h}_t(\mathbf{r}) = j\omega\epsilon(\boldsymbol{\rho})\mathbf{e}_t(\mathbf{r})$$
$$\nabla_t \times \mathbf{h}_t(\mathbf{r}) = j\omega\epsilon(\boldsymbol{\rho})e_z(\mathbf{r})\mathbf{u}_z$$

Separation of variables ⇨

$$e_z \sim f(z) \qquad \mathbf{e}_t \sim g(z)$$
$$h_z \sim g(z) \qquad \mathbf{h}_t \sim f(z)$$

\+

$$\frac{d}{dz}f(z) \sim g(z)$$
$$\frac{d}{dz}g(z) \sim f(z)$$

General eigenmode equations

$$\frac{d}{dz}f(z) \sim g(z) \qquad \frac{d}{dz}g(z) \sim f(z) \quad \Rightarrow \quad \frac{d^2}{dz^2}f(z) \sim f(z) \quad \Rightarrow \quad f(z) = Ae^{-j\gamma z} + Be^{j\gamma z}$$

$\gamma = \beta - j\,\alpha$

propagation factor

propagation constant

attenuation constant

General eigenmode representation

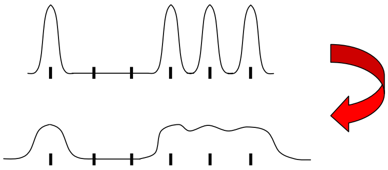
$$\begin{aligned}\mathbf{e}(x,y,z,\omega) &= K^{+}e^{-\alpha z}e^{-j\beta z}[\mathbf{E}_t(x,y,\omega) + E_z(x,y,\omega)\mathbf{u}_z] \\ &\quad + K^{-}e^{+\alpha z}e^{+j\beta z}[\mathbf{E}_t(x,y,\omega) - E_z(x,y,\omega)\mathbf{u}_z] \\ \mathbf{h}(x,y,z,\omega) &= K^{+}e^{-\alpha z}e^{-j\beta z}[\mathbf{H}_t(x,y,\omega) + H_z(x,y,\omega)\mathbf{u}_z] \\ &\quad - K^{-}e^{+\alpha z}e^{+j\beta z}[\mathbf{H}_t(x,y,\omega) - H_z(x,y,\omega)\mathbf{u}_z)\end{aligned}$$

Eigenmode velocity - dispersion

Time domain representation of a mode

$$\mathbf{e}(\mathbf{r},t) = \Re[\mathbf{E}(\boldsymbol{\rho})e^{-j\gamma z + j\omega t}] = \Re[\mathbf{E}(\boldsymbol{\rho})e^{-j\beta z + j\omega t}]e^{-\alpha z}$$

Remember that β and α are in general frequency dependent !!!!!
This is called **mode dispersion**.



effect of attenuation and dispersion on signal propagation

Phase velocity

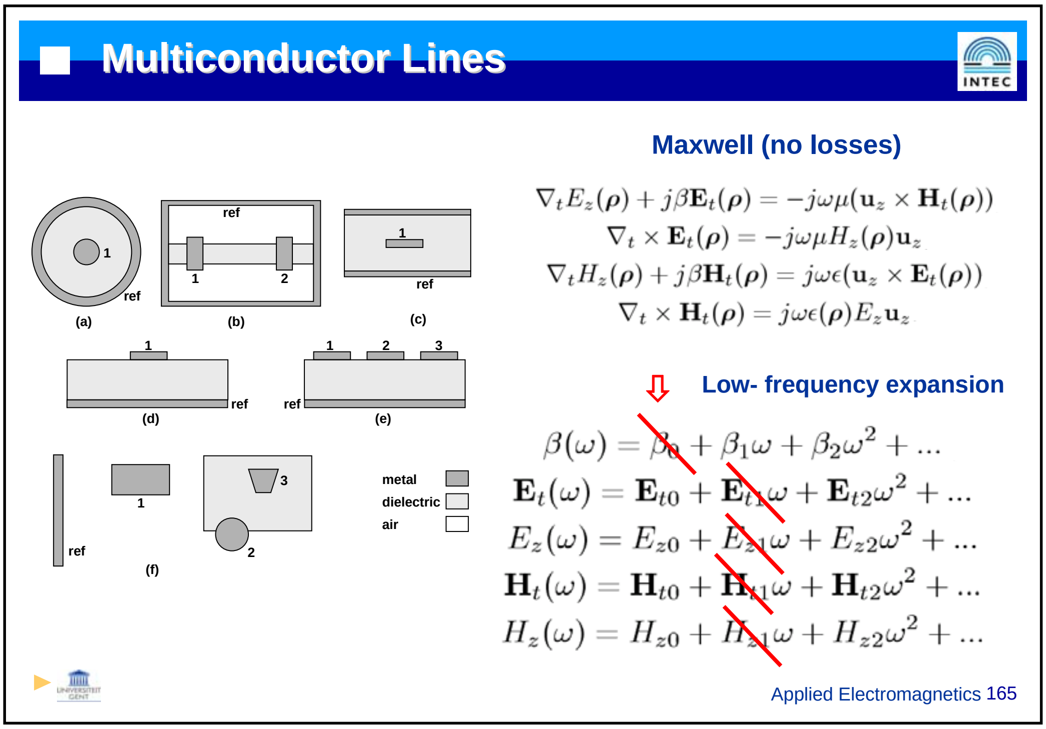
$$v_p = \frac{\omega}{\beta}$$

Group velocity

$$v_g = \frac{1}{\frac{d\beta}{d\omega}} = \frac{d\omega}{d\beta}$$

No dispersion

$v_g = v_p = V\omega$



Multiconductor Lines

Maxwell (no losses)

$$\nabla_t E_z(\boldsymbol{\rho}) + j\beta \mathbf{E}_t(\boldsymbol{\rho}) = -j\omega\mu(\mathbf{u}_z \times \mathbf{H}_t(\boldsymbol{\rho}))$$
$$\nabla_t \times \mathbf{E}_t(\boldsymbol{\rho}) = -j\omega\mu H_z(\boldsymbol{\rho})\mathbf{u}_z$$
$$\nabla_t H_z(\boldsymbol{\rho}) + j\beta \mathbf{H}_t(\boldsymbol{\rho}) = j\omega\epsilon(\mathbf{u}_z \times \mathbf{E}_t(\boldsymbol{\rho}))$$
$$\nabla_t \times \mathbf{H}_t(\boldsymbol{\rho}) = j\omega\epsilon(\boldsymbol{\rho})E_z\mathbf{u}_z$$

⇩ **Low- frequency expansion**

$$\beta(\omega) = \cancel{\beta_0} + \beta_1\omega + \beta_2\omega^2 + \ldots$$
$$\mathbf{E}_t(\omega) = \mathbf{E}_{t0} + \cancel{\mathbf{E}_{t1}}\omega + \mathbf{E}_{t2}\omega^2 + \ldots$$
$$E_z(\omega) = E_{z0} + \cancel{E_{z1}}\omega + E_{z2}\omega^2 + \ldots$$
$$\mathbf{H}_t(\omega) = \mathbf{H}_{t0} + \cancel{\mathbf{H}_{t1}}\omega + \mathbf{H}_{t2}\omega^2 + \ldots$$
$$H_z(\omega) = H_{z0} + \cancel{H_{z1}}\omega + H_{z2}\omega^2 + \ldots$$

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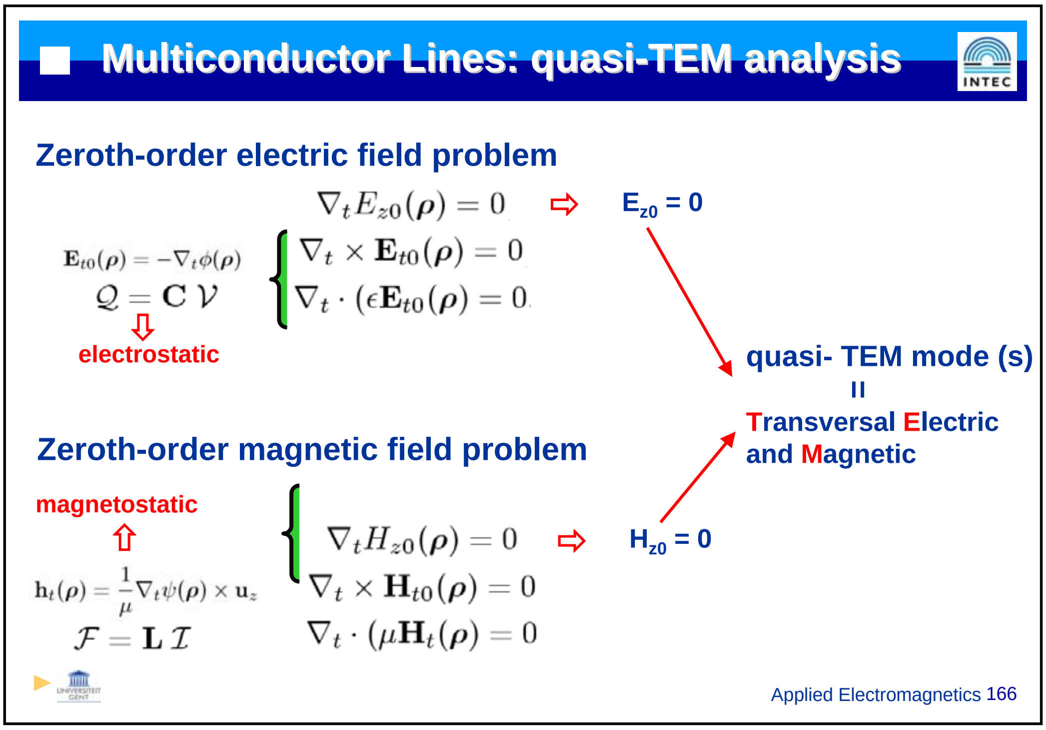

Multiconductor Lines: quasi-TEM analysis

Zeroth-order electric field problem

$$\nabla_t E_{z0}(\boldsymbol{\rho}) = 0 \quad ⇨ \quad E_{z0} = 0$$

$$\mathbf{E}_{t0}(\boldsymbol{\rho}) = -\nabla_t\phi(\boldsymbol{\rho}) \qquad \begin{cases} \nabla_t \times \mathbf{E}_{t0}(\boldsymbol{\rho}) = 0 \\ \nabla_t \cdot (\epsilon\mathbf{E}_{t0}(\boldsymbol{\rho})) = 0 \end{cases}$$

$$\mathcal{Q} = \mathbf{C}\,\mathcal{V}$$

⇩

electrostatic

quasi- TEM mode (s)

=

Transversal Electric and Magnetic

Zeroth-order magnetic field problem

magnetostatic

⇧

$$\begin{cases} \nabla_t H_{z0}(\boldsymbol{\rho}) = 0 \quad ⇨ \quad H_{z0} = 0 \\ \nabla_t \times \mathbf{H}_{t0}(\boldsymbol{\rho}) = 0 \\ \nabla_t \cdot (\mu\mathbf{H}_t(\boldsymbol{\rho})) = 0 \end{cases}$$

$$\mathbf{h}_t(\boldsymbol{\rho}) = \frac{1}{\mu}\nabla_t\psi(\boldsymbol{\rho}) \times \mathbf{u}_z$$

$$\mathcal{F} = \mathbf{L}\,\mathcal{I}$$

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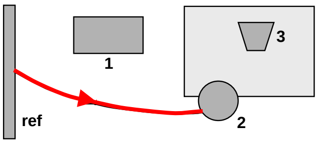
Quasi-TEM analysis: telegrapher's eqns.

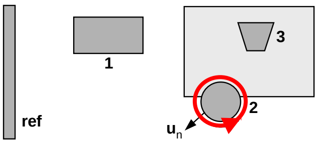


voltage-flux relation

$$\int_{ref}^{m} \nabla_t E_z \cdot \mathbf{dl} + j\beta \int_{ref}^{m} \mathbf{E}_t \cdot \mathbf{dl} = -j\omega\mu \int_{ref}^{m} (\mathbf{u}_z \times \mathbf{H}_t) \cdot \mathbf{dl}$$

$$\Downarrow$$

$$j\beta V_m = j\omega F_m \quad \Rightarrow \quad \frac{\partial}{\partial z} v_m(z,t) = -\frac{\partial}{\partial t} f_m(z,t)$$



conservation of charge

$$\oint_m \nabla_t H_z \cdot \mathbf{dl} + j\beta \oint_m \mathbf{H}_t \cdot \mathbf{dl} = j\omega\epsilon \oint_m (\mathbf{u}_z \times \mathbf{E}_t) \cdot \mathbf{dl}$$

$$\Downarrow$$

$$j\beta I_m = j\omega q_m \quad \Rightarrow \quad \frac{\partial}{\partial z} i_m(z,t) = -\frac{\partial}{\partial t} q_m(z,t)$$

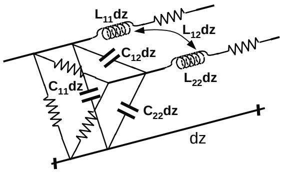
Telegrapher's equations

$$+ \quad \begin{array}{l} \mathcal{Q} = \mathbf{C}\,\mathcal{V} \\ \mathcal{F} = \mathbf{L}\,\mathcal{I} \end{array} \quad \Rightarrow \quad \begin{array}{l} \frac{\partial}{\partial z}\mathcal{V}(z,t) = -\mathbf{L}\frac{\partial}{\partial t}\mathcal{I}(z,t) \\ \frac{\partial}{\partial z}\mathcal{I}(z,t) = -\mathbf{C}\frac{\partial}{\partial t}\mathcal{V}(z,t) \end{array}$$

Quasi-TEM analysis: telegrapher's eqns.

