ADAPTIVE CONTOUR CODING USING WARPED POLYNOMIALS

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ABSTRACT

This paper presents a new method for compactly representing contours. The new method is a generalization of the Fourier descriptor (FD) method. Like the FD method it approximates a complex-valued parameter representation of the contour by a trigonometric polynomial (TP). In the FD method, the TP is the best least mean squares approximation of the parameter representation. However, the mean squared error is not a good a criterion for judging the accuracy of a contour approximation. The paper proposes a better criterion and a method for computing the corresponding optimal TP. It shows that computing this optimal TP is equivalent to fitting a so-called warped TP to the parameter representation in the least mean squares sense. The paper presents preliminary results which show that the new method produces more accurate contour approximations than FD approximations of the same degree.

1. INTRODUCTION

This paper presents a new method for coding closed contours (and open contours, with a small modification). Efficient representation of contours is important in the coding of line drawings [1, 2] and in segmented image coding [3, 4] where contour information accounts for a large fraction of the total bit rate [5]. It is also important in pattern recognition where the object boundaries are used as shape descriptors. Typical applications include motion analysis [6, 7], the identification of aeroplanes, weather maps, machine parts, etc. and the analysis of line drawings [8].

The new method is a generalization of the Fourier descriptor (FD) method, which is popular in pattern recognition because the FDs of a contour change in a simple way when the contour is scaled, translated or rotated. The FD method first models a contour as a complex-valued parameter representation $s(\theta)$; the coefficients in the Fourier series of this functions are the FDs. However, this description is not unique, because the function $s(\theta)$ is not unique. Indeed, given a complex function $s(\theta)$, all functions $s(\gamma(\theta))$, where $\gamma(\theta)$ is a warping function (i.e., a strictly increasing function that maps the domain of $s(\theta)$ onto itself) trace *exactly* the same contour.

In coding applications, the aim is to represent a given contour using the smallest possible number of FDs. This amounts to approximating $s(\theta)$ by a trigonometric sum $s_N(\theta) = \sum_{n=-N}^{N} B_n \exp(jn\theta)$. In the FD method, this sum is the truncated Fourier series of $s(\theta)$, which is also the optimal least mean squares trigonometric sum approximation of $s(\theta)$. However, the FDs are arbitrary in a sense, because of the non-uniqueness of the function $s(\theta)$.

Suppose that C_1 and C_2 are two contours represented by $s_1(\theta)$ and $s_2(\theta)$. In the FD approach, the distance between C_1 and C_2 is defined as the mean squared difference between $s_1(\theta)$ and $s_2(\theta)$, i.e., between *a single* parameter representation of each contour. However, as all warped versions of a complex parameter representation trace the same contour it makes more sense to define the distance between C_1 and C_2 as the smallest mean squared distance between *any* two warped parameter representations $s_1(\gamma_1(\theta))$ and $s_2(\gamma_2(\theta))$ of $C1$ and C_2 [9]. The improved Fourier descriptor (IFD) method described in this paper is based on this idea: it computes the best TP approximation of $s(\theta)$ according to (a simplified version of) the new contour distance measure.

The remainder of this paper is organized as follows: first, section 2 describes the theory of the IFD technique in greater detail. Next, section 3 discusses some algorithmic aspects. Finally 4 presents experimental results which demonstrate the potential of IFDs for contour coding. The examples confirm that approximations using a given number of IFDs are consistently more accurate than those using the same number of FDs. This is true, not only from a numerical point of view but also visually.

2. IMPROVED FOURIER DESCRIPTORS

To compute the FDs of a closed contour, this contour is represented by a periodic, complex-valued function $s(\theta) = x(\theta) + jy(\theta)$, where $(x(\theta), y(\theta))$ are the coordinates of the contour points. Without loss of generality, it can be assumed that $s(\theta)$ is continuous and has the period 2π. The FDs are the coefficients B_n of the Fourier sum $s_N(\theta) = \sum_{n=-N}^{N} B_n \exp(jn\theta)$ of order N of $s(\theta)$. In practice, only samples of $s(\theta)$ are available and the Fourier series is replaced by a discrete Fourier transform (DFT). Nevertheless, it is convenient to think of $s(\theta)$ as being defined for all real values of θ.

