EECE\CS 253 Image Processing

Lecture Notes on Mathematical Morphology: Grayscale Images

Richard Alan Peters II

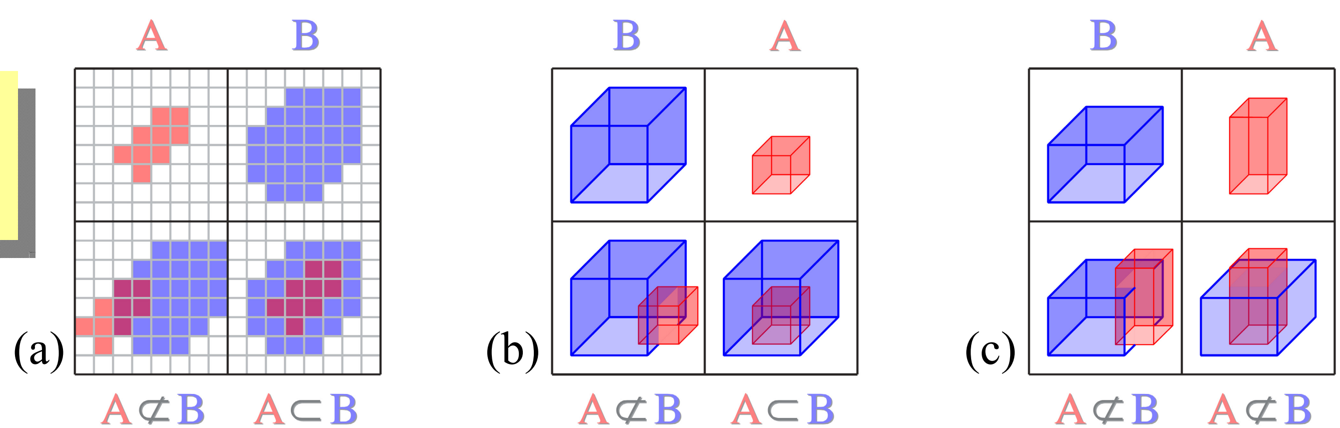
Department of Electrical Engineering and Computer Science

Fall Semester 2007

Grayscale Morphology

Grayscale morphology is a multidimensional generalization of the binary operations. Binary morphology is defined in terms of set-inclusion of pixel sets. So is the grayscale case, but the pixel sets are of higher dimension. In particular, standard *R*×*C,* 1-band intensity images and the associated structuring elements are defined as 3-D solids wherein the 3rd axis is intensity and set-inclusion is volumetric.

set inclusion (explained on p. 11)



(a) binary, (b) & (c) grayscale

Extended Real Numbers

Let $\mathbb{R}$ represent the real numbers.

Define the *extended* real numbers, $\mathbb{R}^*$, as the real numbers plus two symbols, $-\infty$ and ∞ such that

$$-\infty < x < \infty,$$

for all numbers $x \in \mathbb{R}$.

That is if x is any real number, then ∞ is always greater than x and $-\infty$ is always less than x. Moreover,

$$x + \infty = \infty, \quad x - \infty = -\infty, \quad \infty - \infty = 0,$$

for all numbers $x \in \mathbb{R}$.

Real Images

In mathematical morphology a real image, I, is defined as a function that occupies a volume in a Euclidean vector space. I comprises a set, S_p, of coordinate vectors (or pixel locations), **p**, in an *n*-dimensional vector space $\mathbb{R}^n$. Associated with each **p** is a value from $\mathbb{R}^*$. The set of pixel locations together with their associated values form the image – a set in $\mathbb{R}^{n+1}$:

$$\mathrm{I} = \left\{ \left[\mathbf{p}, \mathrm{I}(\mathbf{p}) \right] \middle| \mathbf{p} \in S_{\mathrm{p}} \subseteq \mathbb{R}^n, \;\; \mathrm{I}(\mathbf{p}) \in \mathbb{R}^* \right\}$$

Thus, a conventional, 1-band, $R \times C$ image is a 3D structure with $S_p \subset \mathbb{R}^2$ and $\mathrm{I}(\mathbf{p}) \in \mathbb{R}$. By convention in the literature of MM, $S_p \equiv \mathbb{R}^n$, a real image is defined over all of $\mathbb{R}^n$.

Support of an Image

The support of a real image, I, is

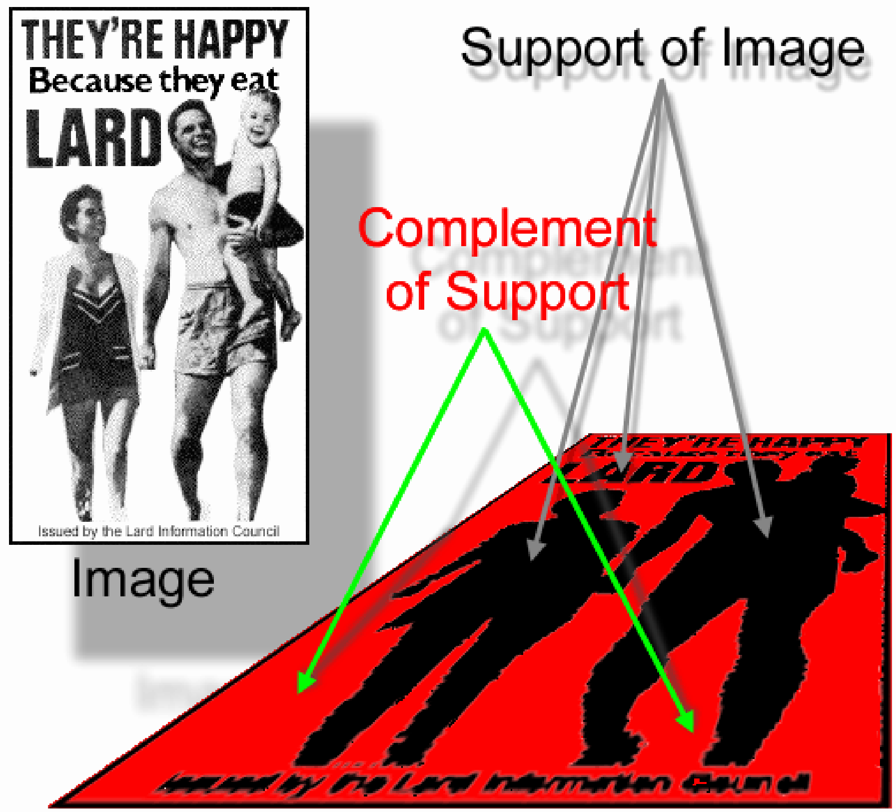
$$\mathrm{supp}(\mathrm{I}) = \{\, \mathbf{p} \in \mathbb{R}^n \mid \mathrm{I}(\mathbf{p}) \in \mathbb{R} \,\}.$$

That is, the support of a real image is the set pixel locations in $\mathbb{R}^n$ such that

$$\mathrm{I}(\mathbf{p}) \neq -\infty \quad \text{and} \quad \mathrm{I}(\mathbf{p}) \neq \infty.$$

The complement of the support is, therefore, the set of pixel locations in $\mathbb{R}^n$ where

$$\mathrm{I}(\mathbf{p}) = -\infty \quad \text{or} \quad \mathrm{I}(\mathbf{p}) = \infty.$$



Grayscale Images

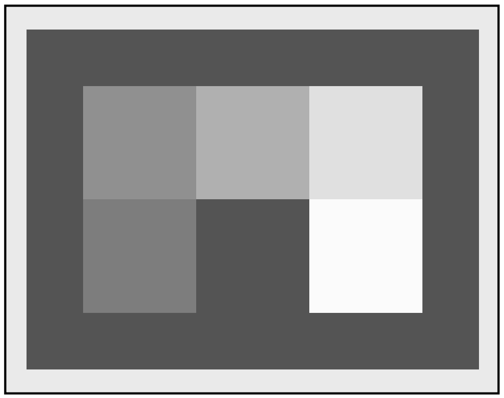
If over its support, I takes on more than one real value, then I is called *grayscale*.

The object commonly known as a black and white photograph is a grayscale image that has support in a rectangular subset of $\mathbb{R}^2$. Within that region, the image has gray values that vary between black and white. If the intensity of each pixel is plotted over the support plane, then

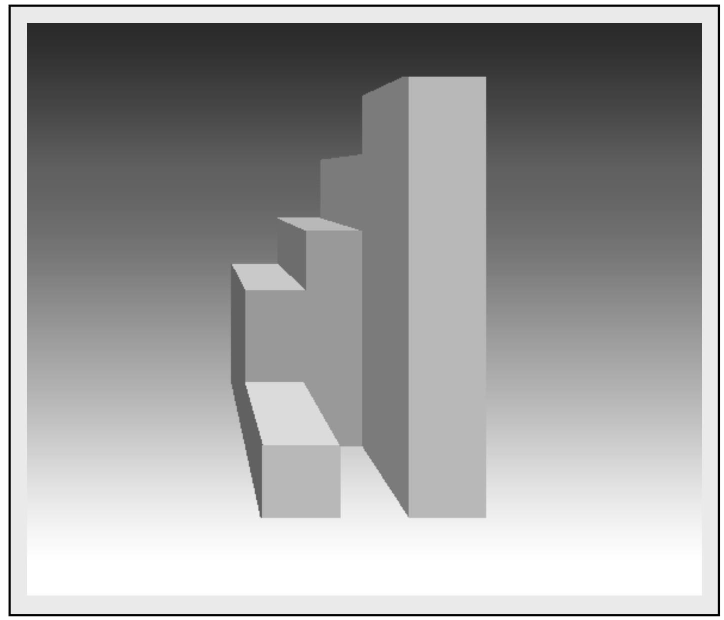
$$\mathrm{I} = \left\{ \left[\mathbf{p}, \mathrm{I}(\mathbf{p}) \right] \mid \mathbf{p} \in \mathrm{supp}(\mathrm{I}) \right\}$$

is a volume in $\mathbb{R}^3$. In the abstraction of MM we assume the image does exist outside the support rectangle, but that $\mathrm{I}(\mathbf{p}) = -\infty$ there.

Grayscale Images



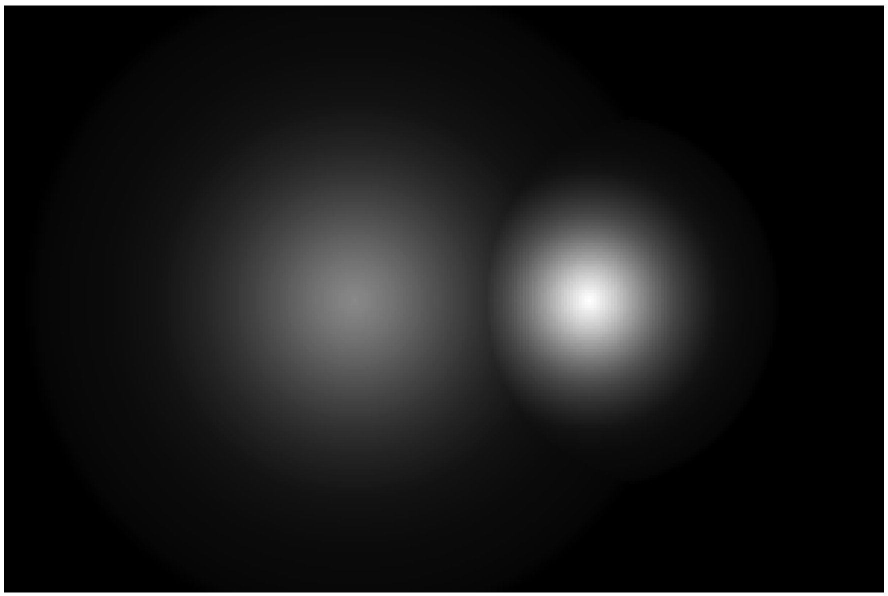
grayscale image



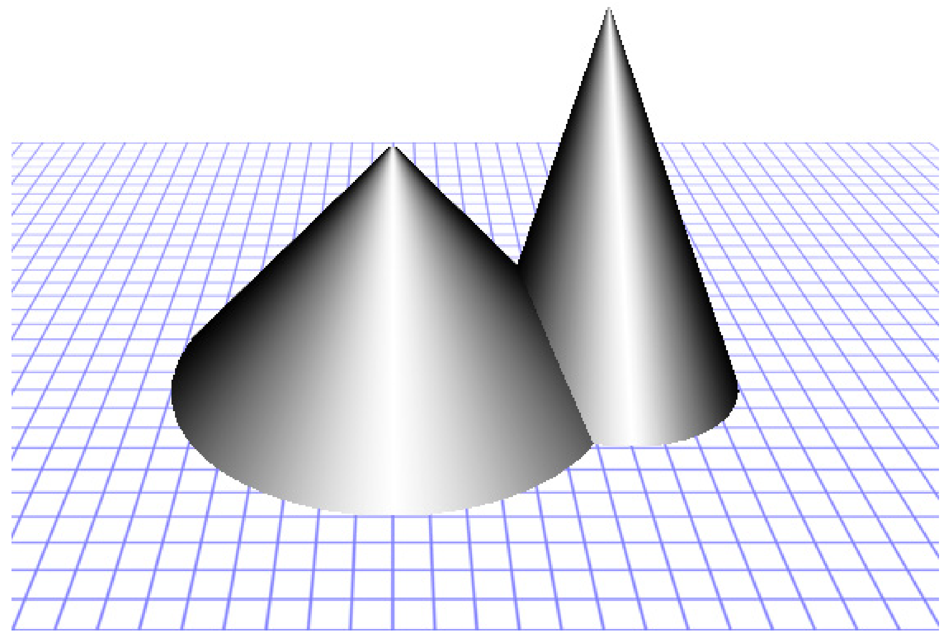
3D solid representation

In MM, a 2D grayscale image is treated as a 3D solid in space – a landscape – whose height above the surface at a point is proportional to the brightness of the corresponding pixel.