Telegrapher's equations
time domain

$$\frac{\partial}{\partial z}\mathcal{V}(z,t) = -\mathbf{L}\frac{\partial}{\partial t}\mathcal{I}(z,t)$$
$$\frac{\partial}{\partial z}\mathcal{I}(z,t) = -\mathbf{C}\frac{\partial}{\partial t}\mathcal{V}(z,t)$$



lumped element section
(2 signal conductors)

Telegrapher's equations
frequency domain

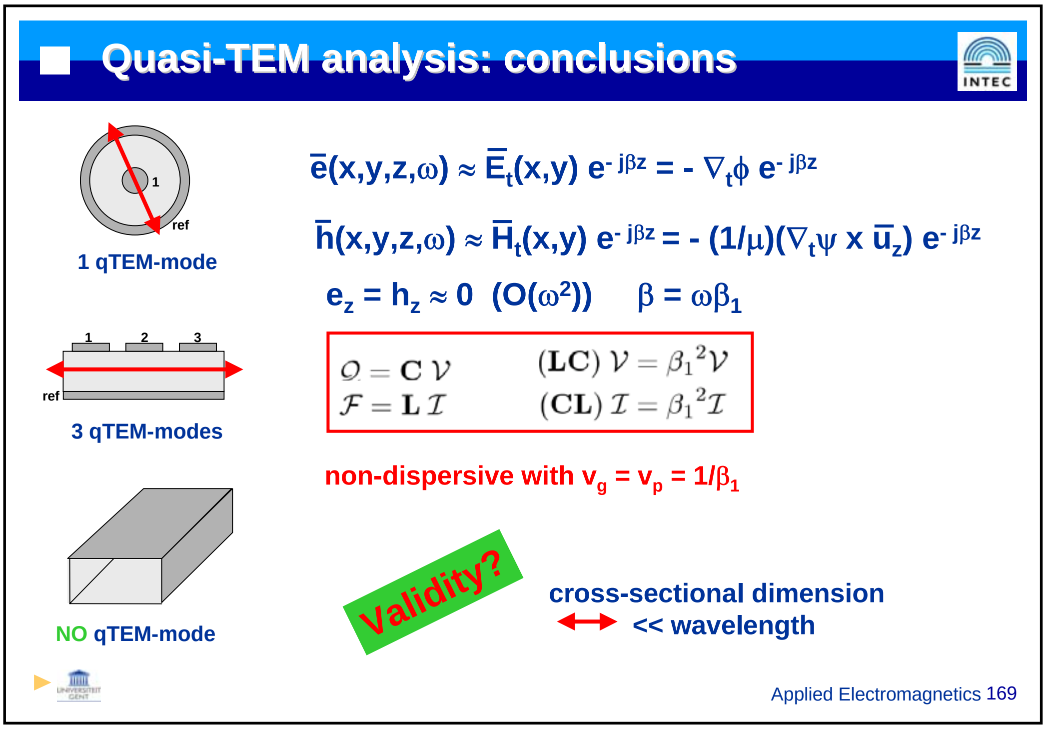
$$\frac{\partial}{\partial z}\mathcal{V}(z,\omega) = -j\omega\mathbf{L}\mathcal{I}(z,\omega)$$
$$\frac{\partial}{\partial z}\mathcal{I}(z,\omega) = -j\omega\mathbf{C}\mathcal{V}(z,\omega)$$

Propagation constants (β = jω β_1)

$$(\mathbf{LC})\,\mathcal{V} = {\beta_1}^2\mathcal{V}$$
$$(\mathbf{CL})\,\mathcal{I} = {\beta_1}^2\mathcal{I}$$

$$v_p = v_g = \frac{1}{\beta_{1m}}$$

M+1 conductors ⇨ M TEM modes



Quasi-TEM analysis: conclusions

1 qTEM-mode

3 qTEM-modes

NO qTEM-mode

$\bar{e}(x,y,z,\omega) \approx \bar{E}_t(x,y)\, e^{-j\beta z} = -\nabla_t\phi\, e^{-j\beta z}$

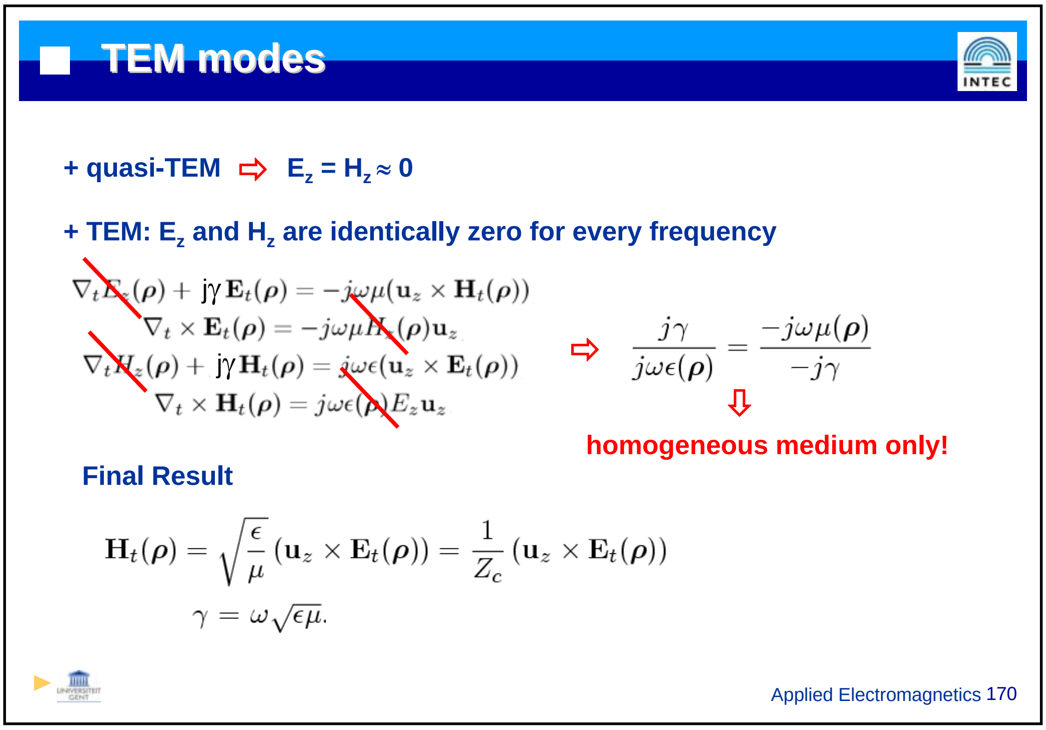
$\bar{h}(x,y,z,\omega) \approx \bar{H}_t(x,y)\, e^{-j\beta z} = -(1/\mu)(\nabla_t\psi \times \bar{u}_z)\, e^{-j\beta z}$

$e_z = h_z \approx 0 \quad (O(\omega^2)) \qquad \beta = \omega\beta_1$

$$\boxed{\begin{array}{ll} \mathcal{Q} = \mathbf{C}\,\mathcal{V} & (\mathbf{LC})\,\mathcal{V} = {\beta_1}^2\mathcal{V} \\ \mathcal{F} = \mathbf{L}\,\mathcal{I} & (\mathbf{CL})\,\mathcal{I} = {\beta_1}^2\mathcal{I} \end{array}}$$

non-dispersive with $v_g = v_p = 1/\beta_1$

cross-sectional dimension
↔ << wavelength



TEM modes

+ quasi-TEM ⇨ $E_z = H_z \approx 0$

+ TEM: E_z and H_z are identically zero for every frequency

$$\begin{aligned} \nabla_t \cancel{E_z}(\boldsymbol{\rho}) + j\gamma\,\mathbf{E}_t(\boldsymbol{\rho}) &= -j\omega\mu(\mathbf{u}_z \times \mathbf{H}_t(\boldsymbol{\rho})) \\ \nabla_t \times \mathbf{E}_t(\boldsymbol{\rho}) &= -j\omega\mu \cancel{H_z(\boldsymbol{\rho})\mathbf{u}_z} \\ \nabla_t \cancel{H_z}(\boldsymbol{\rho}) + j\gamma\,\mathbf{H}_t(\boldsymbol{\rho}) &= j\omega\epsilon(\mathbf{u}_z \times \mathbf{E}_t(\boldsymbol{\rho})) \\ \nabla_t \times \mathbf{H}_t(\boldsymbol{\rho}) &= j\omega\epsilon \cancel{(\boldsymbol{\rho})E_z\mathbf{u}_z} \end{aligned}$$

⇨ $$\frac{j\gamma}{j\omega\epsilon(\boldsymbol{\rho})} = \frac{-j\omega\mu(\boldsymbol{\rho})}{-j\gamma}$$

⇩

homogeneous medium only!

Final Result

$$\mathbf{H}_t(\boldsymbol{\rho}) = \sqrt{\frac{\epsilon}{\mu}}(\mathbf{u}_z \times \mathbf{E}_t(\boldsymbol{\rho})) = \frac{1}{Z_c}(\mathbf{u}_z \times \mathbf{E}_t(\boldsymbol{\rho}))$$

$$\gamma = \omega\sqrt{\epsilon\mu}.$$

TE and TM modes

+ TM modes: $H_z = 0$ for all frequencies but still H_x H_y E_x E_y E_z

+ TE modes: $E_z = 0$ for all frequencies but still E_x E_y H_x H_y H_z

only possible for homogeneous media!

TE modes

$$\mathbf{E}_t = \frac{j\omega\mu}{k^2-\gamma^2}(\mathbf{u}_z \times \nabla_t H_z)$$

$$\mathbf{H}_t = \frac{\gamma}{\omega\mu}(\mathbf{u}_z \times \mathbf{E}_t) \qquad Z_{TE} = \frac{\omega\mu}{\gamma}$$

$$E_z = 0$$

$$\nabla_t^2 H_z + (k^2-\gamma^2)H_z = 0$$

$$\frac{\partial H_z}{\partial n} = 0 \qquad \underbrace{\text{on the conductor surfaces}}_{\text{Neumann problem}}$$

TM modes

$$\mathbf{E}_t = -\frac{\gamma}{\omega\epsilon}(\mathbf{u}_z \times \mathbf{H}_t) \qquad Z_{TM} = \frac{\gamma}{\omega\epsilon}$$

$$\mathbf{H}_t = -\frac{j\omega\epsilon}{k^2-\gamma^2}(\mathbf{u}_z \times \nabla_t E_z)$$

$$H_z = 0$$

$$\nabla_t^2 E_z + (k^2-\gamma^2)E_z = 0$$

$$E_z = 0 \qquad \underbrace{\text{on the conductor surfaces}}_{\text{Dirichlet problem}}$$

Mode orthogonality

The following general orthogonality theorem holds for ALL modes

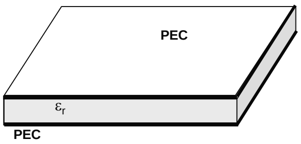
$$\int_S (\mathbf{E}_1 \times \mathbf{H}_2).\mathbf{u}_z dS = \int_S (\mathbf{E}_{1t} \times \mathbf{H}_{2t}).\mathbf{u}_z dS = 0$$

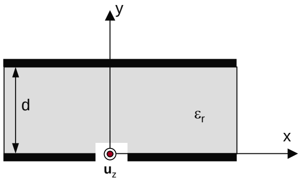
S: cross-section of the waveguide (can be infinite)
E_1: electric field of mode 1 (only the transversal field plays a role)
H_2: magnetic field of mode 2 (only the transversal field plays a role)

(in the notes the example of the TM modes is given -
the general proof uses Lorentz reciprocity theorem)

Parallel-plate waveguide

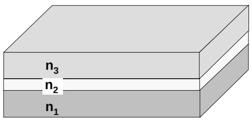
Parellel-plate waveguide





Purpose
determine x-independent modes propagating along z

Optical slab waveguide



homogeneous filling + 2 conductors

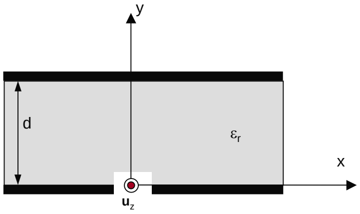
⇩

1 TEM mode
TE modes
TM modes

Parallel-plate waveguide: TEM mode

e(y,z,ω) = - $\nabla_t \phi$ $e^{-j\beta z}$

$$\frac{\mathrm{d}^2}{\mathrm{d}y^2}\phi(y) = 0$$ **ϕ (x=0) = 0   ϕ (x=d) = V**



$$\mathbf{E}_t(y) = -\frac{\mathrm{d}}{\mathrm{d}y}\phi(y)\mathbf{u}_y = -\frac{V}{d}\mathbf{u}_y$$

$$\mathbf{H}_t(y) = \frac{V}{dZ_c}\mathbf{u}_x \quad Z_c = \sqrt{\frac{\mu}{\epsilon}}$$ **= (L / C)$^{1/2}$**

$$C_{pp} = \frac{\epsilon}{d}$$
$$L_{pp} = d\mu$$

⇨ **v = ω / β = 1 / (LC)$^{1/2}$ = 1 / (εμ)$^{1/2}$**

non-dispersive!