It is easily seen that the complex parameter representation of $s(\theta)$ of a contour is not unique. Mathematically, this is reflected by the fact that $s(\gamma(\theta))$ describes the same contour as $s(\theta)$ when $\gamma(\theta)$ is any *warping function*, i.e., any strictly monotonous function that maps the interval $[-\pi, \pi]$ onto itself. Therefore the FDs of a contour are not unique

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either, even though the coefficients of the Fourier series of $s(\theta)$ are. Consequently, the number of FDs needed to describe a contour C with a given accuracy depends on how C is scanned (i.e., how fast and starting from which point).

The FD method computes the trigonometric polynomial (TP) $s_N(\theta) \triangleq \sum_{n=-N}^{N} B_n \exp(jn\theta)$ that is closest (in the mean squares sense) to $s(\theta)$, which is only *one* of the many possible representations of the contour. As mentioned in the introduction it makes more sense to consider all possible warped contour representations $s(\gamma_1(\theta))$ and to compute the warped TP approximation $s'_N(\gamma_2(\theta))$ which is closest to *any* of the functions $s(\gamma_1(\theta))$. This comes down to finding optimal values for $\gamma_1(\theta)$, $\gamma_2(\theta)$ and the coefficients of $s'_N(\theta)$. Unfortunately, this *warped signal and approximation* (WSA) approach is computationally very complex and is also complicated by the fact that $s(\theta)$ is only known for some discrete values θ_k of θ, which means that interpolation is required to compute $s(\gamma_1(\theta_k))$ [9].

Therefore, the IFD method follows a simpler approach: it computes the least mean squares warped TP approximation $s'_N(\gamma(\theta))$ of $s(\theta)$. In other words, it only warps the approximation and not the original contour. In this case, only the optimal value $\gamma^*(\theta)$ of $\gamma(\theta)$ and the optimal values B'_n of the coefficients of $s'_N(\theta)$ need to be computed. In the following, we will use the notation $s^*_N(\theta) \triangleq s'_N(\gamma^*(\theta))$ for the (optimal) IFD approximation.

This *warped approximation* (WA) approach produces an accurate contour approximation whenever the WSA approach does, and vice versa. Indeed, let $s''_N(\theta)$ be the optimal approximating TP in the WSA approach. When $s''_N(\theta)$ is a good approximation of the original contour, then there exists warping functions $\gamma_1(\theta)$ and $\gamma_2(\theta)$ such that $s''_N(\gamma_2(\theta)) \approx s(\gamma_1(\theta))$, with good approximation. This implies that $s(\theta) \approx s''_N(\gamma(\theta))$, where $\gamma(\theta) = \gamma_2(\tau_1(\theta))$ and where $\tau_1(\theta)$ is the inverse of $\gamma_1(\theta)$. In other words, there exists at least one accurate warped TP approximation of $s(\theta)$ so that $s'_N(\theta)$ (being the best warped TP approximation) must also be accurate. It is easily seen that this argument can be reversed, thus demonstrating the "vice versa" part.

Note that even when $s^*_N(\theta)$ is a good *functional* approximation of $s(\theta)$, this need not be the case for $s'_N(\theta)$, i.e., $s^*_N(\theta) \approx s(\theta)$ does not imply that $s'_N(\theta) \approx s(\theta)$. Nevertheless, $s^*_N(\theta)$ and $s'_N(\theta)$ describe the same contour. This fact is important, because $s'_N(\theta)$ can be computed from the optimal coefficients B'_n only, while $s^*_N(\theta)$ can be computed only when $\gamma^*(\theta)$ is known. The coefficients B'_n of $s'_N(\theta)$ are called *improved Fourier descriptors* (IFDs). The previous discussion shows that they unambiguously describe the approximated contour, which can therefore be reconstructed without knowing $\gamma^*(\theta)$.

By definition, the IFD approximation $s'_N(\theta)$ is never less accurate than the FD approximation $s_N(\theta)$ of the same degree. Conversely, generally fewer (and never more) IFDs are required to represent a contour with the same accuracy as with a given number of FDs. This is an important advantage in coding applications and in pattern recognition.