Representation of Grayscale Images



image



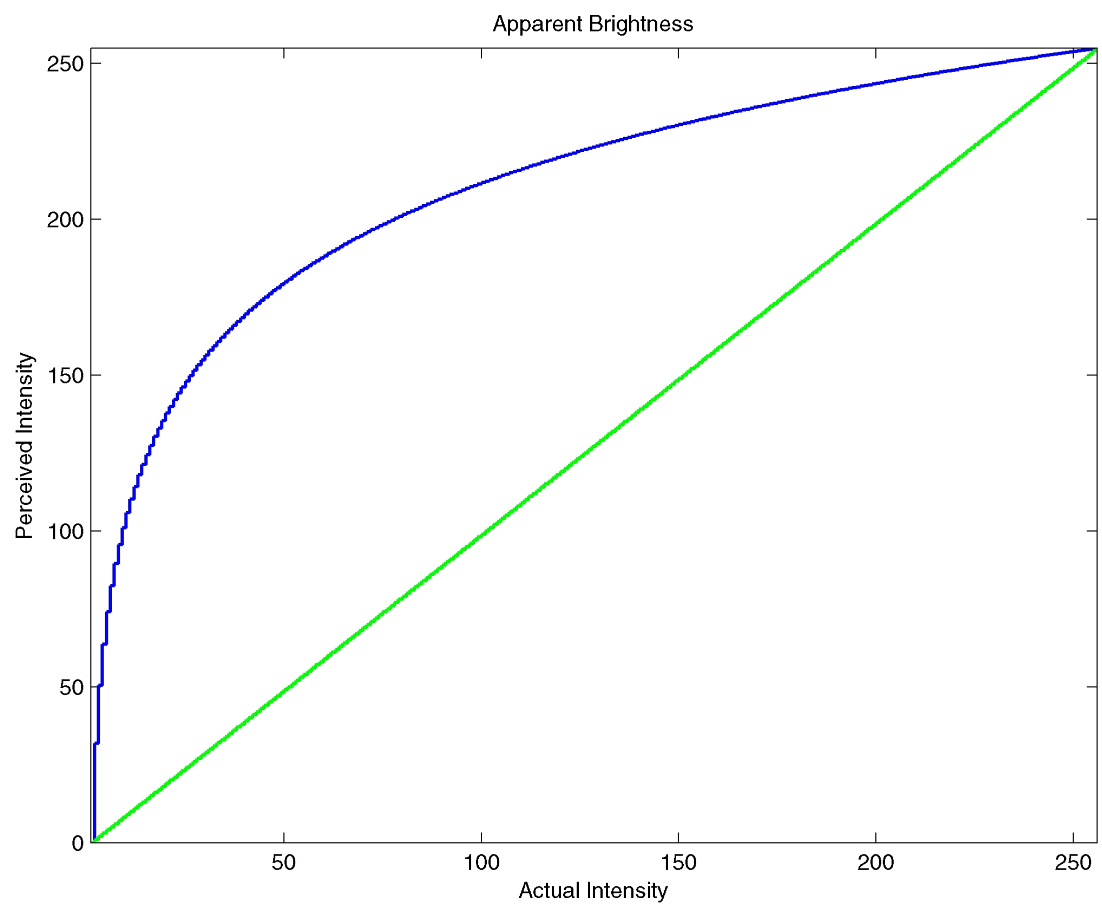
landscape

Example: grayscale cones

Aside: Brightness Perception

The previous slide demonstrates the Weber-Fechner relation. The linear slope of the intensity change is perceived as logarithmic.

The green curve is the actual intensity; the blue curve is the perceived intensity.

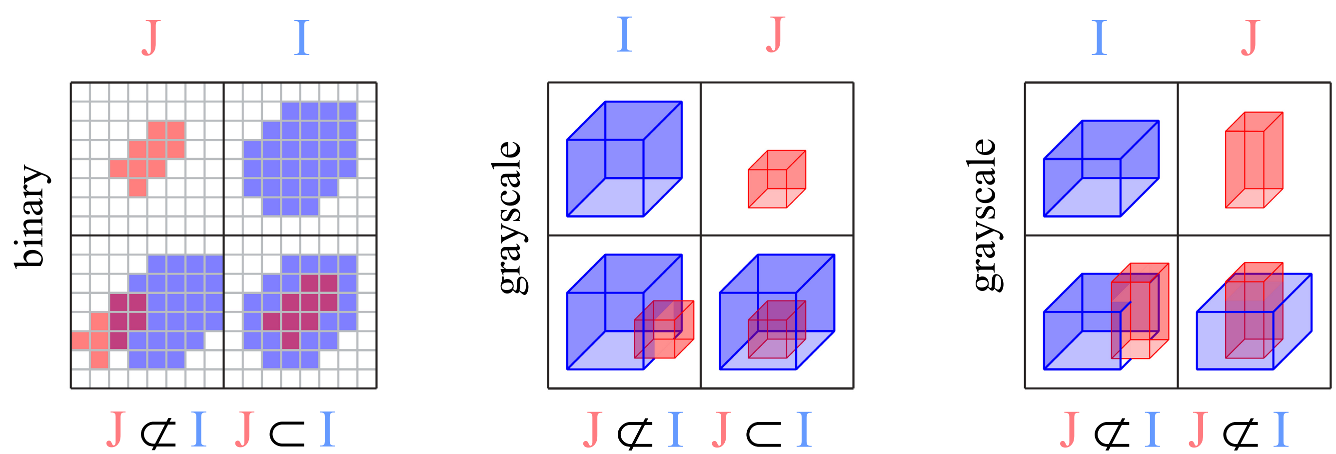


Set Inclusion in Grayscale Images

In grayscale morphology, set inclusion depends on the implicit 3D structure of a 2D image. If I and J are grayscale images then

$$\mathrm{J} \subseteq \mathrm{I} \iff \mathrm{supp}(\mathrm{J}) \subseteq \mathrm{supp}(\mathrm{I}) \text{ AND } \{\mathrm{J}(\mathbf{p}) \leq \mathrm{I}(\mathbf{p}) \mid \mathbf{p} \in \mathrm{supp}(\mathrm{J})\}.$$

That is $\mathrm{J} \subseteq \mathrm{I}$ if and only if the support of J is contained in that of I *and* the value of J is nowhere greater than the value of I on the support of J.



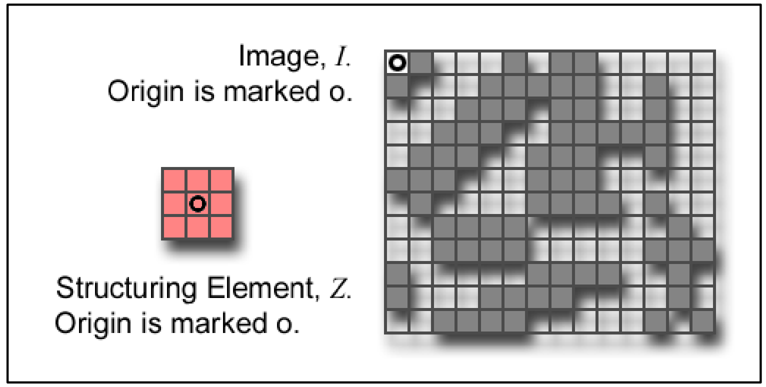
Recall: Binary Structuring Element (SE)

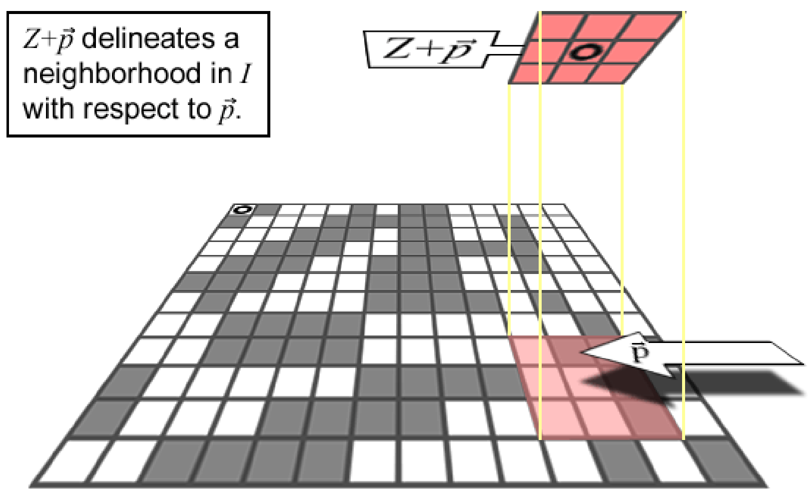
Let I be an image and Z a SE.

Z+**p** means that Z is moved so that its origin coincides with location **p** in S_P.

Z+**p** is the *translate* of Z to location **p** in S_P.

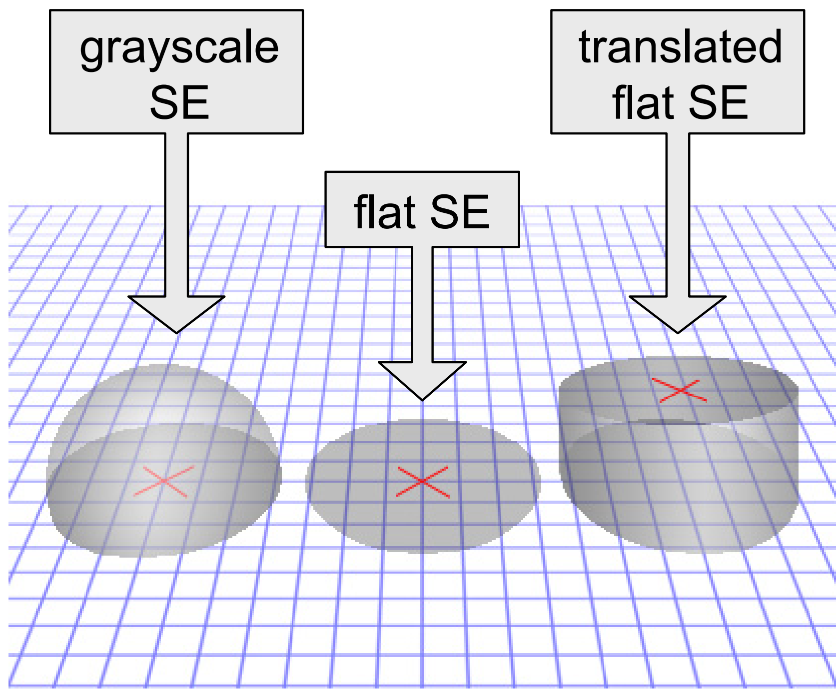
The set of locations in the image delineated by Z+**p** is called the *Z-neighborhood* of **p** in I denoted N{I,Z}(**p**).

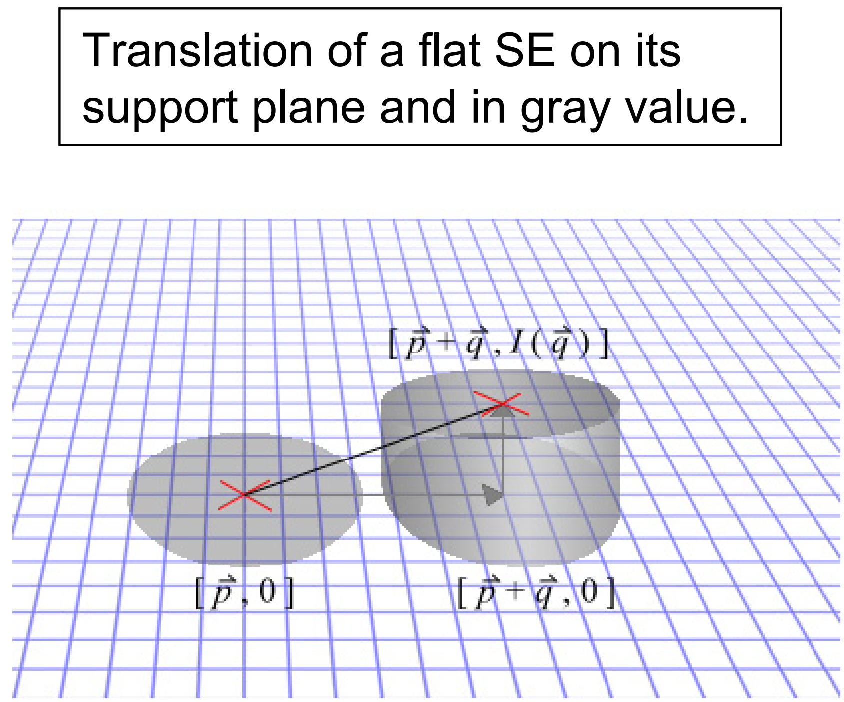




Grayscale Structuring Elements

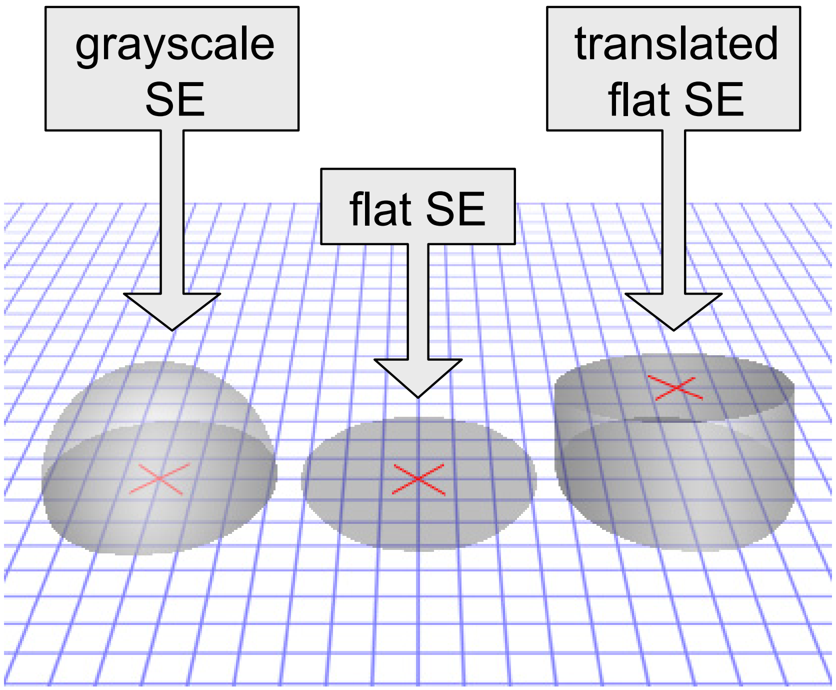
A grayscale structuring element is a small image that delineates a volume at each pixel [**p** , I(**p**)] through out the image volume.