Parallel-plate waveguide: losses

Transmission line equivalent and small losses

$$\gamma = \beta - j\alpha = \sqrt{-(R+j\omega L)(G+j\omega C)} \approx \omega\sqrt{LC} - j\frac{G}{2}\sqrt{\frac{L}{C}} - j\frac{R}{2}/\sqrt{\frac{L}{C}}$$

i.e. the effect of G and R can be evaluated independently

Dielectric losses

+ start from the lossless case: R = G = 0

+ replace C by C + G / (jω) in the transmission line equivalent

+ replace ε by ε + σ /(jω) in the dielectric

$$C_{pp} = \frac{\epsilon}{d} \quad ⇨ \quad C_{pp} + G_{pp} / (j\omega) = \varepsilon/d + \sigma/d\ (1/(j\omega))$$

⇩

$$G_{pp} = \frac{\sigma}{d}$$

Parallel-plate waveguide: losses

Conductor losses: skin-effect losses

+ start from the lossless case (PEC conductors)

+ calculate surface currents: $\mathbf{J}_s(\boldsymbol{\rho}) = \mathbf{u}_n \times \mathbf{H}(\boldsymbol{\rho})$

+ Joule losses per unit of length:

$$P_{losses} = \frac{1}{2}\Re\int_c \mathbf{J}_s \cdot \mathbf{E}^* dc = \frac{1}{2}R_s\int_c |\mathbf{J}_s|^2 dc$$

surface impedance

$$Z_s = R_s + jX_s \qquad R_s = \frac{1}{\sigma\delta} \qquad \delta = \sqrt{\frac{2}{\omega\mu\sigma}}$$

skin depth

+ propagated power:

$$P_{prop} = \frac{1}{2}\Re\int_S (\mathbf{E}(z) \times \mathbf{H}^*(z)) \cdot \mathbf{u}_z dS$$

Parallel-plate waveguide: losses

+ **relationship between decrease of propagated power and losses:**

$$\frac{\mathrm{d}}{\mathrm{d}z}P_{prop} = -P_{losses}$$

+ **proportionality between dissipated power and propagated power:**

$$P_{losses} = 2\alpha P_{prop}$$

Final result

$$\boxed{\begin{gathered} P_{prop}(z) = P_{prop}(z=0)e^{-2\alpha z} \\ \gamma = \beta - j\alpha \\ \alpha = \frac{P_{losses}}{2P_{prop}} \end{gathered}}$$

⇨

power decays with a factor e$^{-2\alpha z}$

fields decay with a factor e$^{-\alpha z}$

$$\alpha_{pp} = \frac{R_s}{dZ_c}$$

Parallel-plate waveguide: losses

Relationship between losses and transmission line equivalent

$$\gamma = \beta - j\alpha = \sqrt{-(R+j\omega L)(G+j\omega C)} \approx \omega\sqrt{LC} - j\underbrace{\frac{G}{2}\sqrt{\frac{L}{C}}}_{\alpha_{diel}} - j\underbrace{\frac{R}{2}/\sqrt{\frac{L}{C}}}_{\alpha_{cond}}$$

$$\begin{aligned} \alpha_{pp} &= \alpha_{cond} + \alpha_{diel} \\ &= \frac{R_s}{dZ_c} + \frac{\sigma Z_c}{2} \end{aligned} \quad ⇨ \quad R_{pp} = \frac{2R_s}{d}$$

Parallel-plate waveguide: TE and TM modes

TE modes

$$\frac{\mathrm{d}^2}{\mathrm{d}y^2}H_z + (k^2-\gamma^2)H_z = 0 \qquad \frac{\mathrm{d}}{\mathrm{d}y}H_z(y=0) = \frac{\mathrm{d}}{\mathrm{d}y}H_z(y=d) = 0.$$

$$\Downarrow$$

$$\left.\begin{aligned} H_{zn}(y) &= B_n \cos\frac{n\pi y}{d} \\ \mathbf{E}_{tn}(y) &= \frac{j\omega\mu d}{n\pi}B_n \sin\frac{n\pi y}{d}\mathbf{u}_x \\ \mathbf{H}_{tn}(y) &= \frac{jd\gamma_n}{n\pi}B_n \sin\frac{n\pi y}{d}\mathbf{u}_y \end{aligned}\right\} e^{-j\gamma_n z}$$

with $\gamma_n = \sqrt{k^2 - \frac{n^2\pi^2}{d^2}}$

n = 1, 2, 3, ….

Parallel-plate waveguide: TE and TM modes

TM modes

$$\frac{\mathrm{d}^2}{\mathrm{d}y^2}E_z + (k^2-\gamma^2)E_z = 0 \qquad E_z(y=0) = E_z(y=d) = 0$$

$$\Downarrow$$

$$\left.\begin{aligned} E_{zn}(y) &= A_n \sin\frac{n\pi y}{d} \\ \mathbf{H}_{tn}(y) &= \frac{j\omega\epsilon d}{n\pi}A_n \cos\frac{n\pi y}{d}\mathbf{u}_x \\ \mathbf{E}_{tn}(y) &= -\frac{jd\gamma_n}{n\pi}A_n \cos\frac{n\pi y}{d}\mathbf{u}_y \end{aligned}\right\} e^{-j\gamma_n z}$$

with $\gamma_n = \sqrt{k^2 - \frac{n^2\pi^2}{d^2}}$

n = 1, 2, 3, ….

Remark: orthogonality and completeness

Cut-off frequency

z-dependence: $e^{-j\gamma_n z}$ **with** $\gamma_n = \sqrt{k^2 - \frac{n^2\pi^2}{d^2}}$ **and** $\gamma_n = \beta_n - j\alpha_n$

Case 1: $n\pi/d < k$ **or** $\lambda < 2d / n$ ⇨ $\gamma_n = \beta_n = \sqrt{k^2 - \frac{n^2\pi^2}{d^2}}$ **> 0** ⇨ $e^{-j\beta_n z}$

the mode with index n is a propagating mode!

Case 2: $n\pi/d > k$ **or** $\lambda > 2d / n$ ⇨ $\alpha_n = \sqrt{\frac{n^2\pi^2}{d^2} - k^2}$ **> 0** ⇨ $e^{-\alpha_n z}$

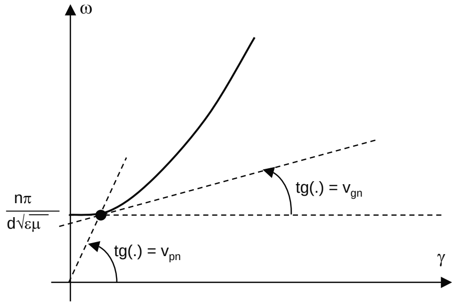
the mode with index n is an evanescent mode!

Cut-off frequency of mode n: $f_{cn} = \frac{n}{2d\sqrt{\epsilon\mu}}$ **(zero for a TEM mode)**

Phase and group velocity

dispersion: **the propagation factor γ of a mode is frequency dependent**

$\gamma_n = \beta_n = \sqrt{k^2 - \frac{n^2\pi^2}{d^2}}$ **for the TE and TM modes**



dispersion relation i.e. γ (ω)

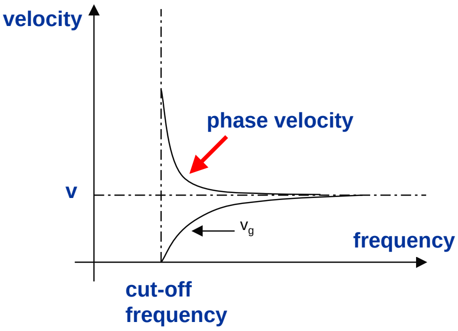
Phase and group velocity

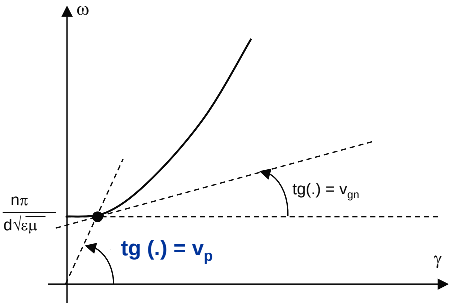
Phase velocity of mode n

velocity in infinite space

$$v_{pn} = \frac{\omega}{\gamma_n} = \frac{\omega}{\sqrt{k^2 - \frac{n^2\pi^2}{d^2}}} = \frac{v}{\sqrt{1-(\frac{n\pi}{kd})^2}}$$

> v !





Phase and group velocity

Group velocity of mode n

+ consider a frequency modulated wave: $\Re[e^{j(\omega(t)t-\gamma_n(\omega)z)}]$

+ consider a modulation around central frequency ω_0

$$\omega(t) = \omega_0 + \Delta\omega(t)$$

$$\gamma_n = \gamma_{n0} + [\frac{d\gamma_n}{d\omega}]_{\omega=\omega_0}\Delta\omega(t)$$

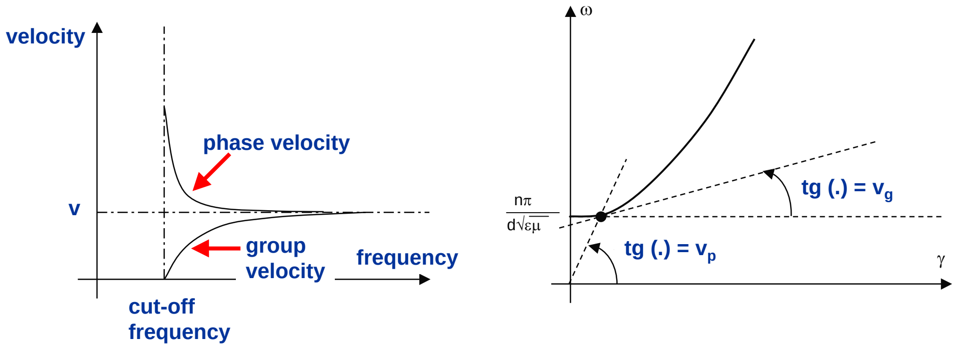
+ result: $\Re[e^{j(\omega_0 t-\gamma_{n0}z)}e^{-j\Delta\omega(z/v_g-t)}]$

$$v_{gn} = v\sqrt{1-(\frac{n\pi}{kd})^2} \quad \Longleftrightarrow \quad v_{pn} = \frac{v}{\sqrt{1-(\frac{n\pi}{kd})^2}}$$

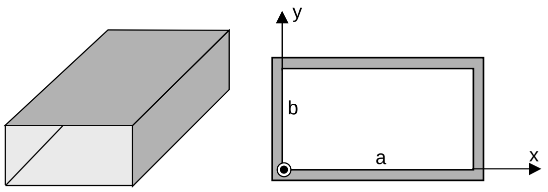
Phase and group velocity

Group velocity of mode n

$$v_{gn} = v\sqrt{1-(\frac{n\pi}{kd})^2}$$



Rectangular waveguide



type	width a (cm)	height b (cm)	cut-off frequency (GHz)	frequency band (GHz)
EIA-WR-340	8.636	4.318	1.737	2.17 - 3.30
EIA-WR-184	7.214	3.404	2.079	2.60 - 3.95

+ no TEM mode

+ TE modes

+ TM modes

+ in practical cases: a = 2b

Rectangular waveguide

TE modes

$$H_{z,nm}(x,y) = B_{nm}\cos\frac{n\pi x}{a}\cos\frac{m\pi y}{b}$$

$$\mathbf{E}_{t,nm}(x,y) = \frac{j\omega\mu B_{nm}}{\frac{n^2\pi^2}{a^2}+\frac{m^2\pi^2}{b^2}}[-\frac{n\pi}{a}\sin\frac{n\pi x}{a}\cos\frac{m\pi y}{b}\mathbf{u}_y + \frac{m\pi}{b}\cos\frac{n\pi x}{a}\sin\frac{m\pi y}{b}\mathbf{u}_x]$$

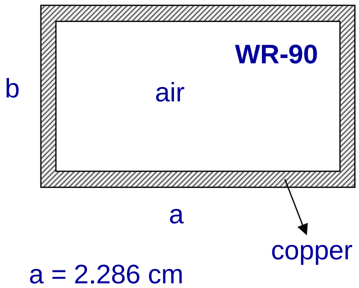
$$\mathbf{H}_{t,nm}(x,y) = \frac{j\gamma_{nm} B_{nm}}{\frac{n^2\pi^2}{a^2}+\frac{m^2\pi^2}{b^2}}[\frac{n\pi}{a}\sin\frac{n\pi x}{a}\cos\frac{m\pi y}{b}\mathbf{u}_x + \frac{m\pi}{b}\cos\frac{n\pi x}{a}\sin\frac{m\pi y}{b}\mathbf{u}_y]$$

cut-off frequencies

$$\gamma_{nm} = \sqrt{k^2 - \frac{n^2\pi^2}{a^2} - \frac{m^2\pi^2}{b^2}} \qquad f_{c,nm} = \frac{1}{2\sqrt{\epsilon\mu}}\sqrt{\frac{n^2}{a^2}+\frac{m^2}{b^2}}$$

Rectangular waveguide: example

X-band waveguide: 8 - 12 GHz



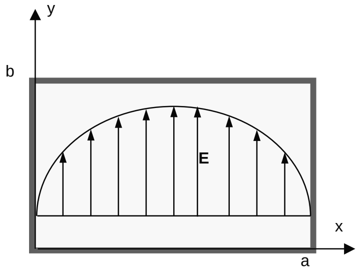
a = 2.286 cm
b = 1.016 cm

(attenuation at 10 GHz: 0.11 dB/m)

Mode	m	n	f_c (GHz)
TE	1	0	6.562
TE	2	0	13.123
TE	0	1	14.764
TE, TM	1	1	16.156
TE, TM	1	2	30.248
TE, TM	2	1	19.753

Rectangular waveguide: TE$_{10}$ mode

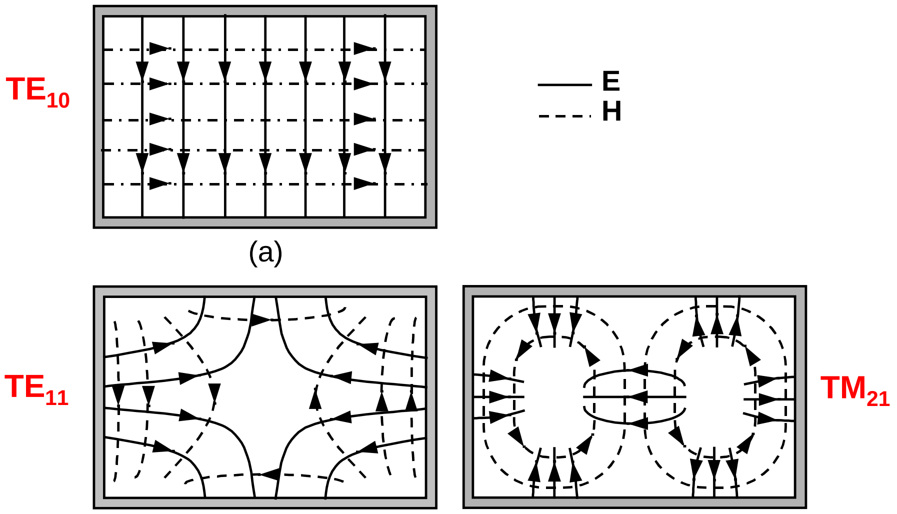
The TE$_{10}$ mode is the only one used in practice ⇨ monomode operation !



$$\left.\begin{aligned} E_{x,10} = E_{z,10} = H_{y,10} &= 0 \\ E_{y,10} &= -\frac{j\omega\mu}{\pi/a} B_{10} \sin\frac{\pi x}{a} \\ H_{x,10} &= \frac{j\sqrt{k^2-(\frac{\pi}{a})^2}}{(\pi/a)} B_{10} \sin\frac{\pi x}{a} \\ H_{z,10} &= B_{10} \cos\frac{\pi x}{a} \end{aligned}\right\} e^{-j\sqrt{k^2-(\frac{\pi}{a})^2}z}$$

a = 2b ⇨ monomode operation for $\frac{1}{2a\sqrt{\epsilon\mu}} < f < \frac{1}{a\sqrt{\epsilon\mu}}$