Finally, like FDs, IFDs may also be used to represent open contours by first transforming these into closed ones. This is done by scanning the contours from one end point ($\theta = -\pi$) to the other ($\theta = 0$) and then back again ($\theta = \pi$). Of course, each contour point is scanned twice, but this does not double the number of IFDs (or the number of FDs for that matter). Indeed, the parameter representation $s(\theta)$ of an open contour is even, which implies that the FDs B_n satisfy $B_{-n} = \overline{B_n}$, where $\overline{x}$ denotes the complex conjugate of x. Therefore, only half of them need to be stored. On the other hand, in the IFD approach, it is easily seen that for an open contour the optimal warping function $\gamma^*(\theta)$ will be odd, i.e., $\gamma^*(-\theta) = -\gamma^*(\theta)$ and the optimal approximation $s'_N(\theta)$ will be even. Therefore, the IFDs B'_n also satisfy $B'_{-n} = \overline{B'_n}$ and only half of them need to be stored.

3. COMPUTING THE OPTIMAL WARPED TP APPROXIMATION

The previous section showed that computing the IFDs of a contour is equivalent to computing the warped TP $s^*_N(\theta) \triangleq s'_N(\gamma^*(\theta))$ that is closest to $s(\theta)$ in the least mean squares sense. In turn, this is equivalent to minimizing the mean squared error between a general warped TP $f_N(\gamma;\theta) \triangleq \sum_{n=-N}^{N} C_n \exp(jn\gamma(\theta))$ and $s(\theta)$ w.r.t. both the coefficients C_n and the warping function $\gamma(\theta)$. In this section, we address this minimization problem.

In practice, contours are usually extracted from images, which consist of a discrete number of sample points. Therefore, only a finite number $L+1$ of points $s(\theta_k)$, $k = 0, \ldots, L$ of $s(\theta)$ are known; consequently it makes sense to define the mean squared distance $d(s, f_N(\gamma)) \triangleq \|s - f_N(\gamma)\|$ between $s(\theta)$ and $f_N(\gamma(\theta))$ in terms of the discrete inner product $f \cdot g \triangleq (1/L) \sum_{k=0}^{L} f(\theta_k)\overline{g(\theta_k)}$ and the discrete norm $\|f\| \triangleq \sqrt{f \cdot f}$.

Minimizing $d(s, f_N(\gamma))$ is simplified by introducing orthonormal warped TP base functions $\phi_n(\gamma;\theta)$, $n = 0, 1, 2, 3, \ldots, 2N+1$ which span the same space as the functions $\exp(jn\gamma(\theta))$, $n = 0, 1, \ldots, N$. There exist many orthonormal TPs which satisfy this requirement, but the particular base functions $\phi_n(\gamma;\theta)$ that we will now introduce have some very desirable properties:

- they are real-valued and can be computed without resorting to complex arithmetic,
- they can be generated quickly (and more quickly than most other orthonormal TP base functions) using a recurrence with a small number of terms,
- they are independent of N; therefore N may be increased without having to recompute the base functions that have already been generated.

The base functions $\phi_n(t)$ are defined using the following equations, in which we abbreviated $\phi_n(\gamma;\theta)$ by $\phi(\theta)$:

$$\phi'_0(\theta) = 1, \qquad \phi_0(\theta) = \phi'_0(\theta)/N_0, \tag{1}$$

and, for $n \geq 1$:

$$\phi'_n(\theta) = \psi_n(\theta) - \sum_{i=1}^{4} \alpha_i^n \phi_{n-i}(\theta), \quad \phi_n(\theta) = \phi'_n(\theta)/N_n. \tag{2}$$

In the previous equations, $\psi_1(\theta) \triangleq \cos(\gamma(\theta))\,\phi_0(\theta)$ and $\psi_2(\theta) \triangleq \sin(\gamma(\theta))\,\phi_0(\theta)$, while $\psi_n(\theta) \triangleq \cos(\gamma(\theta))\,\phi_{n-2}(\theta)$

for $n \geq 3$; furthermore, $\alpha_i^n \triangleq \psi_n \cdot \phi_{n-i}$ when $i \leq n$ and $\alpha_i^n \triangleq 0$ when $i > n$; finally, $N_n \triangleq \|\phi_n'\|$.

Note that the base functions defined by the above equations depend on the warping function $\gamma(\theta)$; for this reason we will now denote them by the longer but more accurate notation $\phi_n(\gamma; \theta)$. By induction, it can be shown that they are indeed normalized and mutually orthogonal, i.e., that $\phi_n(\gamma) \cdot \phi_m(\gamma) = \delta_{i,j}$, where $\delta_{i,j}$ is Kronecker's delta. The functions $\phi_n(\gamma; \theta)$ are closely related to the so-called Jackson polynomials [10, 11].