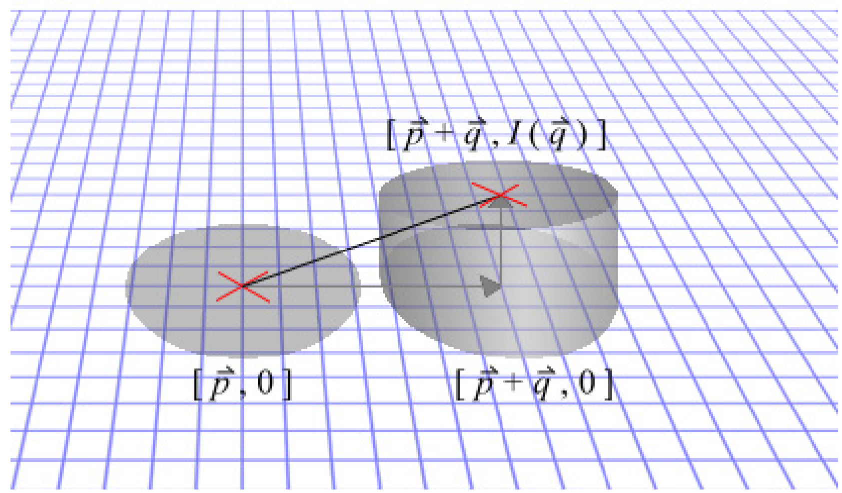


SE Translation: × marks the location of the structuring element origin.

Grayscale Structuring Elements

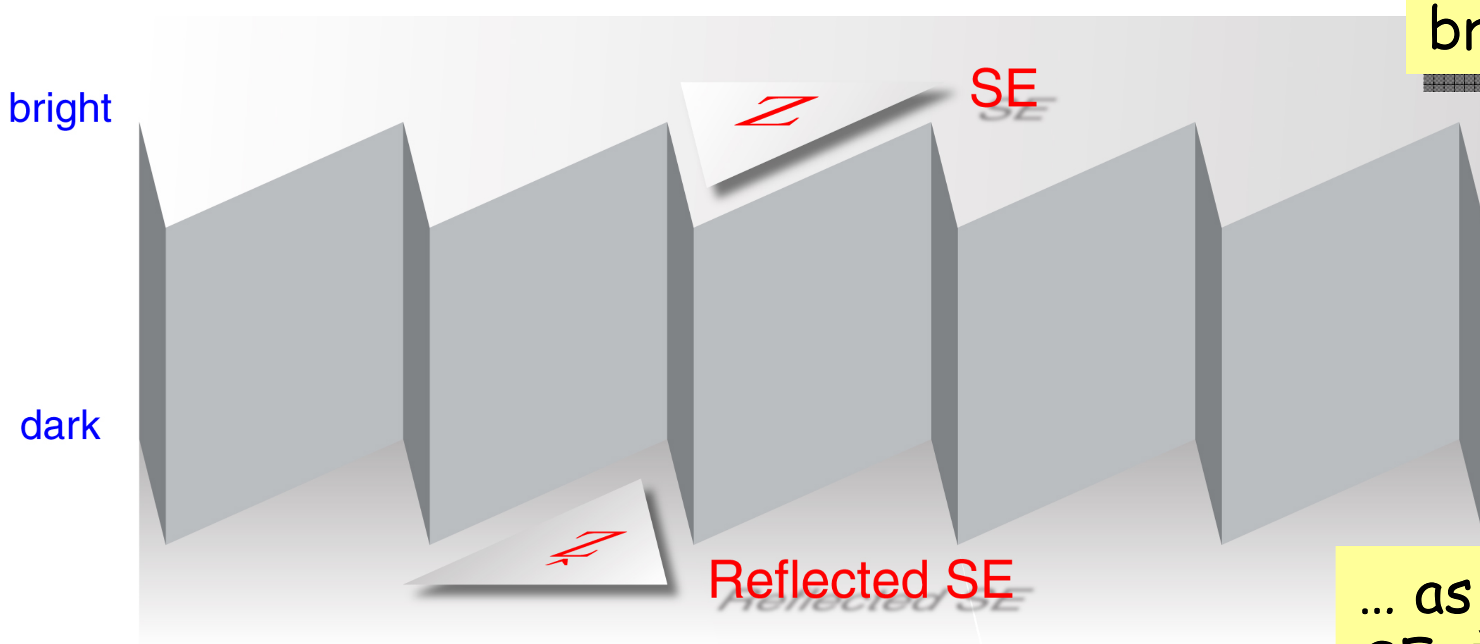


Translation of a flat SE on its support plane and in gray value.



If $Z = [\ \mathbf{p}, Z(\mathbf{p})\ ]$ is a structuring element and if $\mathbf{q} = [\ \mathbf{q}_s, q_g\ ]$ is a pixel [location, value] then $Z+\mathbf{q} = [\ \mathbf{p}+\mathbf{q}_s, Z(\mathbf{p})+q_g\ ]$ for all $\mathbf{p} \in \text{supp}\{Z\}$.

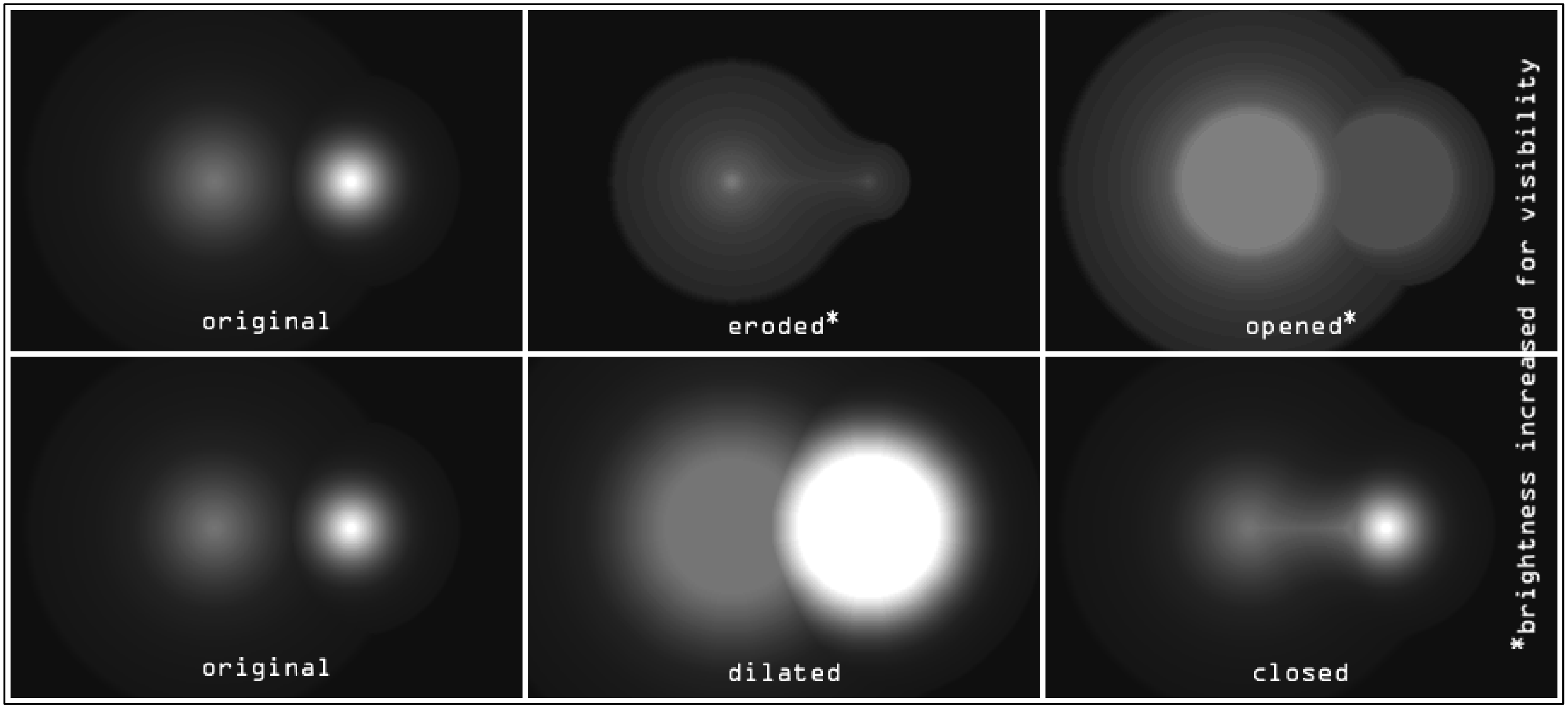
Reflected Structuring Elements



Note that the SE, **Z**, is to the bright regions…

… as the reflected SE, **Ž**, is to the dark regions.

Grayscale Morphology: Basic Operations



Dilation: General Definition

The *dilation* of an image I by structuring element Z at coordinate $\vec{p} \in \Re^n$ is defined by

$$[I \oplus Z](\vec{p}) = \max_{\vec{q} \in \text{supp}(\check{Z}+\vec{p})} \left\{ I(\vec{q}) - \check{Z}(\vec{q}-\vec{p}) \right\} = \max_{\vec{q} \in \text{supp}(\check{Z}+\vec{p})} \left\{ I(\vec{q}) + Z(\vec{p}-\vec{q}) \right\}.$$

This can be computed as follows:

1. Translate $\check{Z}$ to $\vec{p}$.
2. Trace out the $\check{Z}$-neighborhood of I at $\vec{p}$.
3. Let $\vec{p}$ be the origin of I temporarily, during the operation.
4. Compute the set of numbers

$$\mathcal{D} = \{ I(\vec{q}) - \check{Z}(\vec{q}) \mid \vec{q} \in \text{supp}\{\check{Z}\}\} = \{ I(\vec{q}) + Z(-\vec{q}) \mid \vec{q} \in \text{supp}\{\check{Z}\}\}.$$

5. The output value, $[I \oplus Z](\vec{p})$, is the maximum value in the set $\mathcal{D}$.

Fast Computation of Dilation

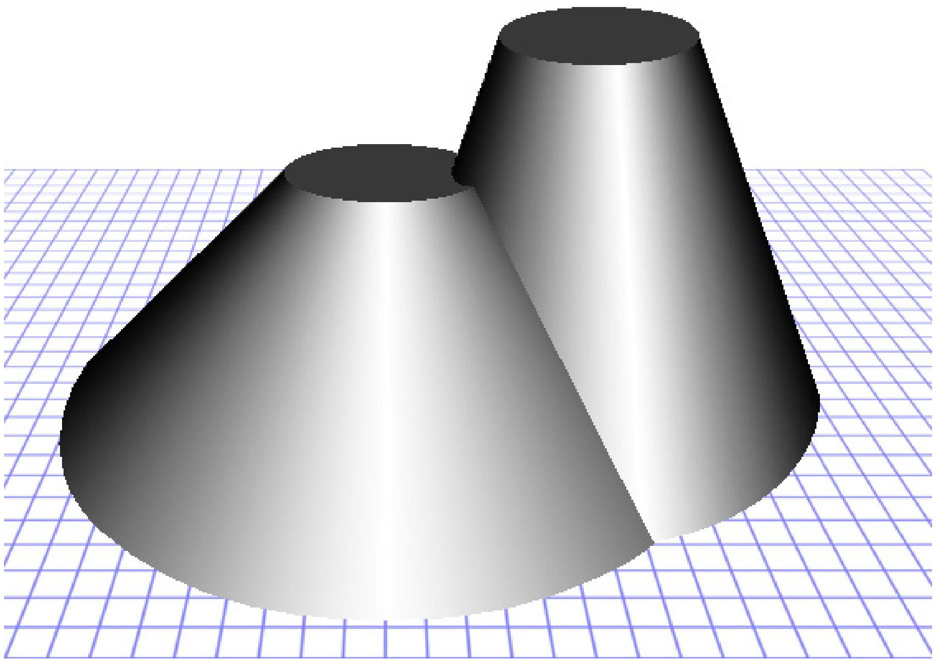
The fastest way to compute *grayscale* dilation is to use the translates-of-the-image definition of dilation. That is, use

$$\mathrm{J} = \mathrm{J} \oplus Z = \max_{q \in \mathrm{supp}\{Z\}} \left\{ \left[\mathrm{I} + \mathbf{q} \right] + Z(\mathbf{q}) \right\}.$$