Field patterns for other waveguide modes

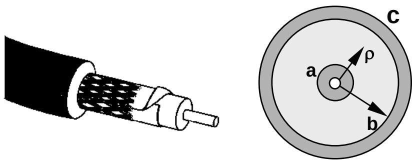


(a)

(b)

(c)

Coaxial cable: TEM mode



example: RG-142/U cable

R_c = 50Ω
a = 0.94mm   b = 2.95mm
c = 4.34mm
teflon filling with ε_r = 2.041

TEM mode

$$\nabla_t^2 \phi(\boldsymbol{\rho}) = 0$$

$$\phi(\rho = a) = 0 \qquad \phi(\rho = b) = V$$

$$\phi(\rho) = V \frac{\ln \frac{\rho}{a}}{\ln \frac{b}{a}}$$

$$\mathbf{E}_t(\rho) = -\nabla_t \phi(\rho) = -\frac{d}{d\rho}\phi(\rho)\mathbf{u}_\rho = -\frac{V}{\rho \ln \frac{b}{a}}\mathbf{u}_\rho$$

$$\mathbf{H}_t(\rho) = -\frac{V}{Z_c \rho \ln \frac{b}{a}}\mathbf{u}_\phi$$

$$e^{-j\gamma z} = e^{-jkz} = e^{-j\omega\sqrt{\epsilon\mu}z}$$

Coaxial cable: transmission line parameters

$$C_{coax} = \frac{2\pi\epsilon}{\ln \frac{b}{a}}$$

$$L_{coax} = \frac{\mu \ln \frac{b}{a}}{2\pi}$$

$$R_{c,coax} = \sqrt{\frac{L}{C}} = \sqrt{\frac{\mu}{\epsilon}} \frac{\ln \frac{b}{a}}{2\pi}$$

e.g. 50Ω

Losses

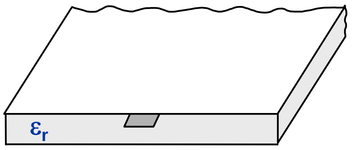
$$G_{coax} = \frac{2\pi\sigma}{\ln \frac{b}{a}}$$

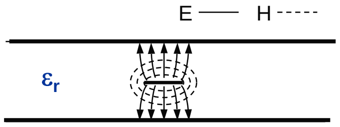
$$R_{coax} = \frac{R_s}{2\pi}\left(\frac{1}{a} + \frac{1}{b}\right)$$

$$\alpha_{coax} = \alpha_{cond} + \alpha_{diel} = \frac{R_s}{2Z_c \ln \frac{b}{a}}\left(\frac{1}{a} + \frac{1}{b}\right) + \frac{\sigma Z_c}{2}$$

cut-off frequency: determined by the TE_{11} mode

Stripline





+ a pure TEM mode exists

+ its capacitance C_{strip} per unit of length is found by solving the appropriate potential problem

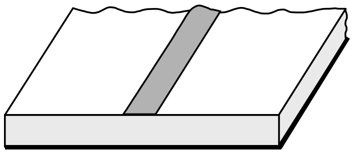
+ $L_{strip} = \frac{\epsilon_r}{C_{strip} c^2}$

c: velocity of light in free space

+ $R_{c,strip} = \frac{\sqrt{\epsilon_r}}{C_{strip} c}$

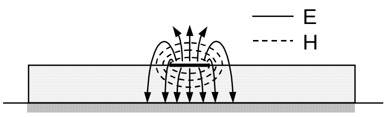
+ signal velocity: $c/\sqrt{\epsilon_r}$

Microstrip



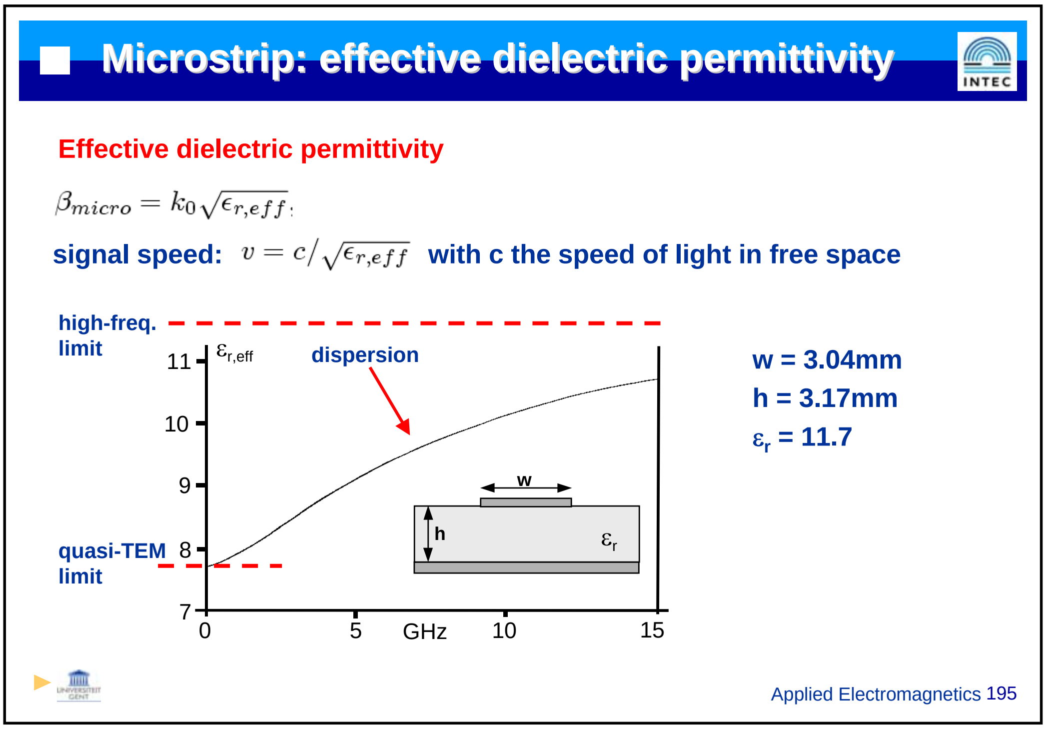
+ no pure TEM, TE or TM modes

+ for sufficiently low frequencies a quasi-TEM mode exists



$$R_{c,micro} = \sqrt{\frac{L_{micro}}{C_{micro}}}$$

$$\beta_{micro} = \omega\sqrt{L_{micro} C_{micro}}$$



Microstrip: effective dielectric permittivity

Effective dielectric permittivity

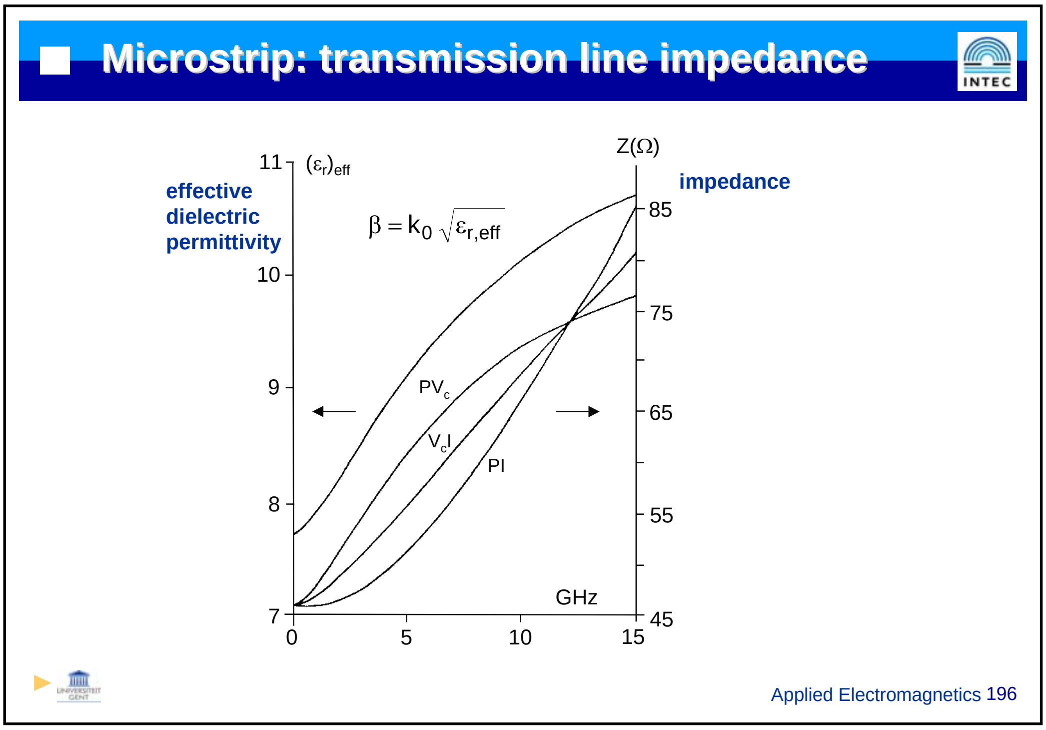
$\beta_{micro} = k_0\sqrt{\epsilon_{r,eff}}$:

signal speed: $v = c/\sqrt{\epsilon_{r,eff}}$ **with c the speed of light in free space**

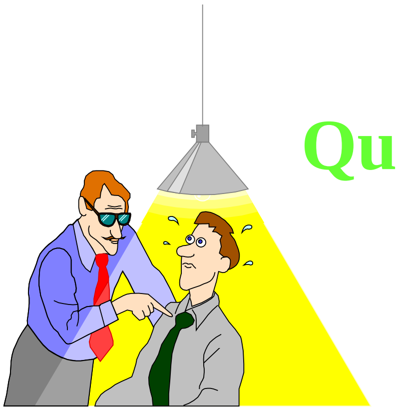
w = 3.04mm

h = 3.17mm

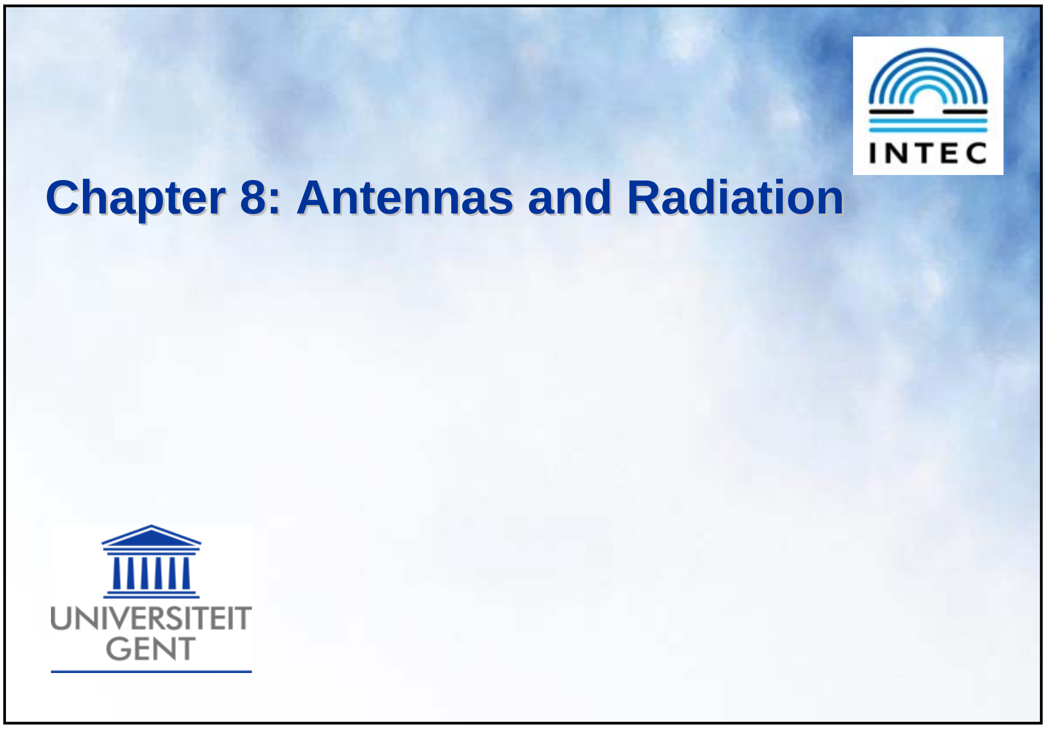
ε_r = 11.7



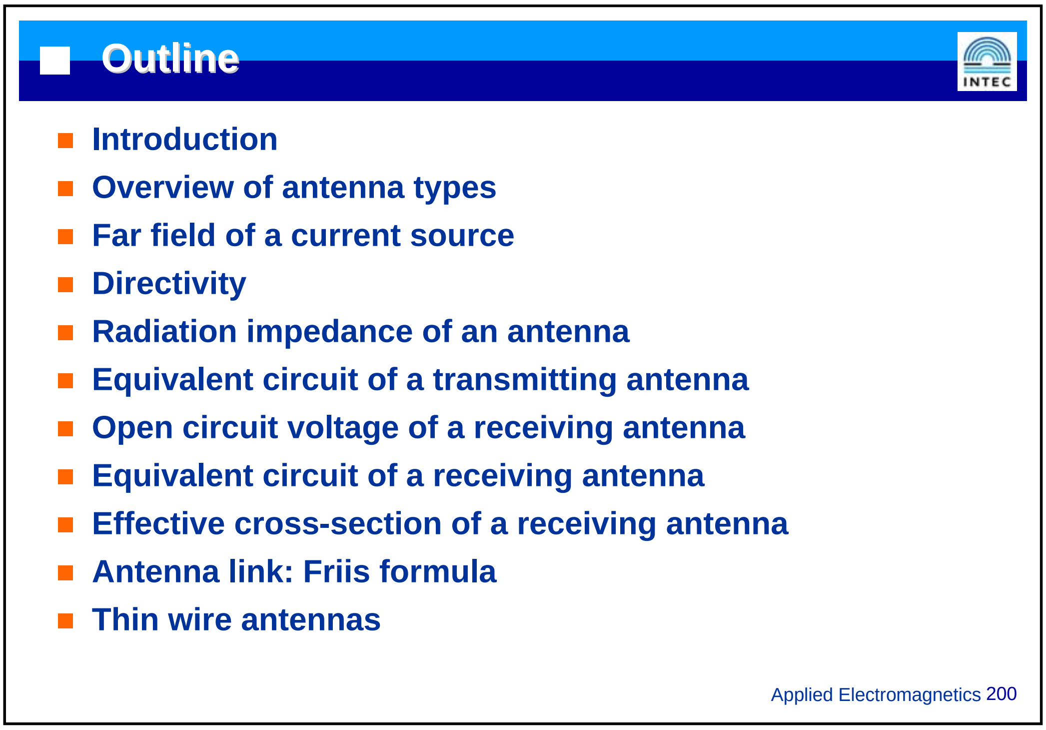
Microstrip: transmission line impedance



Questions?