Using eq. (2), the base functions can be computed recursively: for each n, $\psi_n(\theta)$ is computed first, followed by the coefficients α_i^n, the function $\phi_n'(\theta)$ and its norm N_n, and finally $\phi_n(\theta)$ itself. Some of these computations can be avoided by exploiting the following relations: $\alpha_3^n = \alpha_1^{n-1}$ for $n > 3$, $\alpha_4^n = N_{k-2}$ for $n > 4$, and $\alpha_4^4 = 0$.

The minimization problem can now be simplified by expressing $f_N(\gamma; \theta)$ in terms of the base functions $\phi_n(\gamma; \theta)$. First, suppose that $\gamma(\theta)$ is fixed; because of the orthonormality of the base functions, the warped TP $f_N(\gamma; \theta)$ that is closest to $s(\theta)$ is then simply given by

$$f_N(\gamma; \theta) = \sum_{n=-N}^{N} F_n(\gamma) \phi_n(\gamma; \theta), \tag{3}$$

where $F_n(\gamma) = s \cdot \phi_n(\gamma)$; the corresponding mean squared error $d^2(s, f_N(\gamma))$ equals $e_N^2(\gamma) \triangleq \|s\|^2 - \sum_{n=-N}^{N} F_n^2(\gamma)$. The notations again reflect the dependency on the warping function $\gamma(\theta)$.

The above discussion shows that minimizing $d(s, f_N(\gamma))$, w.r.t. both the coefficients C_n and the warping function $\gamma(\theta)$ is equivalent to minimizing $e_N(\gamma)$ w.r.t. $\gamma(\theta)$ alone; furthermore $e_N(\gamma)$ can be computed easily and quickly for any given $\gamma(\theta)$, using eqs. (1-3).

To minimize $e_N(\gamma)$ w.r.t. $\gamma(\theta)$, an algorithm similar to the one developed earlier for time-warped polynomial approximation of real-valued signals [12] may be employed (and is indeed employed in this paper). The algorithm produces not only the optimal warping function $\gamma^*(\theta)$ (i.e., the one which minimizes $e_N(\gamma)$), but also the corresponding optimal coefficients $F_n^* \triangleq F_n(\gamma^*)$, and the corresponding optimal approximation

$$s_N^*(\theta) = f_N(\gamma^*; \theta) = \sum_{n=-N}^{N} F_n^* \phi_n(\gamma^*; \theta). \tag{4}$$

Finally, in order to obtain the desired IFDs B_n', the right-hand side of eq. (4) must be expanded as a linear combination of the powers of $\exp(\gamma^*(\theta))$. This expansion can be computed quickly using an algorithm that is related to the one proposed by Clenshaw [13] for evaluating a linear combination of orthogonal polynomials.

4. RESULTS AND DISCUSSION

In this section, we compare warped TP contour approximations to their non-warped counterparts of the same order N and we show that the former are more accurate. Fig. 1 shows some original contours which were extracted from segmented images and the approximated versions of these contours using 15 IFDs and 15 FDs, respectively. Fig. 1 also shows that the rms approximation error $d(s, s_N^*)$ of the IFD-reconstructions are 10% to 30% lower than the corresponding error $d(s, s_N)$ of the FD-reconstructions. Therefore, from an rms point of view, IFDs are to be preferred.

In all of the examples, the number of descriptors is small compared to the number of contour points. Nevertheless, the IFD approximations reconstruct the major features of the original contours relatively faithfully. As expected, the FD approximations are less accurate than their IFD counterparts of the same order. In fact, they lack some of the important features by which a human observer differentiates one contour from another. For instance, some of the "U-bends" in the original contours deform into V-shaped features (fig. 1, left) or are missing (fig. 1, right) in the FD-reconstructions. Also rough contour parts are smoothed much more in the FD approximations.

The examples demonstrate the potential of IFDs for data compression: complex contours are represented faithfully even if the number of IFDs is 10 to 40 times smaller than the number of contour points. Note that in the examples, we are interested in approximating only the gross contour features. However, if desired, it is possible to approximate smaller features by using more descriptors.