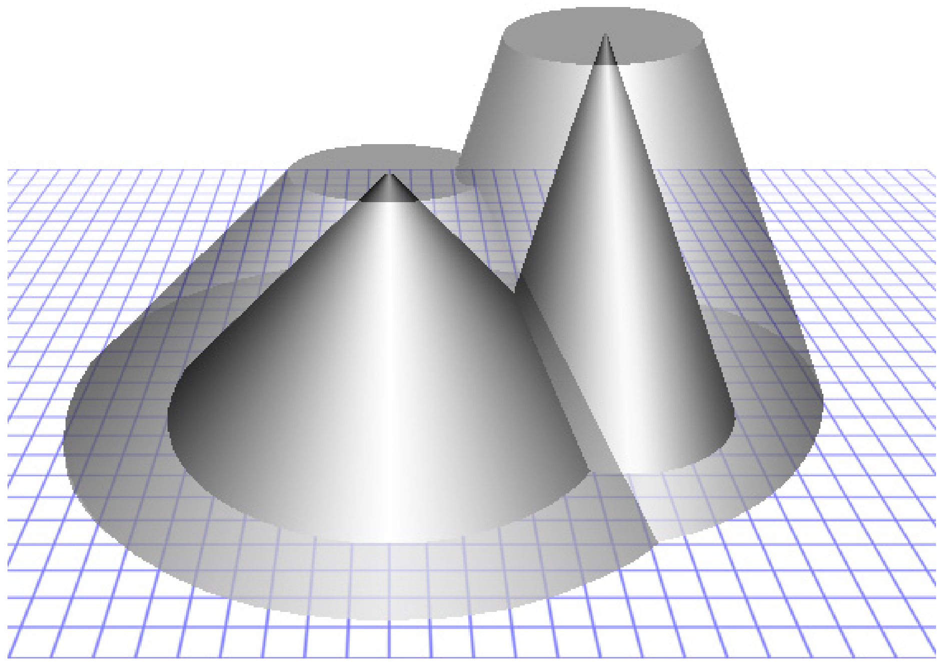
Note that if Z is flat -- all its foreground elements are 0 -- then step (3) is unnecessary. Then it is a *maximum* filter.

That is,
(1) Make a copy of I for each foreground element, **q**, in *Z*.
(2) Translate the *q*th copy so that its ULHC (origin) is at position **q** in *Z*.
(3) Add *Z*(**q**) to every pixel in the *q*th copy.
(4) Take the pixelwise maximum of the resultant stack of images.
(5) Copy out the result starting at the SE origin in the maximum image.

Grayscale Morphology: Dilation

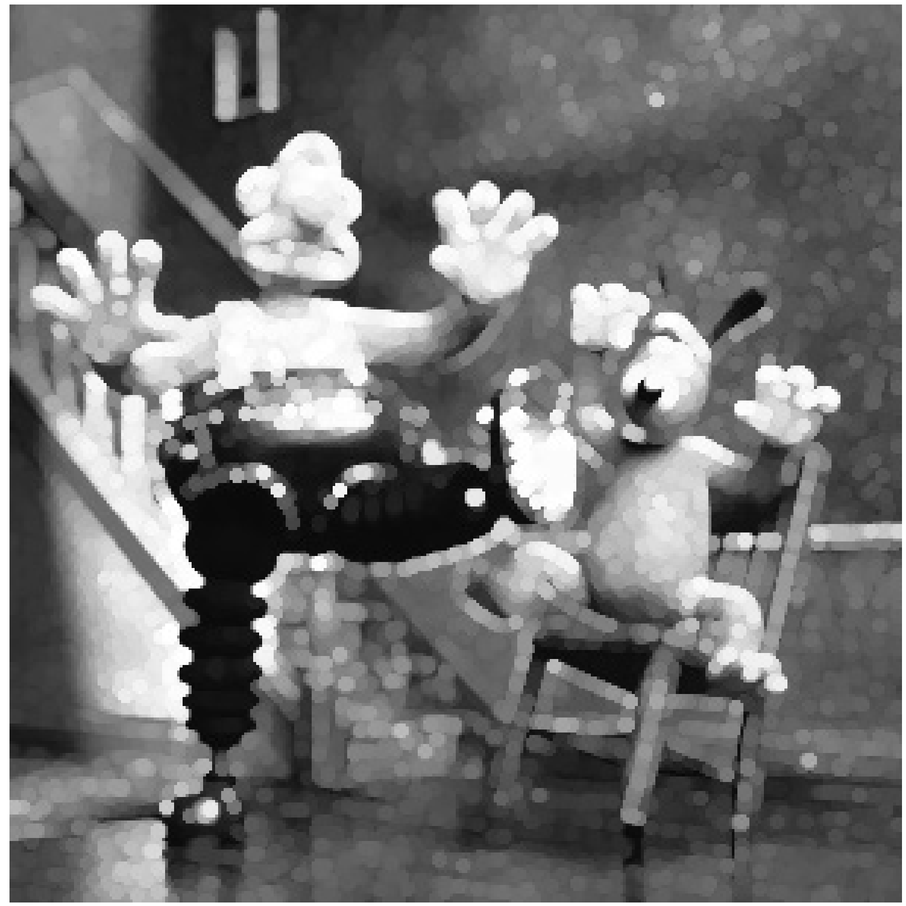


dilation



dilation over original

Grayscale Morphology: Dilation



Erosion: General Definition

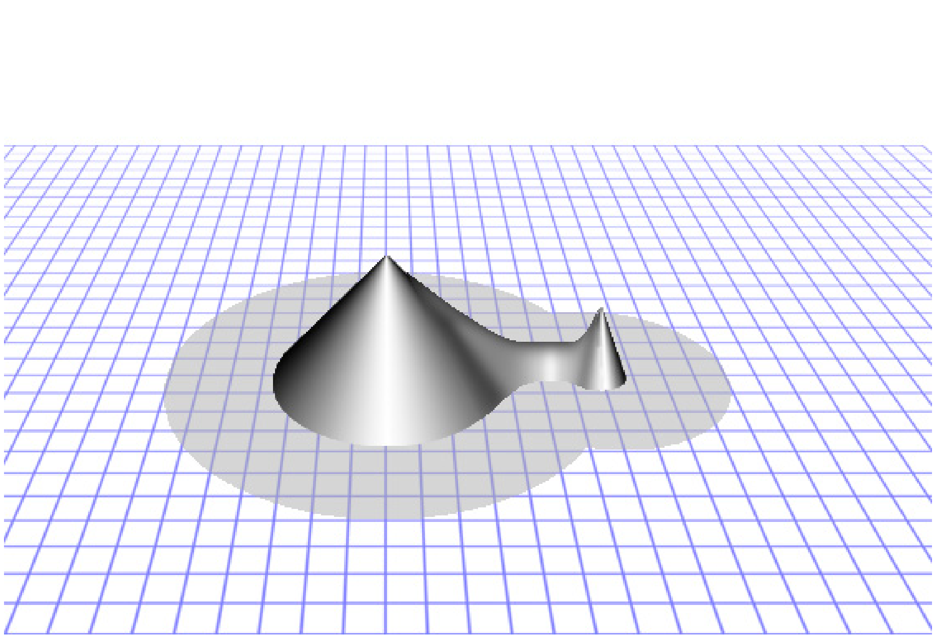
Erosion is the fundamental operation of structural morphology (MM defined using SEs). The other operators can be defined in terms of erosions, complements, unions, and intersections.

The *erosion* of image I by structuring element Z at coordinate $\vec{p} \in \Re^n$ is defined (most generally) by

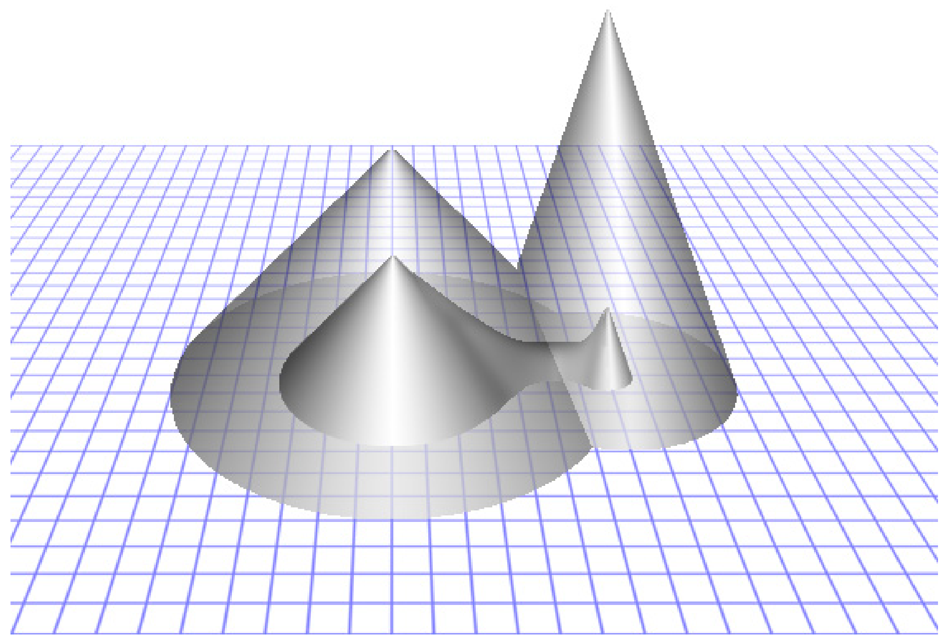
$$[I \ominus Z](\vec{p}) = \min_{\vec{q} \in \mathrm{supp}(Z+\vec{p})} \{I(\vec{q}) - Z(\vec{q} - \vec{p})\}\,.$$

In general, the value of the erosion of I by Z at $\vec{p}$ is the minimum pixel-wise difference between the image values in the Z-neighborhood of $\vec{p}$, and the corresponding SE pixel values.

Grayscale Morphology: Erosion



erosion



erosion under original

Fast Computation of Erosion

The fastest way to *grayscale* erosion is to create a stack of images translated to minus the values of the reflected SE then take the pixelwise minimum:

$$J = I \ominus Z = \min_{\vec{q} \in \breve{Z}} \left\{ \left[I + \vec{q} \right] + \breve{Z}(\vec{q}) \right\}$$

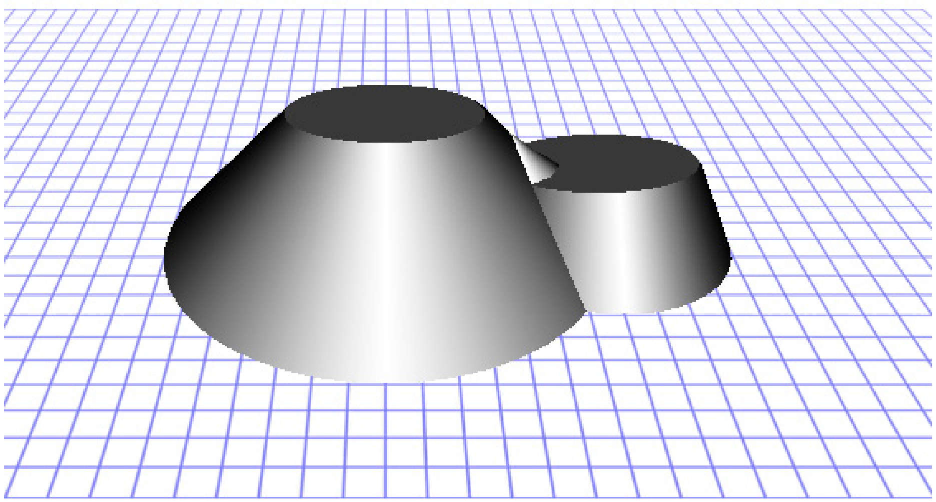
$$\breve{Z} = \left\{ -Z(-\vec{q}) \,\middle|\, \vec{q} \in \Re^2 \right\}$$

Note that if Z is symmetric and if all the foreground elements are 0, then Ž=Z and step (3) is unnecessary. Then it is a *minimum* filter

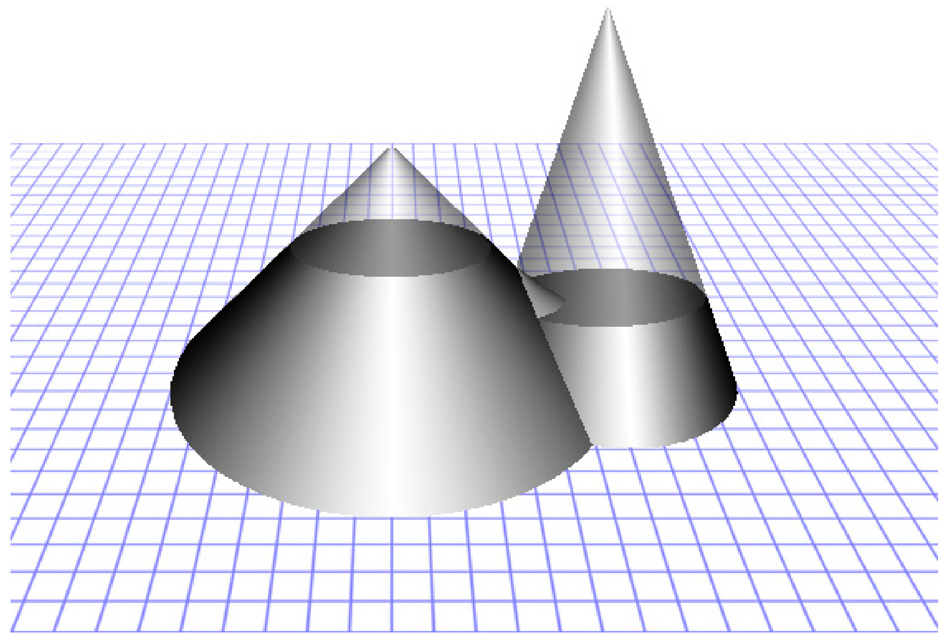
That is, (1) make a copy of I for each foreground element, q, in $\check{Z}$. (Note that if q is a foreground element in $\check{Z}$ then $-q$ is a foreground element in Z.) (2) Translate each copy so that its ULHC (origin) is at position q in $\check{Z}$ (or $-q$ in Z). (3) Then add $\check{Z}(q)$ (or subtract $Z(-q)$) to every pixel in the qth copy. Finally, (4) take the pixelwise minimum of the resultant stack of images.