Chapter 8: Antennas and Radiation



Outline

- Introduction
- Overview of antenna types
- Far field of a current source
- Directivity
- Radiation impedance of an antenna
- Equivalent circuit of a transmitting antenna
- Open circuit voltage of a receiving antenna
- Equivalent circuit of a receiving antenna
- Effective cross-section of a receiving antenna
- Antenna link: Friis formula
- Thin wire antennas

Introduction

The Concise Oxford Dictionary defines an antenna or aerial as

“ a metal rod, wire or other structure by which signals are transmitted or received as part of a radio transmission or receiving system ”

Purposes

+ to understand the physics of antennas

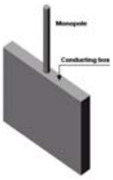
+ to obtain equivalent circuit representations for transmitting and receiving

+ to know more about wire antennas

+ to get a short overview of other antenna types

Overview of antenna types

monopole on a ground plane



monopole on a box

folded dipole

biconical antenna

wire grid antenna

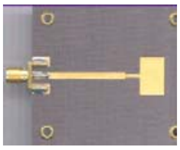
Yagi antenna

horn antennas

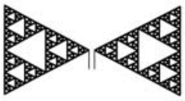
Overview of antenna types

spiral antenna

biconical patch antenna



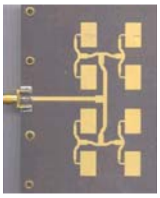
patch antenna with feed line



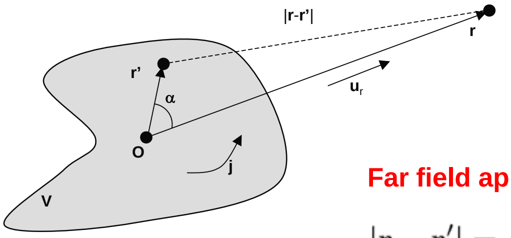
broadband antenna

radio telescope

patch antenna array



Far field of a current source



Vector potential of the source

$$\mathbf{a}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int_V \frac{e^{-jk|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}\mathbf{j}(\mathbf{r}')dV'$$

$$k = \omega\sqrt{\epsilon_0\mu_0}$$

Far field approximation

$$|\mathbf{r}-\mathbf{r}'| = \sqrt{r^2 + r'^2 - 2rr'\cos\alpha}$$
$$\approx r - r'\cos\alpha = r - \mathbf{r}'\cdot\mathbf{u}_r$$

$$\lim_{kr\to\infty}\mathbf{a}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{e^{-jkr}}{r}\int_V e^{jk\mathbf{u}_r\cdot\mathbf{r}'}\mathbf{j}(\mathbf{r}')dV' = \frac{e^{-jkr}}{r}\mathbf{N}(\theta,\phi)$$

Far field of a current source - cont.

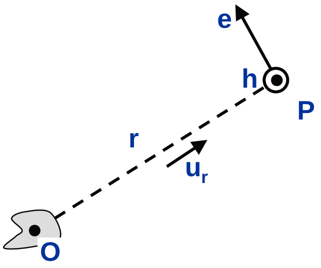
Far field at P (r >> λ)

$$\lim_{kr\to\infty} \mathbf{e}(\mathbf{r}) = \mathbf{F}(\theta,\phi)\frac{e^{-jkr}}{r}$$

$$\lim_{kr\to\infty} \mathbf{h}(\mathbf{r}) = \frac{1}{Z_c}[\mathbf{u}_r \times \mathbf{F}(\theta,\phi)]\frac{e^{-jkr}}{r}$$

$$\mathbf{F}(\theta,\phi) = j\omega\mathbf{u}_r \times [\mathbf{u}_r \times \mathbf{N}(\theta,\phi)]$$

$$Z_c = \sqrt{\frac{\epsilon_0}{\mu_0}}$$



Far field region

$$r \geq \frac{2d^2}{\lambda}$$

locally plane wave ⟷ globally spherical wave

Directivity

Poynting's vector for a particular direction $\mathbf{u}_r$

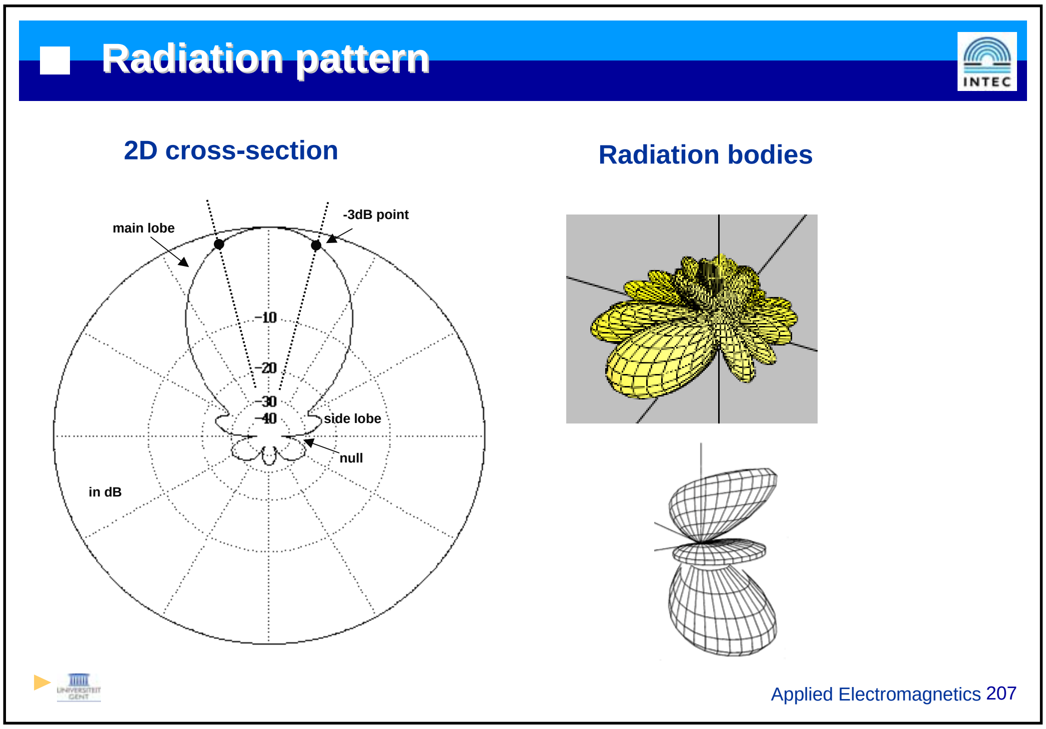
$$\mathbf{p}(\mathbf{r}) = \frac{|\mathbf{F}(\theta,\phi)|^2}{2r^2 Z_c}\mathbf{u}_r$$

Total radiated power (integration over a large sphere)

$$P_{tot} = \frac{1}{2Z_c}\int_0^{2\pi}\int_0^{\pi} |\mathbf{F}(\theta,\phi)|^2 \sin\theta d\theta d\phi$$

Directivity: ratio of radiation in space angle dΩ to mean value

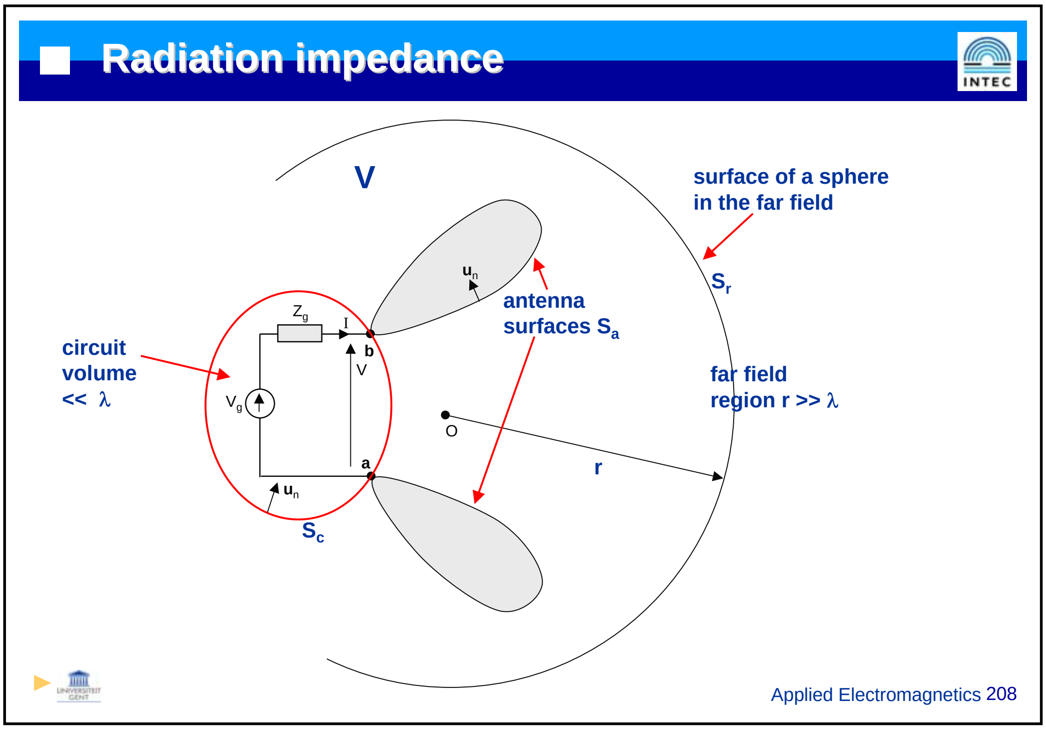
$$D(\theta,\phi) = \frac{\frac{|\mathbf{F}(\theta,\phi)|^2}{2Z_c}}{\frac{P_{tot}}{4\pi}} = \frac{4\pi|\mathbf{F}(\theta,\phi)|^2}{\int_\Omega |\mathbf{F}(\theta,\phi)|^2 d\Omega}$$



Radiation pattern

2D cross-section

Radiation bodies



Radiation impedance

Radiation impedance - cont.

Starting point: energy balance equation in V

$$\int_{S_a+S_c+S_r} (\mathbf{e}\times\mathbf{h}^*)\cdot\mathbf{u}_n dS = \int_V (-j\omega\mu|\mathbf{h}|^2 + j\omega\epsilon|\mathbf{e}|^2)dV$$

PEC

$$\int_{S_a} (\mathbf{e}\times\mathbf{h}^*)\cdot\mathbf{u}_n dS = 0$$

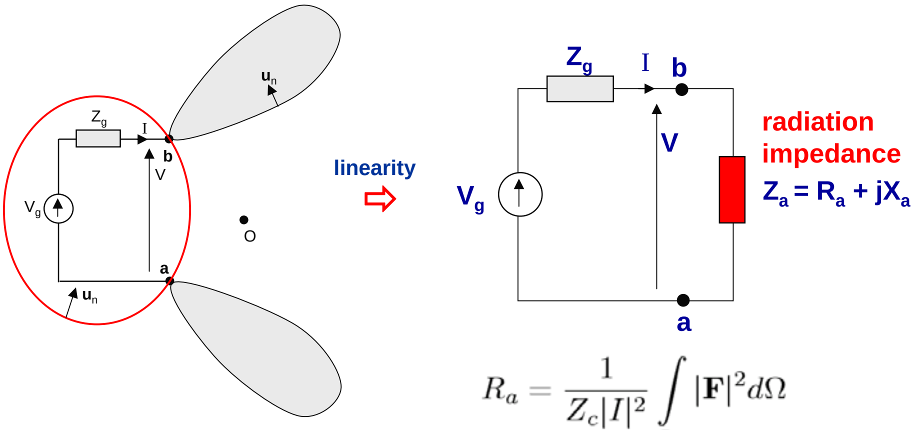
far field expressions

$$\int_{S_r} (\mathbf{e}\times\mathbf{h}^*)\cdot\mathbf{u}_n dS = \frac{1}{Z_c}\int_{\Omega} |\mathbf{F}|^2 d\Omega$$

low-frequency approximation

$$\int_{S_c} (\mathbf{e}\times\mathbf{h}^*)\cdot\mathbf{u}_n dS = -VI^*$$

Radiation impedance & equivalent circuit



linearity ⇨

radiation impedance

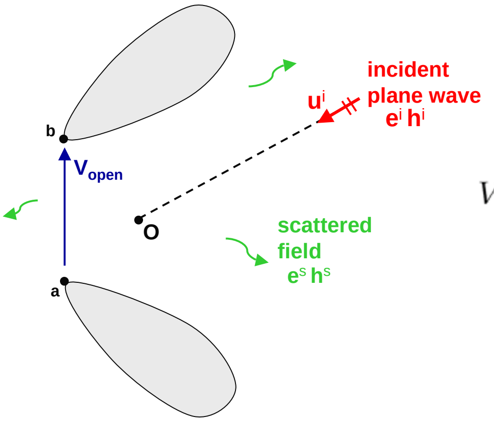
$Z_a = R_a + jX_a$

$$R_a = \frac{1}{Z_c|I|^2}\int_{\Omega} |\mathbf{F}|^2 d\Omega$$

$$X_a = -\frac{\omega}{|I|^2}\int_V (\epsilon|\mathbf{e}|^2 - \mu|\mathbf{h}|^2)dV$$

Open circuit voltage of a receiving antenna

What is the relation between an antenna as transmitter and the same antenna as a receiver?