5. CONCLUSION

This paper introduces the IFD contour descriptors, which represent contours more compactly than ordinary FDs because they are optimal w.r.t. a more realistic error criterion. The paper shows that a representation in terms of IFDs is equivalent to an approximation of a contour by a warped TP. The examples in the paper confirm that IFDs reconstruct the major features of even large contours much more faithfully than the same number of FDs. This is reflected in a lower rms error and in a better accuracy, as judged by human observers. The IFD representation is a promising tool for efficient contour coding; as such it is useful in segmented image coding. IFDs may also be used as shape descriptors in pattern recognition.

6. REFERENCES

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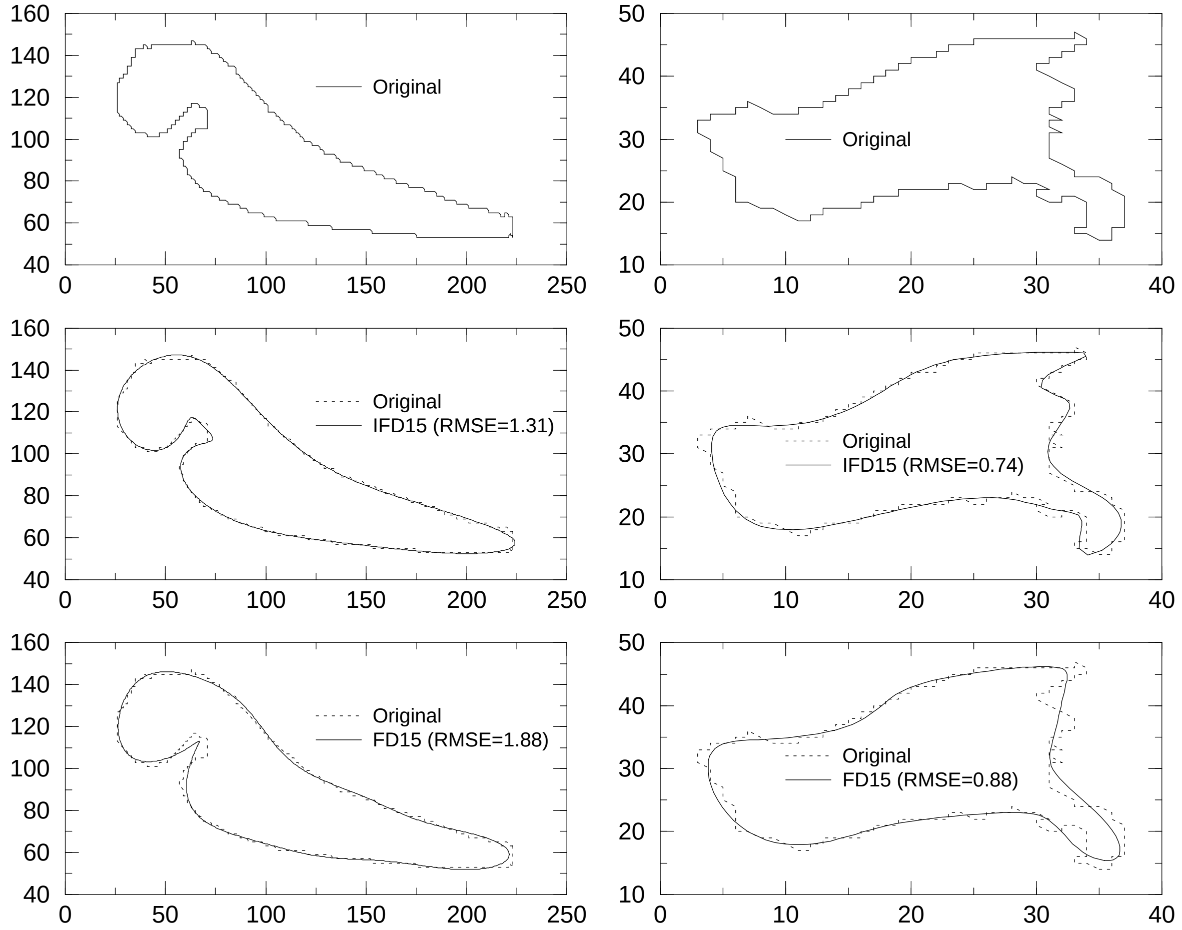

Figure 1: Approximation of two contours using 15 ordinary Fourier descriptors (FD15) and the same number of improved Fourier descriptors (IFD15). The contours were extracted from segmented versions of an mr-image (left) and of the "peppers" image (right) and consist of 587 and 152 points, respectively.

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