Grayscale Morphology: Erosion

Grayscale Morphology: Opening

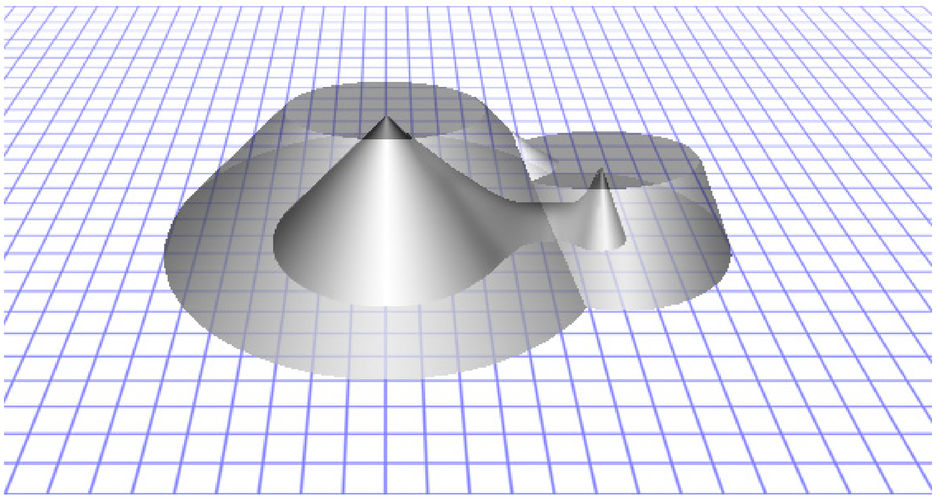


opening: erosion then dilation

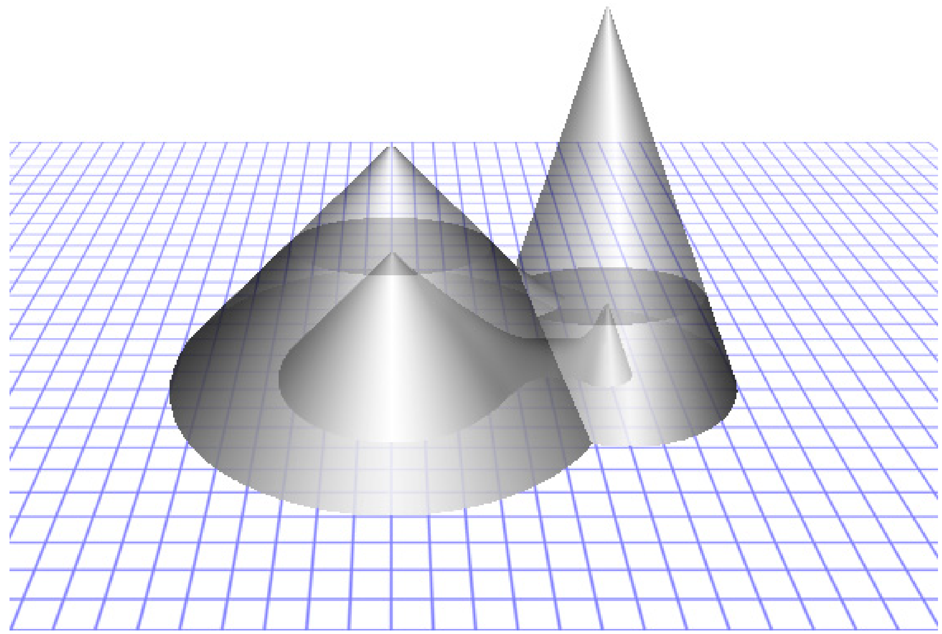


opened & original

Grayscale Morphology: Opening



erosion & opening



erosion & opening & original

Opening and Closing

Opening is erosion by *Z* followed by dilation by *Z*.

$$I \circ Z = (I \ominus Z) \oplus Z$$

The opening is the best approximation of the image FG that can be made from copies of the SE, given that the opening is contained in the original. I ○ Z contains no FG features that are smaller than the SE.

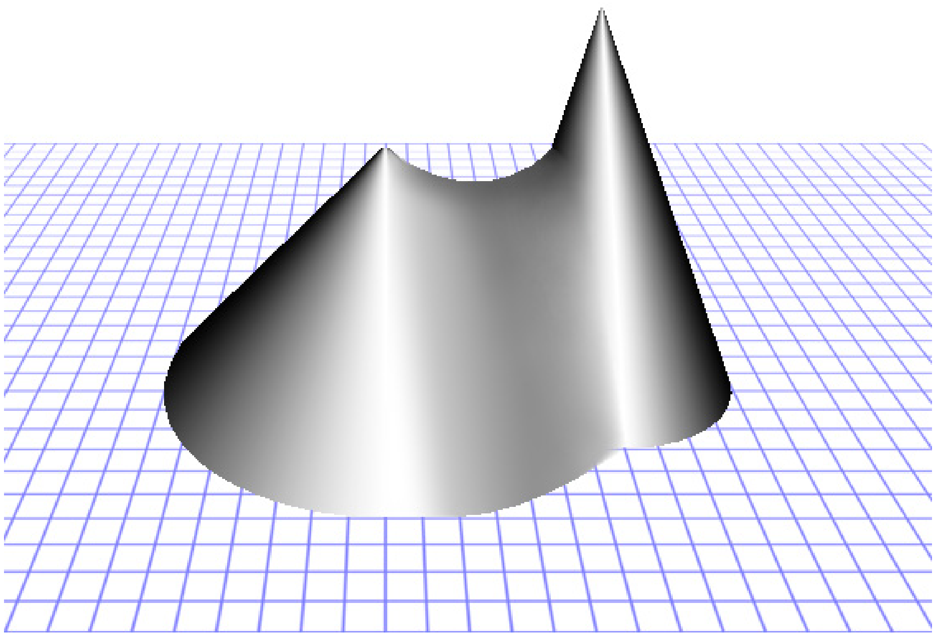
Closing is dilation by *Ž* followed by erosion by *Ž*.

$$I \bullet Z = (I \oplus \breve{Z}) \ominus \breve{Z}$$

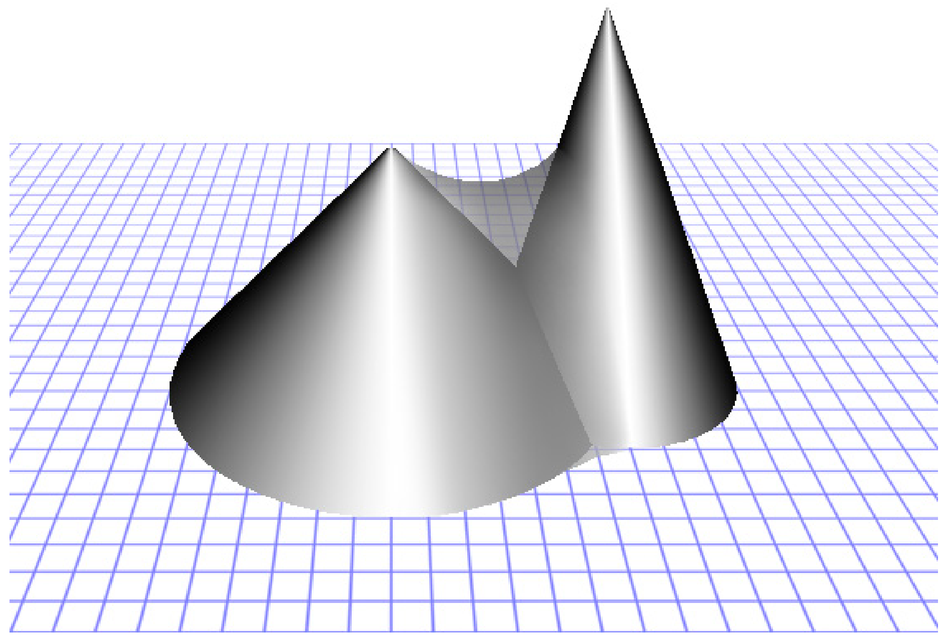
The closing is the best approximation of the image BG that can be made from copies of the SE, given that the closing is contained in the image BG. I • Z contains no BG features that are smaller than the SE.