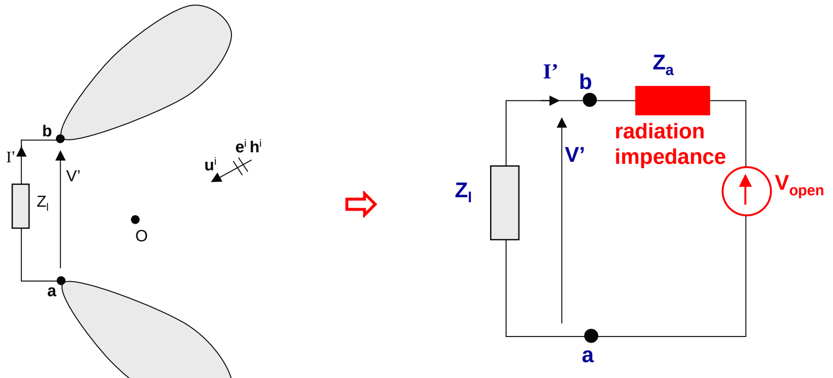


$$\mathbf{e}^i(\mathbf{r}) = \mathbf{E}_0 e^{-jk\mathbf{u}^i\cdot\mathbf{r}}$$

$$V_{open} = -\frac{2j\lambda}{Z_c I}\mathbf{E}_0 \cdot \mathbf{F}(-\mathbf{u}^i)$$

radiation property in the same direction

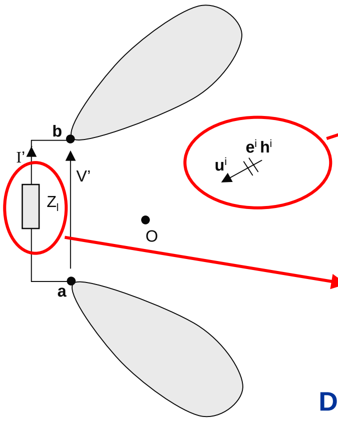
Equivalent circuit of a receiving antenna



Thevenin equivalent

+ **Thevenin source: the open circuit voltage**
+ **Thevening impedance: the radiation impedance of the (transmitter) antenna**

Effective cross-section



incident power P_{in}: $\frac{|\mathbf{E}_0|^2}{2Z_c}$ in W/m^2

field at origin O

received power at load: $P_{rec} = 1/2\ |I'|^2 R_l$ in W

DEFINITION

effective cross-section: $\sigma_{eff} = \frac{P_{rec}}{P_{in}}$ in m^2

Effective cross-section - cont.

Final result

wavelength

$$\sigma_{eff}(\mathbf{u}^i) = \frac{\lambda^2 G(-\mathbf{u}^i)}{4\pi} MQ(-\mathbf{u}^i)$$

gain of the antenna when radiating in the direction (-u$_i$)

||

$$G(\theta,\phi) = \eta(\theta,\phi)D(\theta,\phi)$$

empirical factor to account for metal and dielectric losses

directivity

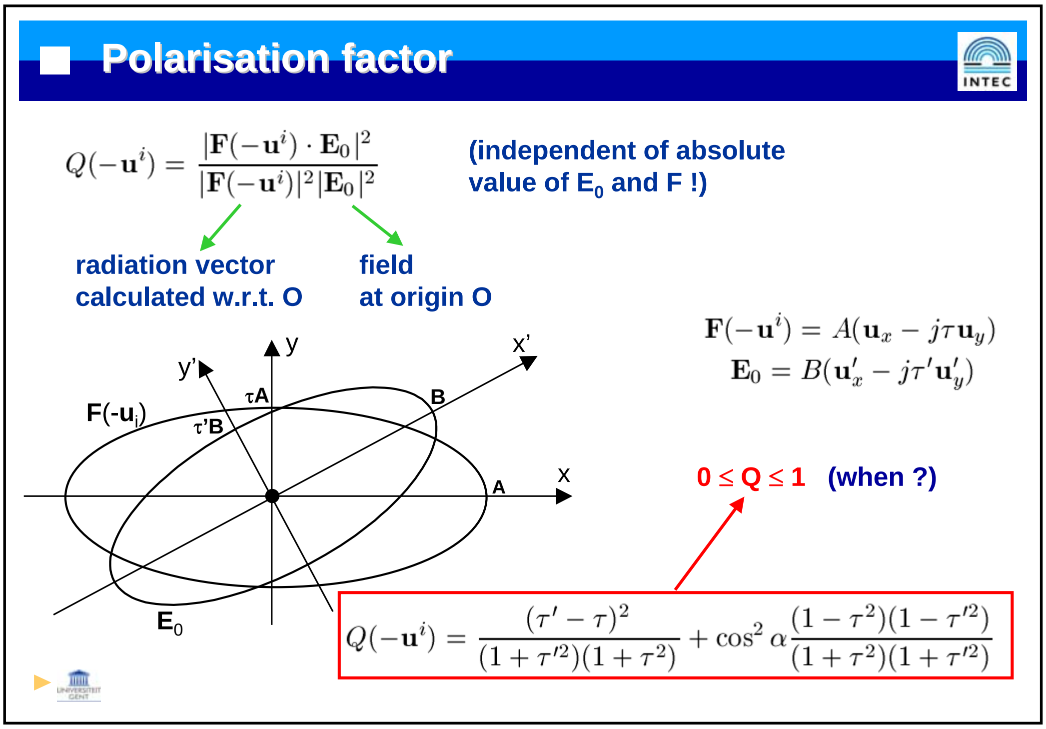
mismatch factor

$$= \frac{4R_l R_a}{|Z_l + Z_a|^2}$$

accounts for mismatch between Z_l and Z_a

is 1 for $Z_l = Z^*_a$

polarisation factor



Polarisation factor

$$Q(-\mathbf{u}^i) = \frac{|\mathbf{F}(-\mathbf{u}^i)\cdot\mathbf{E}_0|^2}{|\mathbf{F}(-\mathbf{u}^i)|^2|\mathbf{E}_0|^2}$$

(independent of absolute value of E_0 and F !)

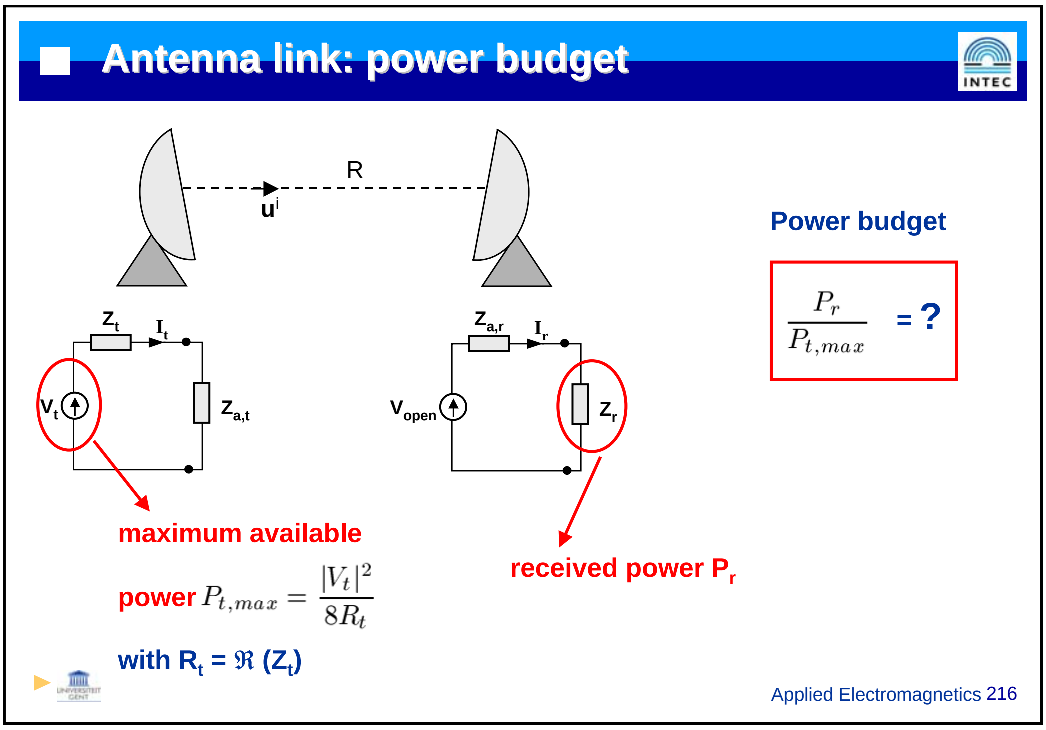
radiation vector calculated w.r.t. O

field at origin O

$$\mathbf{F}(-\mathbf{u}^i) = A(\mathbf{u}_x - j\tau\mathbf{u}_y)$$
$$\mathbf{E}_0 = B(\mathbf{u}'_x - j\tau'\mathbf{u}'_y)$$

$0 \leq Q \leq 1$ (when ?)

$$\boxed{Q(-\mathbf{u}^i) = \frac{(\tau'-\tau)^2}{(1+\tau'^2)(1+\tau^2)} + \cos^2\alpha\frac{(1-\tau^2)(1-\tau'^2)}{(1+\tau^2)(1+\tau'^2)}}$$



Antenna link: power budget

Power budget

$$\boxed{\frac{P_r}{P_{t,max}} = ?}$$

maximum available power $P_{t,max} = \frac{|V_t|^2}{8R_t}$

with $R_t = \Re(Z_t)$

received power P_r

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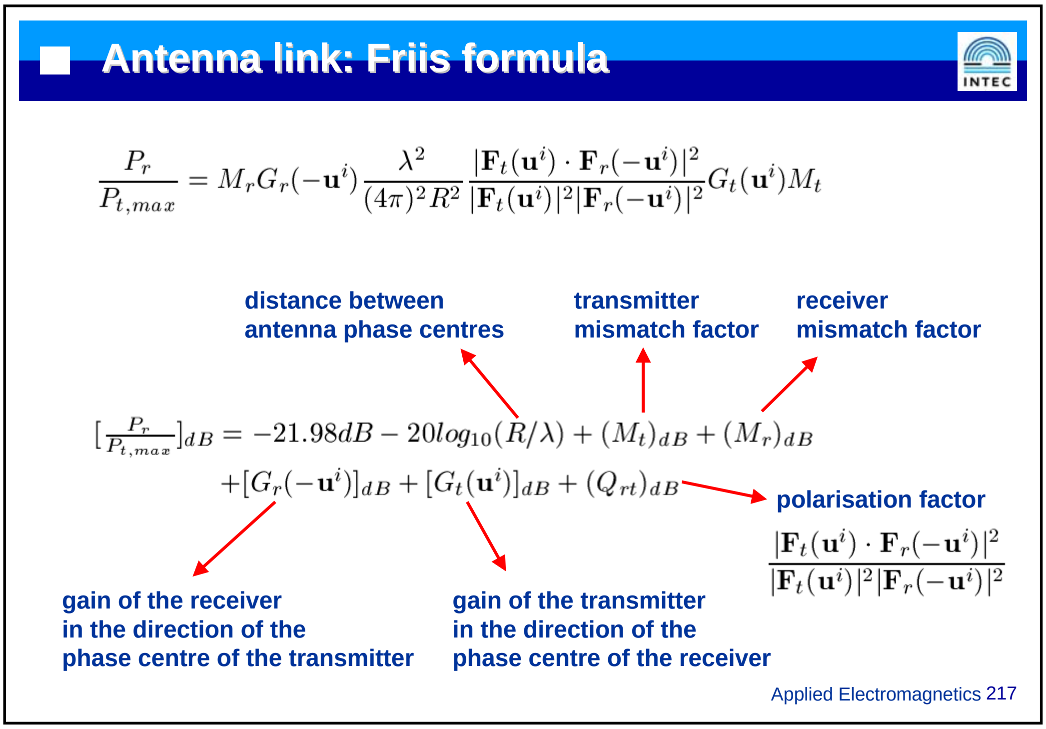


Antenna link: Friis formula

$$\frac{P_r}{P_{t,max}} = M_r G_r(-\mathbf{u}^i)\frac{\lambda^2}{(4\pi)^2 R^2}\frac{|\mathbf{F}_t(\mathbf{u}^i)\cdot\mathbf{F}_r(-\mathbf{u}^i)|^2}{|\mathbf{F}_t(\mathbf{u}^i)|^2|\mathbf{F}_r(-\mathbf{u}^i)|^2}G_t(\mathbf{u}^i)M_t$$

distance between
antenna phase centres

transmitter
mismatch factor

receiver
mismatch factor

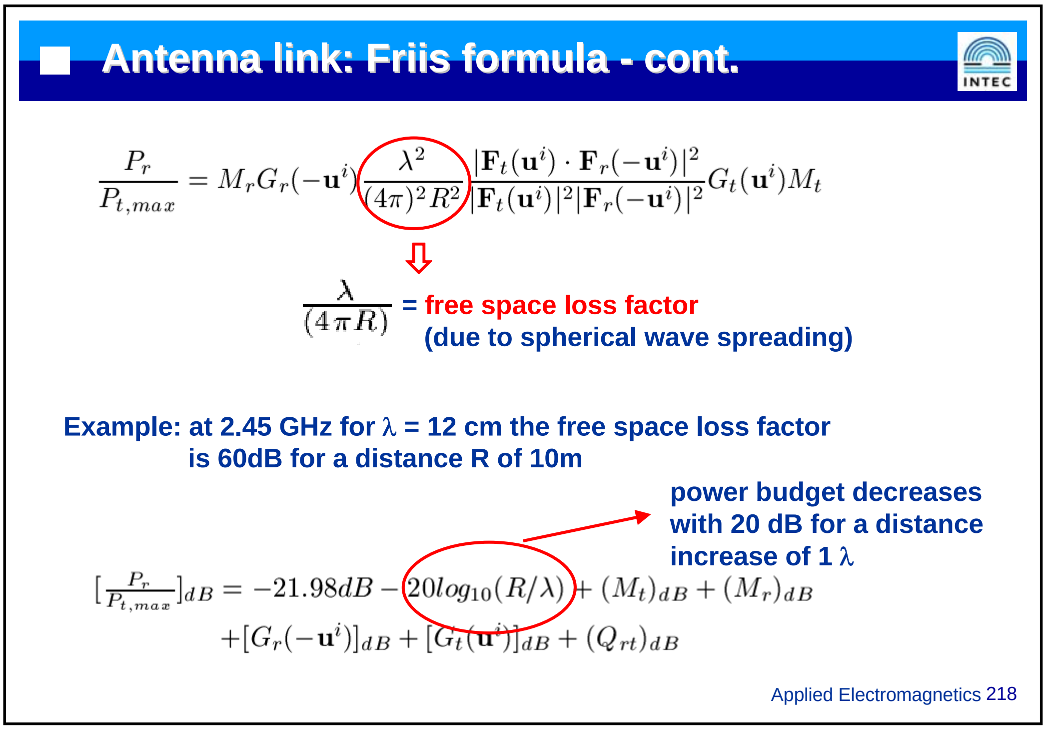
$$[\frac{P_r}{P_{t,max}}]_{dB} = -21.98dB - 20log_{10}(R/\lambda) + (M_t)_{dB} + (M_r)_{dB}$$
$$+[G_r(-\mathbf{u}^i)]_{dB} + [G_t(\mathbf{u}^i)]_{dB} + (Q_{rt})_{dB}$$

polarisation factor

$$\frac{|\mathbf{F}_t(\mathbf{u}^i)\cdot\mathbf{F}_r(-\mathbf{u}^i)|^2}{|\mathbf{F}_t(\mathbf{u}^i)|^2|\mathbf{F}_r(-\mathbf{u}^i)|^2}$$

gain of the receiver
in the direction of the
phase centre of the transmitter

gain of the transmitter
in the direction of the
phase centre of the receiver



Antenna link: Friis formula - cont.

$$\frac{P_r}{P_{t,max}} = M_r G_r(-\mathbf{u}^i)\boxed{\frac{\lambda^2}{(4\pi)^2 R^2}}\frac{|\mathbf{F}_t(\mathbf{u}^i)\cdot\mathbf{F}_r(-\mathbf{u}^i)|^2}{|\mathbf{F}_t(\mathbf{u}^i)|^2|\mathbf{F}_r(-\mathbf{u}^i)|^2}G_t(\mathbf{u}^i)M_t$$

⇩

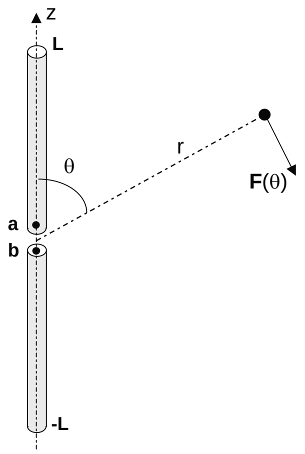
$\frac{\lambda}{(4\pi R)}$ = free space loss factor
(due to spherical wave spreading)

Example: at 2.45 GHz for λ = 12 cm the free space loss factor
is 60dB for a distance R of 10m

power budget decreases
with 20 dB for a distance
increase of 1 λ

$$[\frac{P_r}{P_{t,max}}]_{dB} = -21.98dB - \boxed{20log_{10}(R/\lambda)} + (M_t)_{dB} + (M_r)_{dB}$$
$$+[G_r(-\mathbf{u}^i)]_{dB} + [G_t(\mathbf{u}^i)]_{dB} + (Q_{rt})_{dB}$$

Wire antenna



Far Field

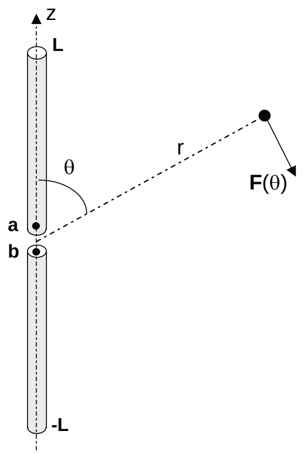
$$\lim_{kr\to\infty} \mathbf{e}(\mathbf{r}) = \mathbf{F}(\theta,\phi)\frac{e^{-jkr}}{r}$$

$$\lim_{kr\to\infty} \mathbf{h}(\mathbf{r}) = \frac{1}{Z_c}[\mathbf{u}_r \times \mathbf{F}(\theta,\phi)]\frac{e^{-jkr}}{r}$$

$$\mathbf{F}(\theta,\phi) = j\omega\mathbf{u}_r \times [\mathbf{u}_r \times \mathbf{N}(\theta,\phi)]$$

$$\mathbf{N}(\theta,\phi) = \frac{\mu_0}{4\pi}\int_V e^{jk\mathbf{u}_r\cdot\mathbf{r}'}\mathbf{j}(\mathbf{r}')dV'$$

Thin wire antenna



ad hoc current

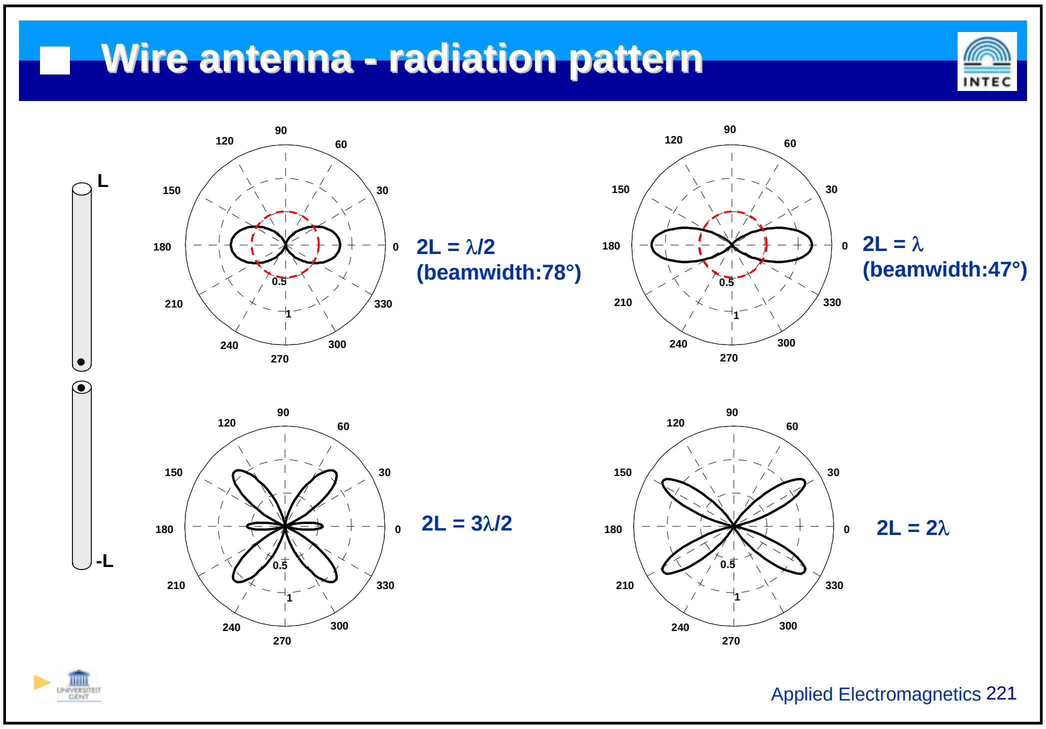
$$I(z) = I_m \sin k(L - |z|)$$

$$\Downarrow$$

$$\mathbf{F}(\theta) = \frac{j\omega\mu_0}{4\pi}\sin\theta\int_{-L}^{+L} e^{jkz'\cos\theta}I(z')dz'\mathbf{u}_\theta$$

$$\Downarrow$$

$$\mathbf{F}(\theta) = \frac{jI_m Z_c}{2\pi}\frac{\cos(kL\cos\theta) - \cos kL}{\sin\theta}\mathbf{u}_\theta$$



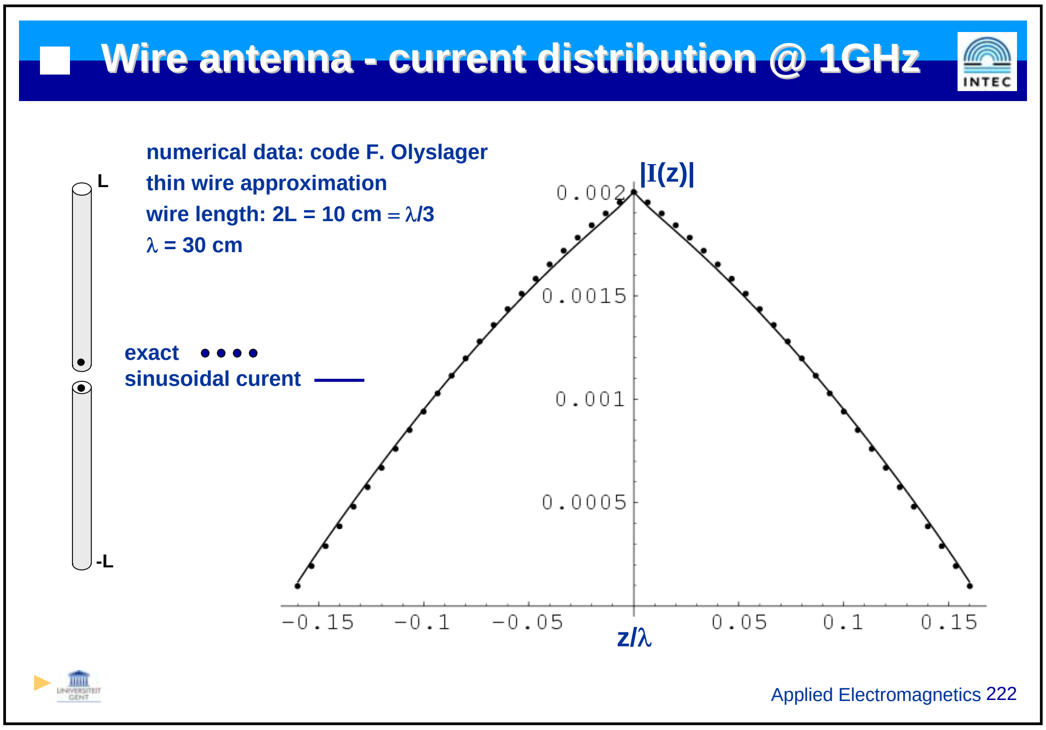
Wire antenna - radiation pattern

$2L = \lambda/2$
(beamwidth:78°)

$2L = \lambda$
(beamwidth:47°)

$2L = 3\lambda/2$

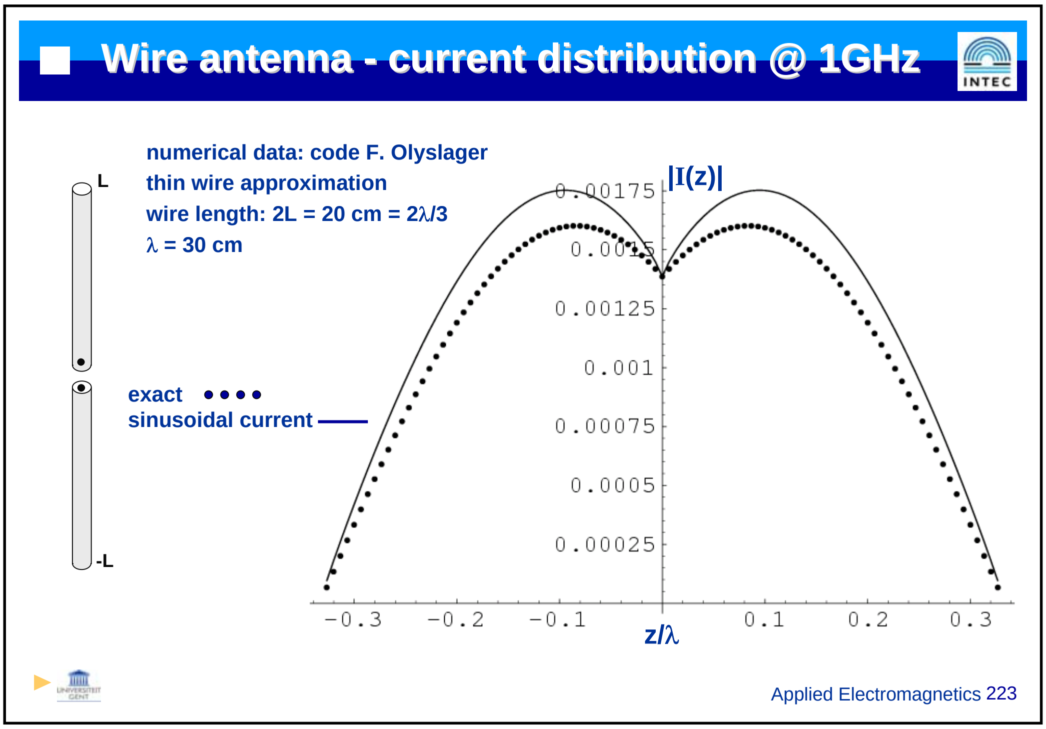
$2L = 2\lambda$



Wire antenna - current distribution @ 1GHz

numerical data: code F. Olyslager
thin wire approximation
wire length: $2L = 10$ cm $= \lambda/3$
$\lambda = 30$ cm

exact ● ● ● ●
sinusoidal curent ——



Wire antenna - current distribution @ 1GHz

numerical data: code F. Olyslager
thin wire approximation
wire length: 2L = 20 cm = 2λ/3
λ = 30 cm

exact ●●●●
sinusoidal current ——

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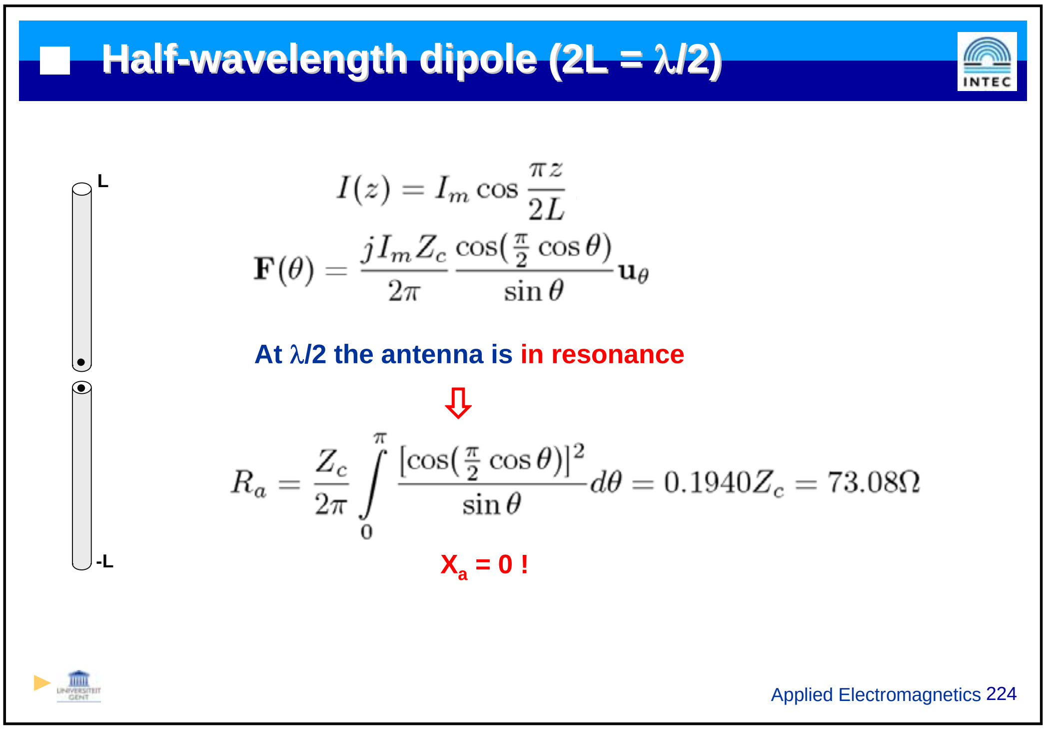