Grayscale Morphology: Opening

Grayscale Morphology: Closing

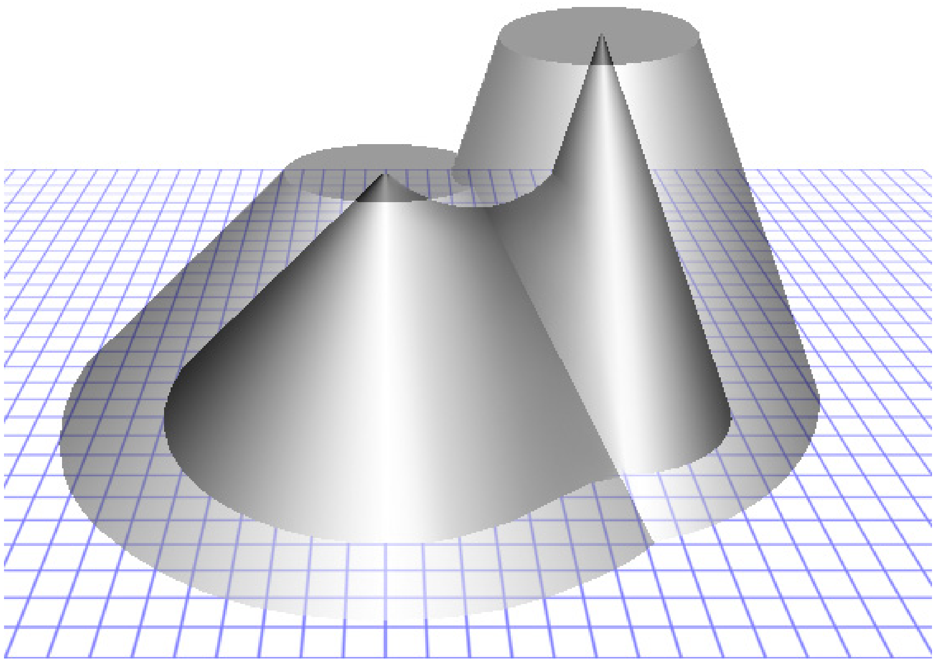


closing: dilation then erosion

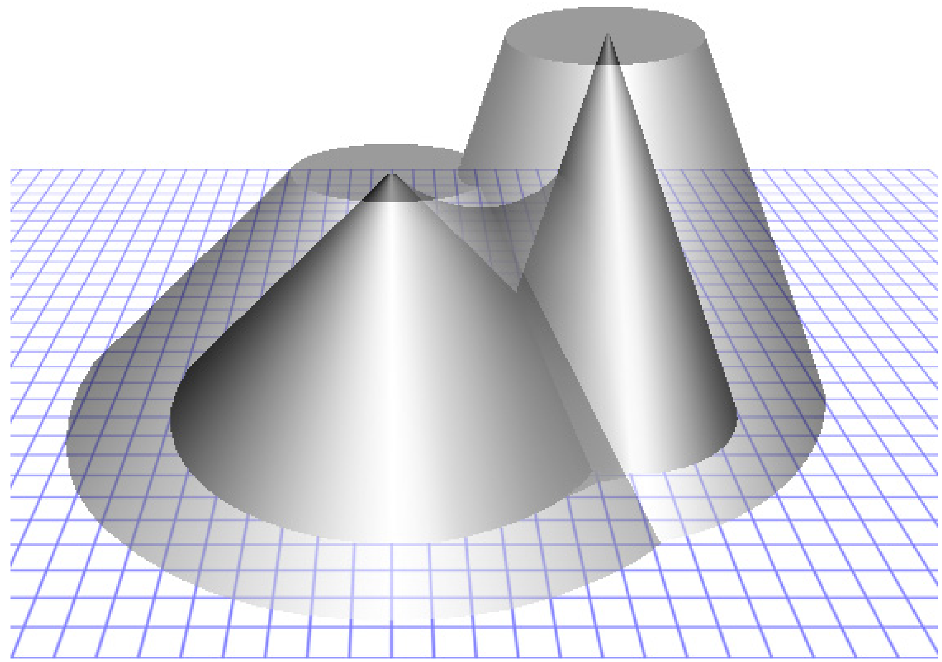


closing & original

Grayscale Morphology: Closing

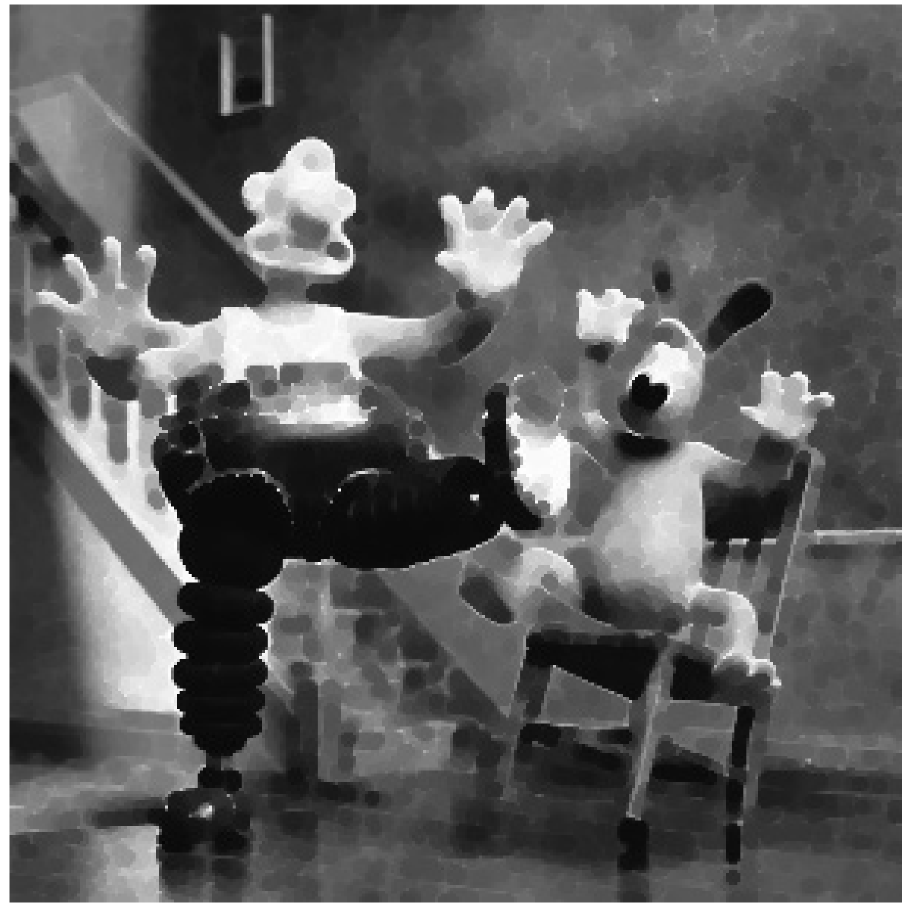


dilation over closing



dilation & closing & original

Grayscale Morphology: Closing



Duality Relationships

Erosion in terms of dilation: $\mathrm{I} \ominus Z = \left[\mathrm{I}^C \oplus \breve{Z}\right]^C$

Dilation in terms of erosion: $\mathrm{I} \oplus Z = \left[\mathrm{I}^C \ominus \breve{Z}\right]^C$

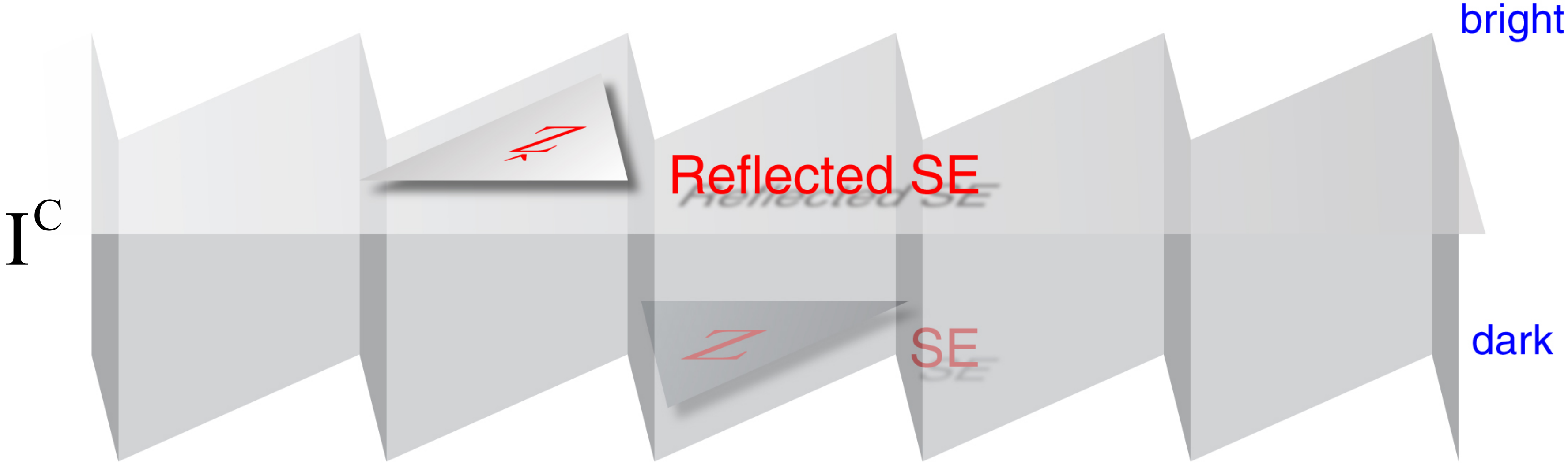
Opening in terms of closing: $\mathrm{I} \circ Z = \left[\mathrm{I}^C \bullet Z\right]^C$

Closing in terms of opening: $\mathrm{I} \bullet Z = \left[\mathrm{I}^C \circ Z\right]^C$

I^C is the complement of I and $\breve{Z}$ is the reflected SE.

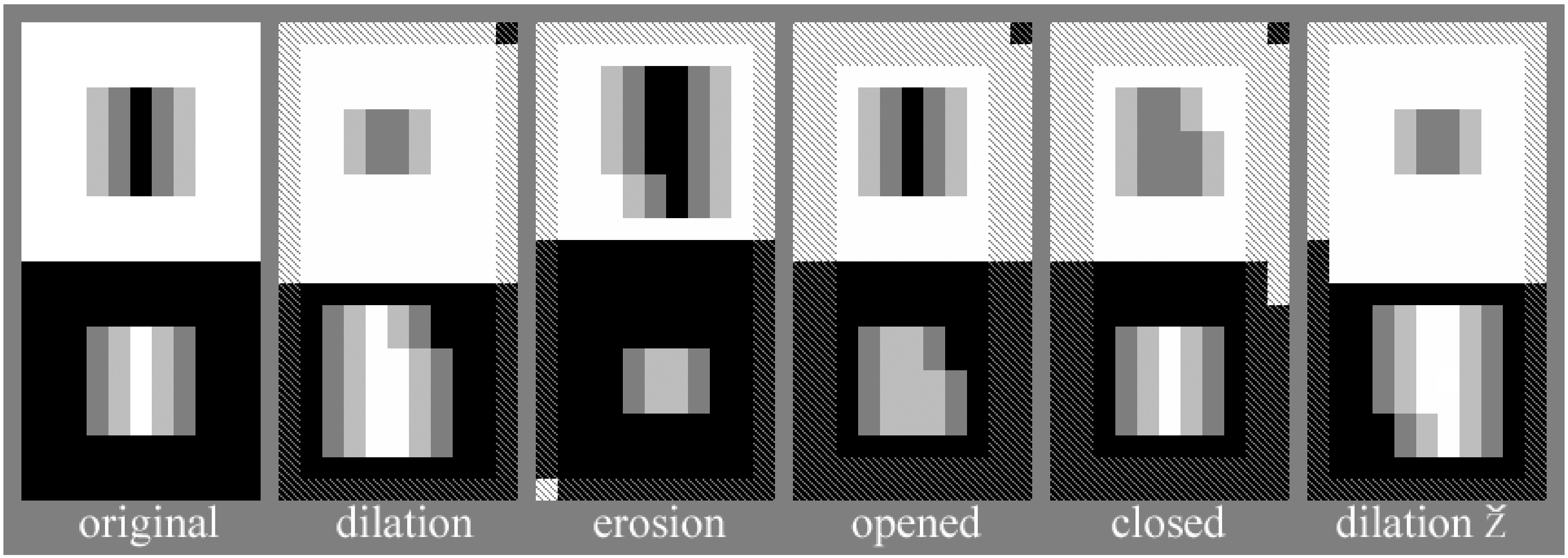
Duality Relationships

SE, Ž, operates on $\mathbf{I}^C$ as if it were Z operating on $\mathbf{I}$.



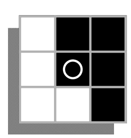
SE, Z, operates on $\mathbf{I}^C$ as if it were Ž operating on $\mathbf{I}$.

Gray Ops with Asymmetric SEs



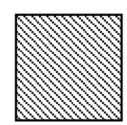
"L" shaped SE

O marks origin



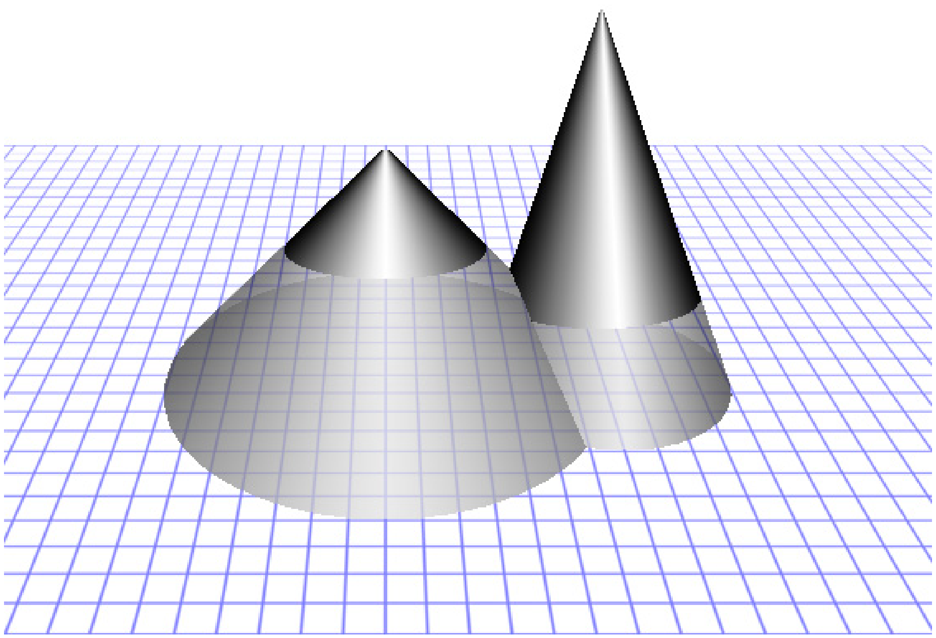
Foreground: white pixels

Background: black pixels

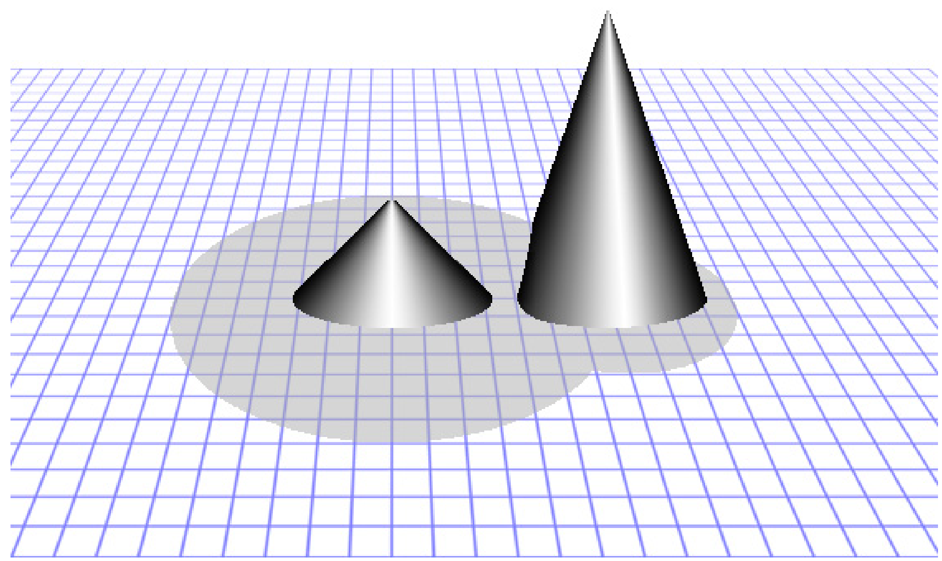


Cross-hatched pixels are indeterminate.

Grayscale Morphology: Tophat



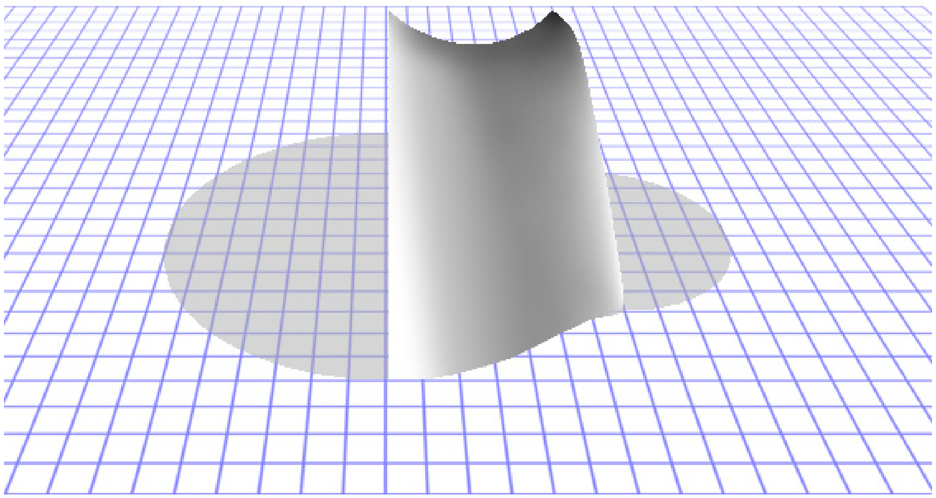
tophat + opened = original



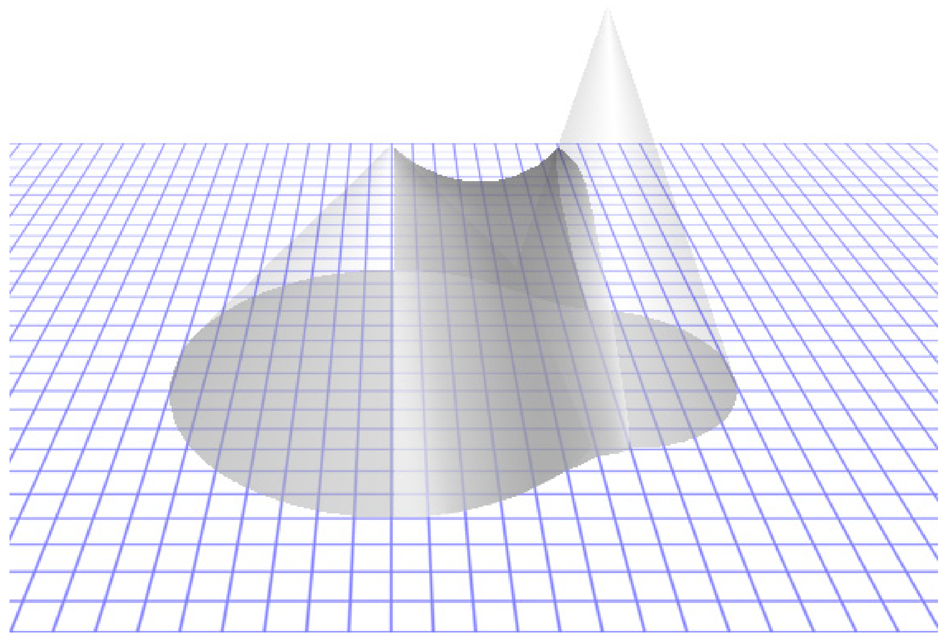
tophat: original - opening

Grayscale Morphology: Tophat

Grayscale Morphology: Bothat

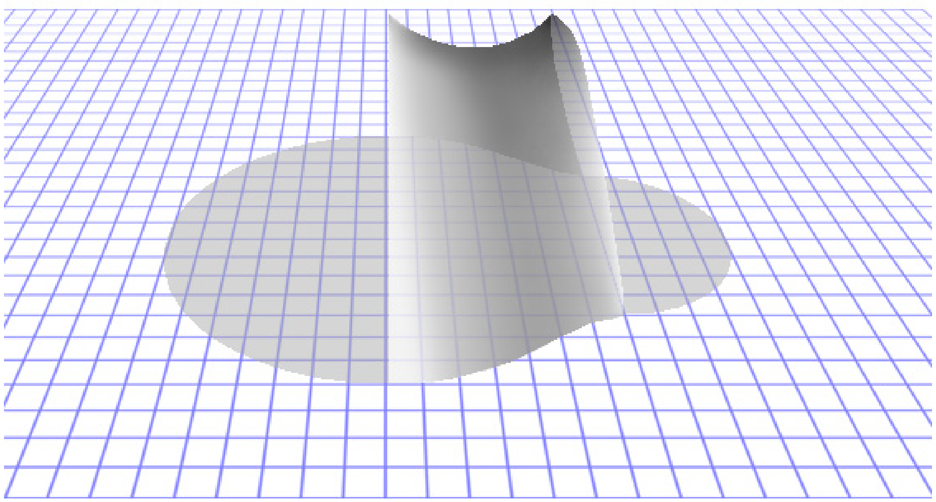


region added by dilation

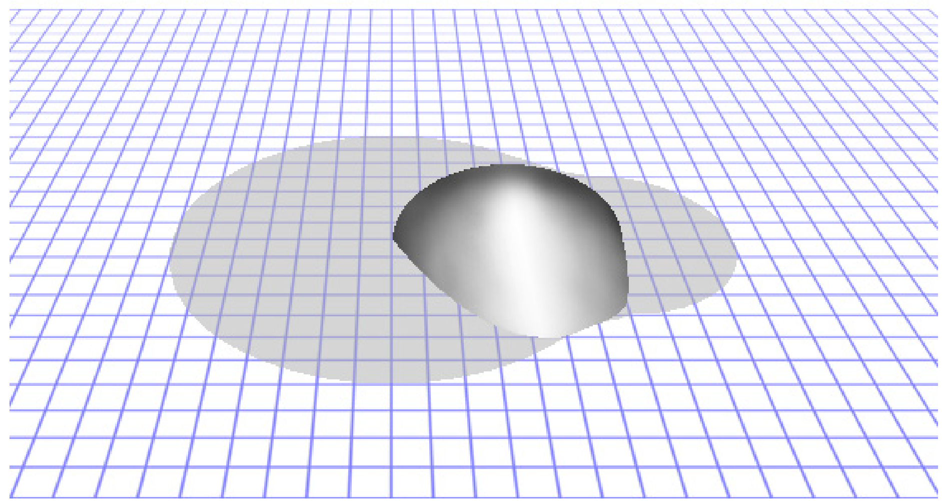


superimposed on original

Grayscale Morphology: Bothat

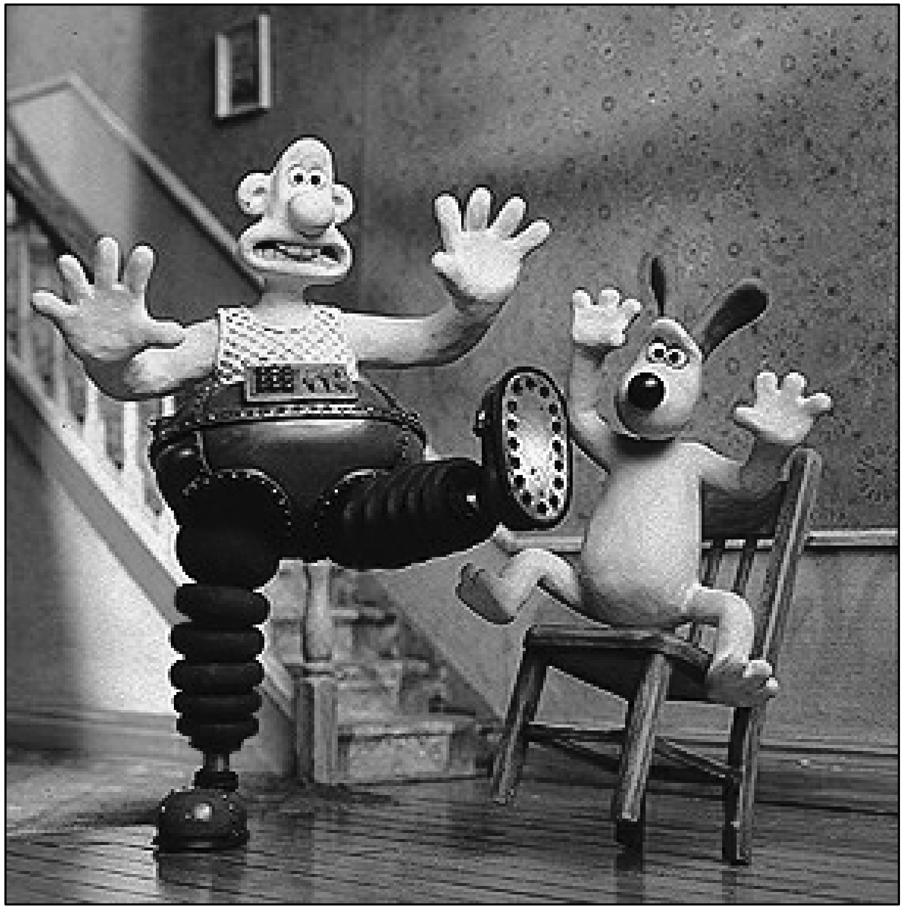


region added by dilation

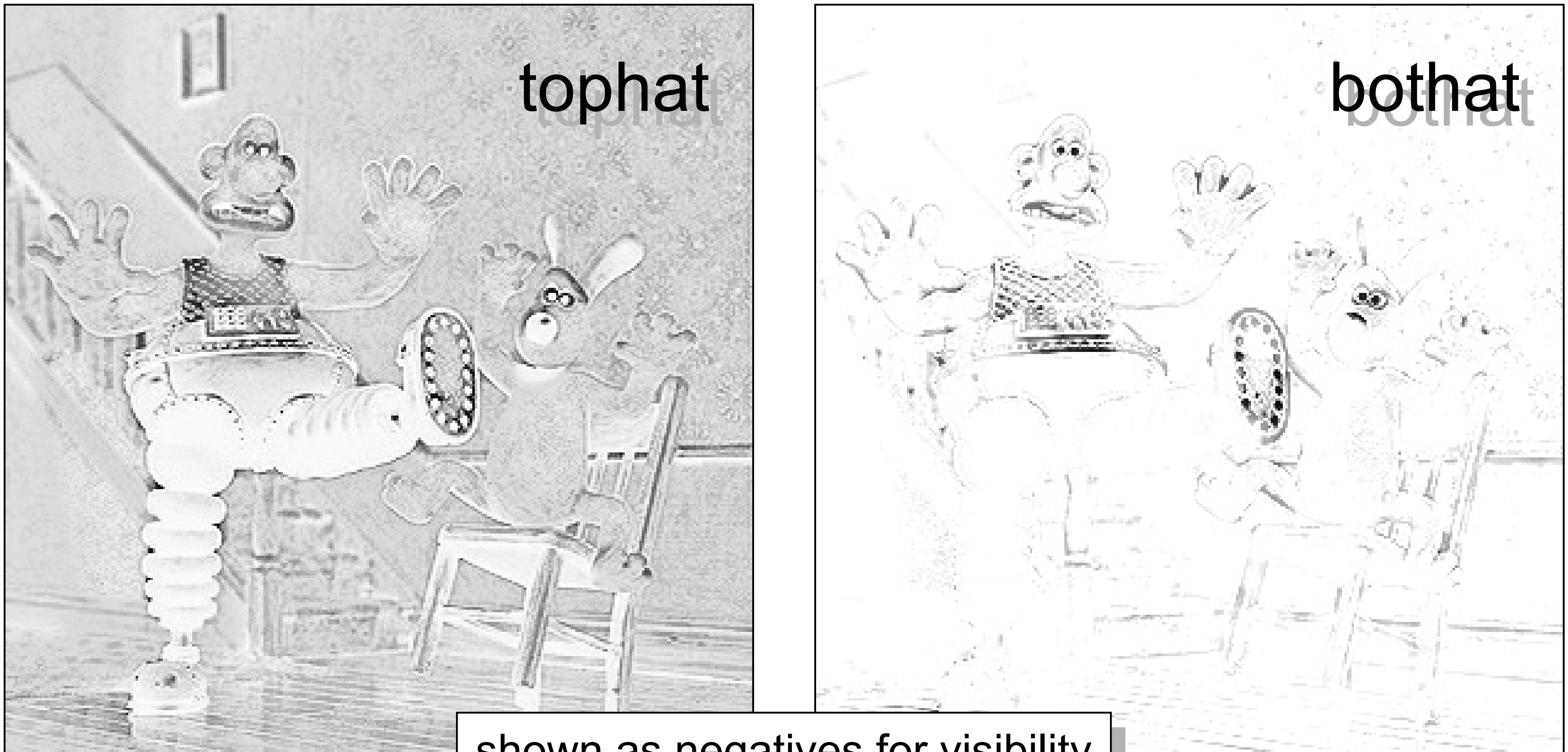


Bothat: closing - original

Grayscale Morphology: Bothat

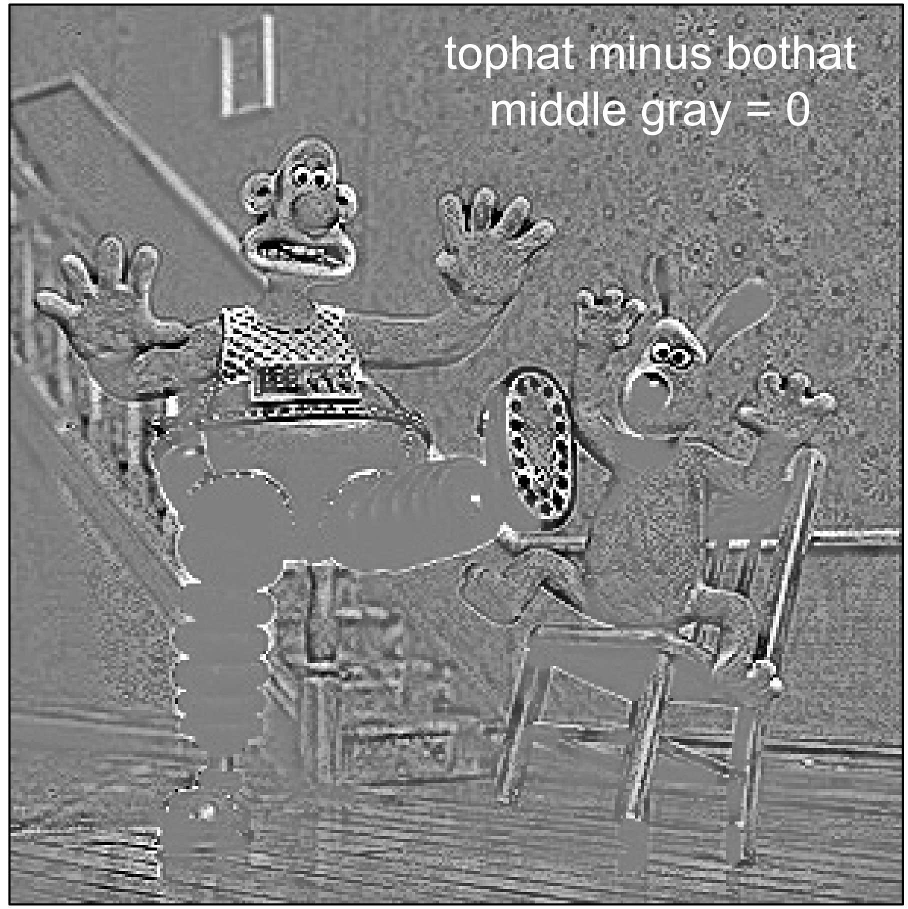


Grayscale Morphology: Tophat and Bothat



shown as negatives for visibility

Grayscale Morphology: Small Feature Detection



Algorithm for Grayscale Reconstruction

1. $J = I \circ Z$, where Z is any SE.
2. $T = J$,
3. $J = J \oplus Z_k$, where $k=4$ or $k=8$,
4. $J = \min\{I, J\}$, [pixelwise minimum of I and J.]
5. if $J \neq T$ then go to 2,
6. else stop; [J is the reconstructed image.]

This is the same as binary reconstruction but for grayscale images $J(r,c) \in I$ if and only if $J(r,c) \leq I(r,c)$.

Algorithm for Graysc

Usually a program for reconstruction will take both J and I as inputs. E.g,