Half-wavelength dipole (2L = λ/2)

$$I(z) = I_m \cos\frac{\pi z}{2L}$$

$$\mathbf{F}(\theta) = \frac{jI_m Z_c}{2\pi}\frac{\cos(\frac{\pi}{2}\cos\theta)}{\sin\theta}\mathbf{u}_\theta$$

At λ/2 the antenna is in resonance

⇩

$$R_a = \frac{Z_c}{2\pi}\int_0^\pi \frac{[\cos(\frac{\pi}{2}\cos\theta)]^2}{\sin\theta}d\theta = 0.1940 Z_c = 73.08\Omega$$

$X_a = 0$!

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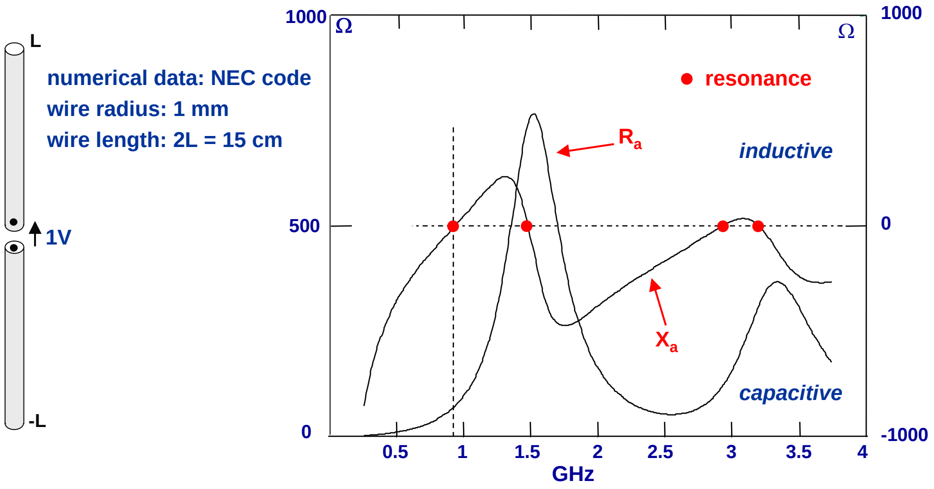
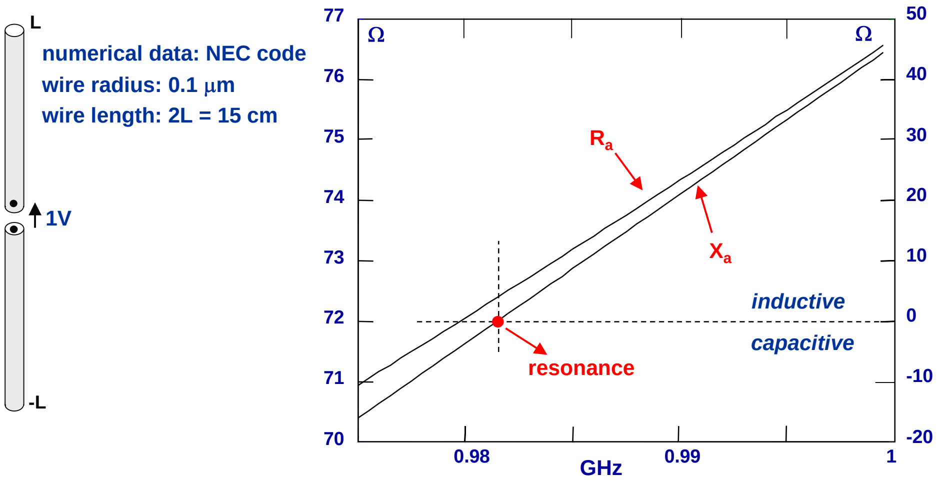
Wire antenna - radiation impedance

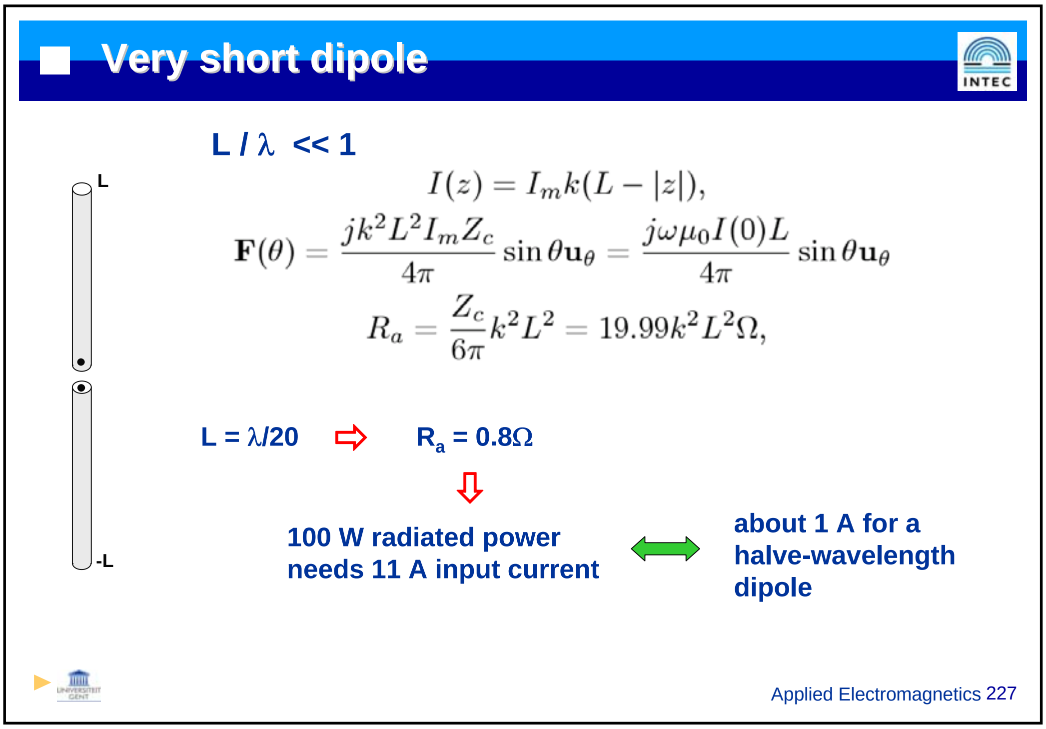


numerical data: NEC code
wire radius: 1 mm
wire length: 2L = 15 cm

Wire antenna - radiation impedance



numerical data: NEC code
wire radius: 0.1 μm
wire length: 2L = 15 cm



Very short dipole

L / λ << 1

$$I(z) = I_m k(L - |z|),$$

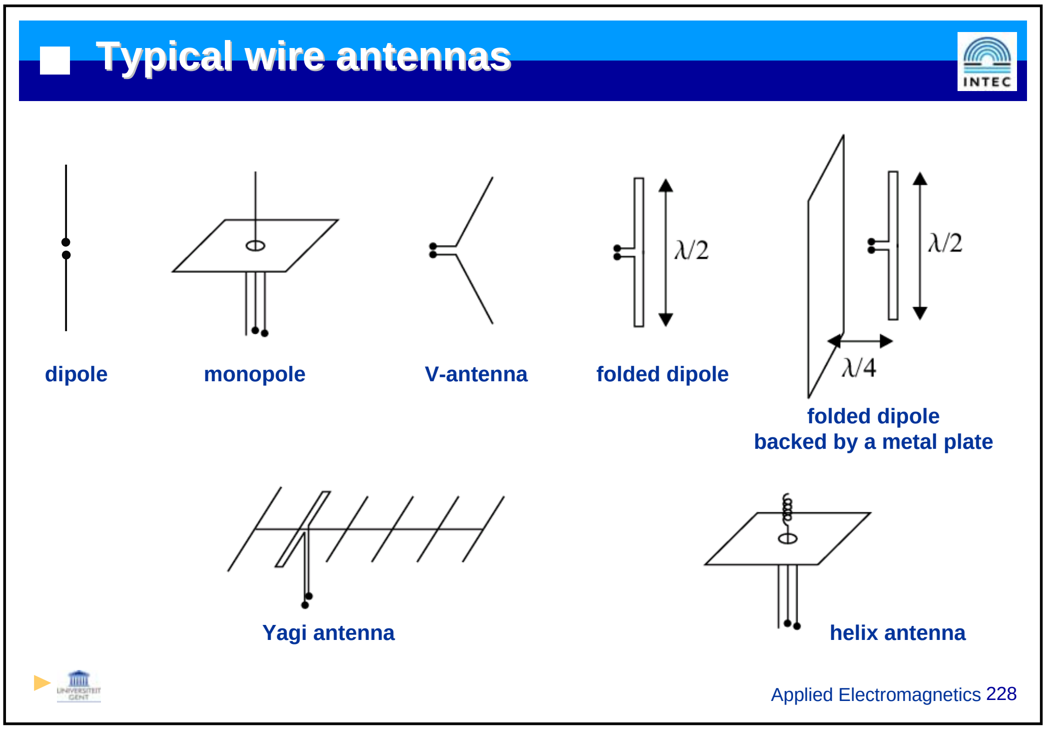
$$\mathbf{F}(\theta) = \frac{jk^2L^2I_mZ_c}{4\pi}\sin\theta\mathbf{u}_\theta = \frac{j\omega\mu_0 I(0)L}{4\pi}\sin\theta\mathbf{u}_\theta$$

$$R_a = \frac{Z_c}{6\pi}k^2L^2 = 19.99k^2L^2\Omega,$$

L = λ/20 ⇨ R_a = 0.8Ω

⇩

100 W radiated power needs 11 A input current ⟺ about 1 A for a halve-wavelength dipole



Typical wire antennas

dipole

monopole

V-antenna

folded dipole

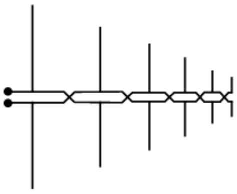
folded dipole
backed by a metal plate

Yagi antenna

helix antenna

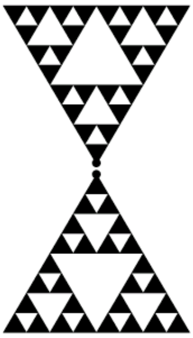
Broadband wire antennas

spiral antenna
$r = e^{A\phi}$



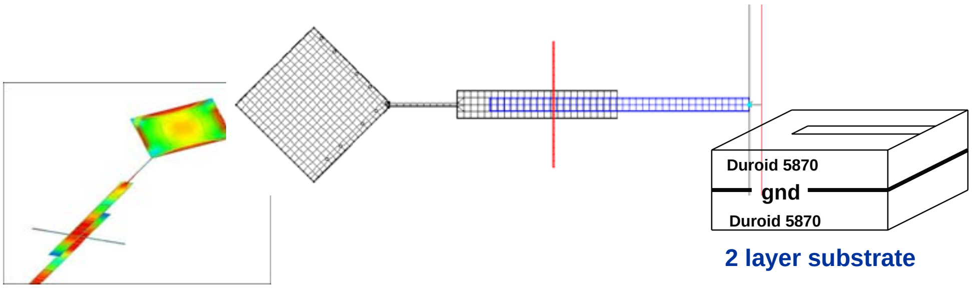
log-periodic antenna

$$\frac{l_i}{l_{i-1}} = \frac{d_i}{d_{i-1}} = \tau$$

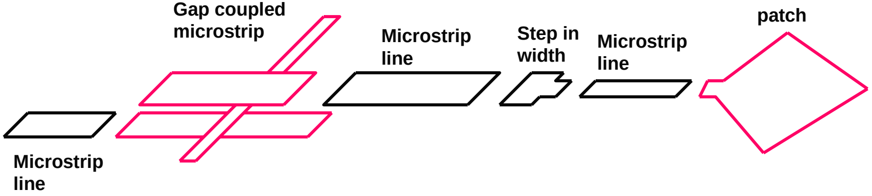


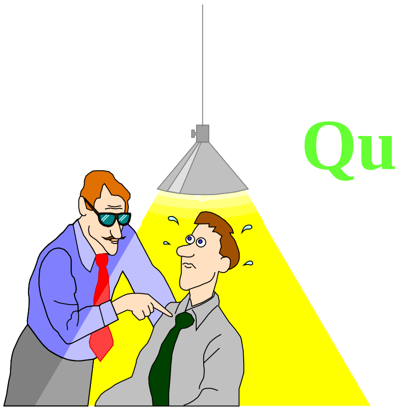
fractal antenna

Microstrip-fed patch antenna @ 10GHz



2 layer substrate





Questions?

The End