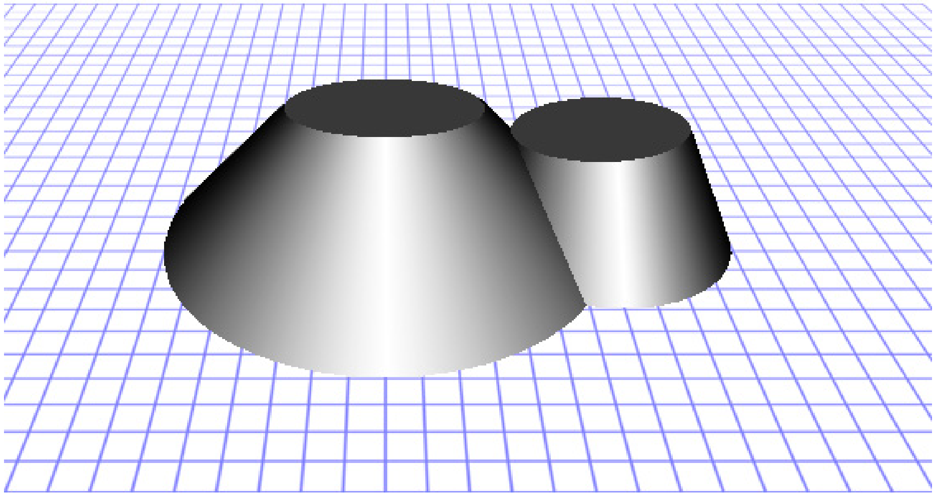
```
K = ReconGray(I,J,Z);
```

Then the algorithm starts at step 2.

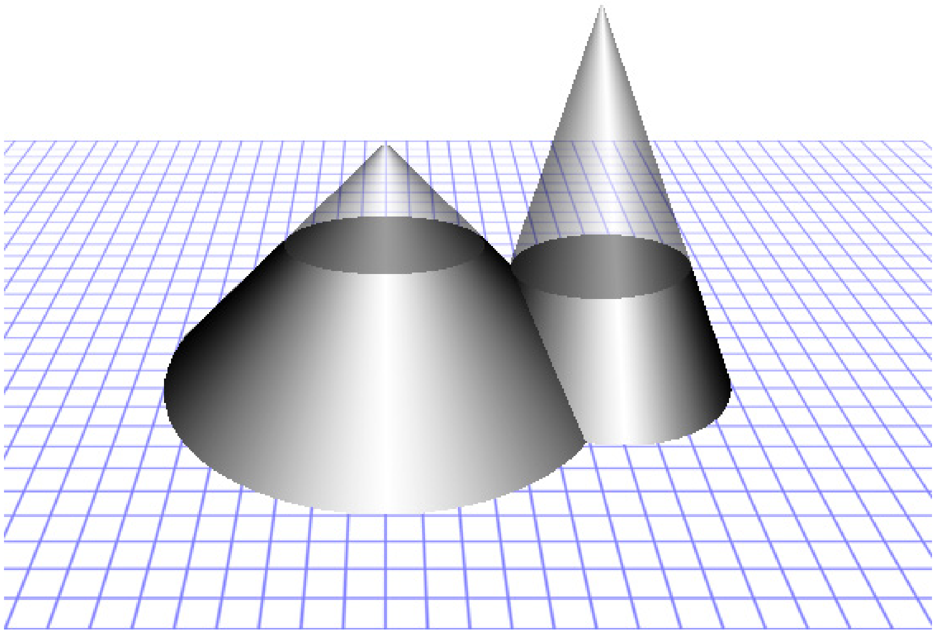
1. $J = I \circ Z$, where Z i
2. $T = J$,
3. $J = J \oplus Z_k$, where $k=4$ or $k=8$,
4. $J = \min\{I, J\}$, [pixelwise minimum of I and J.]
5. if $J \neq T$ then go to 2,
6. else stop; [J is the reconstructed image.]

This is the same as binary reconstruction but for grayscale images $J(r,c) \in I$ if and only if $J(r,c) \leq I(r,c)$. It also works on binary images.

Grayscale Reconstruction

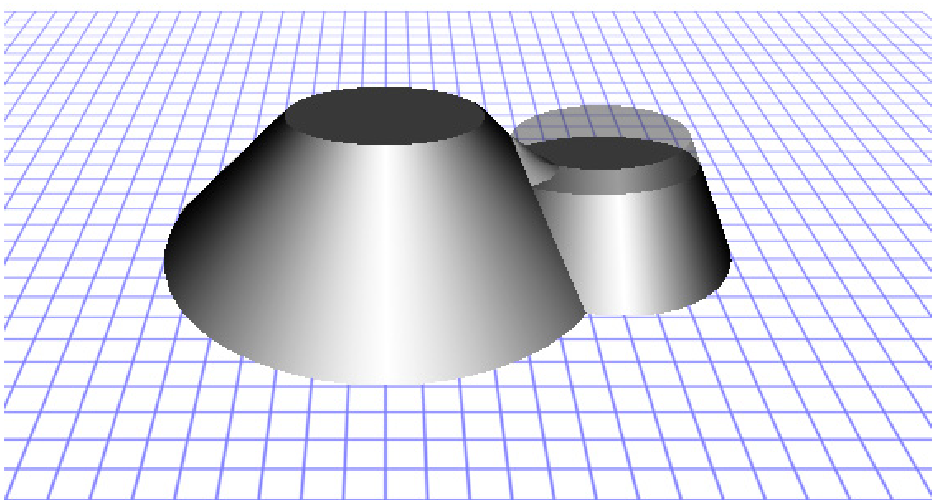


opened image

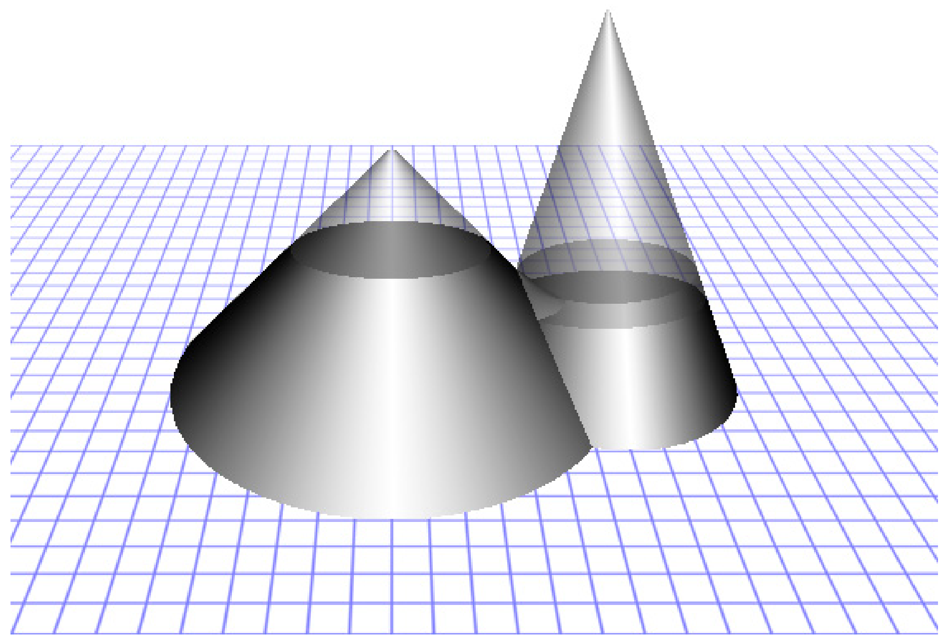


opened image & original

Grayscale Reconstruction

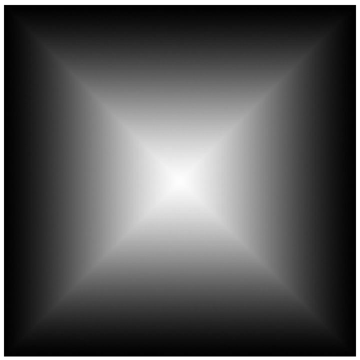


opened & recon. image

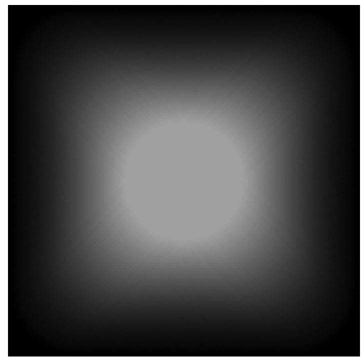


opened, recon., & original

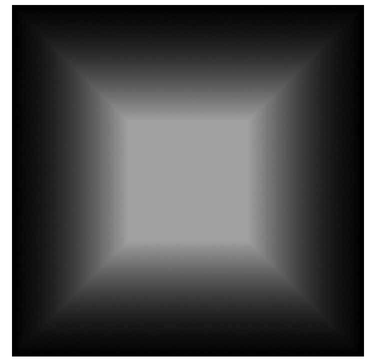
Grayscale Morphology: Reconstruction



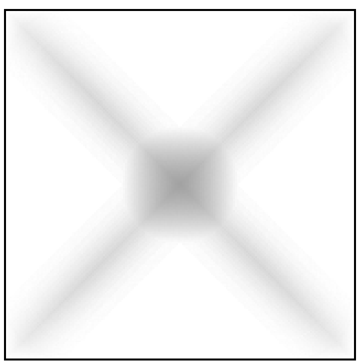
original



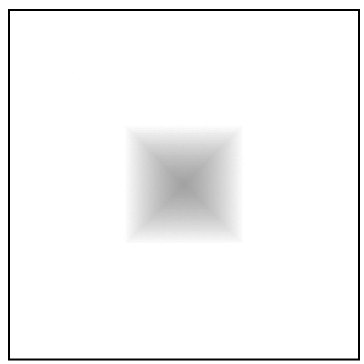
opened



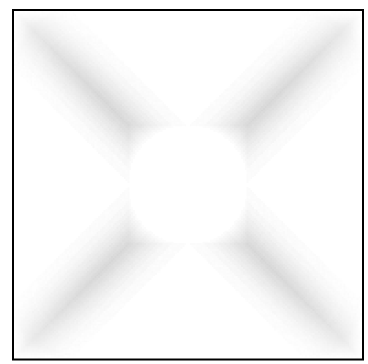
reconstructed



orig - opened



orig - recon



recon - opened

Grayscale Reconstruction