Exact and Approximate Queueing Analysis of the Selective-Repeat ARQ Protocol

Stijn De Vuyst, Sabine Wittevrongel, Herwig Bruneel

SMACS Research Group
Department of Telecommunications and Information Processing
Ghent University, Belgium

Introduction and motivation
  ARQ Retransmission Protocols
  The Classical Variants of ARQ

Exact queueing analysis of SR-ARQ
  Modelling assumptions
  Analysis: transmit and transit queue

Approximate analysis: ‘Ideal SR-ARQ’
  Principle
  Ideal SR-ARQ adjustments

Extension to a channel with Markov-modulated errors

Further work
ARQ retransmission protocols

‘Automatic Repeat reQuest’

Reliable transmission of packets over *unreliable* channel

- Receiver checks packets for errors and returns **ACK** or **NACK** accordingly. Transmitter **retransmits** packet if necessary.

![Diagram showing ARQ retransmission](image)
Classical Variants of ARQ

Stop and Wait ARQ

sender

channel

receiver

feedback delay $N$

error

Classical Variants of ARQ

Stop and Wait ARQ
Classical Variants of ARQ

Go-back-$N$ ARQ (GBN)

- Has window size $M$: maximum number of outstanding acknowledgements
- On NACK of packet $i$, all packets $i, i+1, \ldots$ in window are retransmitted

- Usually, $M = N$. 
Classical Variants of ARQ

Selective Repeat ARQ (SR)

- Only the NACKed packets are retransmitted
- Most efficient, but requires resequencing!
Modelling assumptions

Discrete-time model of the sender queue for SR-ARQ

- number of arrivals per slot: iid with common pgf $A(z)$
  \[ \lambda = P'(1) = E[a] \]
- $\infty$ queue capacity
- 1 slot transmission time per packet
- fixed feedback delay: $N$ slots
- channel errors are uncorrelated
  for each packet: NACK with probability $e$

Exact analysis of this model:

  Analysis in transformation domain
  Lacks efficient procedure for calculating the $2^{N-1}$ unknowns

  State transition matrix only

\[ \delta \triangleq 1 - e \]
Queueing analysis (Konheim)

Sender queue:
**TRANSMIT queue**: packets waiting for first transmission
\[ h_k = \text{content of transmit queue in slot } k \]
**TRANSIT queue**: transmitted packets waiting for ACK/NACK
\[ m_{i,k} = 0, 1 \text{ is 1 if a packet was transmitted in slot } k - i + 1 \]

\[ \Rightarrow \text{ Markovian description of the system state in slot } k: \]
\[ \langle m_1,k, m_2,k, \ldots, m_N,k; h_k \rangle \]

Let \( \Sigma_N \) be the set of all bit-sequences of length \( N \).
For all \( \sigma = (\sigma_1, \sigma_2, \ldots, \sigma_N) \in \Sigma_N \), let
\[ \pi_{\sigma,k}(z) = E[z^{h_k} \mathbb{1}(m_{1,k} = \sigma_1, \ldots, m_{N,k} = \sigma_N)] \]
Queueing analysis

transmit queue

transit queue

h_k
m_k,1
m_k,2
m_k,3
m_k,4
m_k,5

0 2 1 0 0 0 1 1 1 1 1 1 1 1 1 0 0
m_k,1
1 1 1 1 0 0 1 1 1 1 1 1 1 1 1 0 0
m_k,2
0 0 1 1 1 0 0 1 1 1 1 1 1 1 1 0 0
m_k,3
0 0 0 1 1 1 0 0 1 1 1 1 1 1 1 0 0
m_k,4
0 0 0 0 1 1 0 0 1 1 1 1 1 1 1 0 0
m_k,5
0 0 0 0 0 0 1 1 0 0 1 1 1 1 1 0 0

ACK ACK ACK ACK
NACK NACK NACK

1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
**System equations**

Assume equilibrium \((k \to \infty)\) \ldots drop index \(k\)

\[2^N\text{ system equations}\]

\[\nu(z) = \delta + (1 - \delta)z\]

For \(N = 4\):

\[
\begin{align*}
\pi_{0000}(z) &= A(z)\alpha_{000} & \pi_{1000}(z) &= A(z)/z \left( \pi_{0000}(z) + \nu(z)\pi_{0001}(z) - \alpha_{000} \right) \\
\pi_{0001}(z) &= A(z)\alpha_{001} & \pi_{1001}(z) &= A(z)/z \left( \pi_{0010}(z) + \nu(z)\pi_{0011}(z) - \alpha_{001} \right) \\
\pi_{0010}(z) &= A(z)\alpha_{010} & \pi_{1010}(z) &= A(z)/z \left( \pi_{0100}(z) + \nu(z)\pi_{0101}(z) - \alpha_{010} \right) \\
\pi_{0011}(z) &= A(z)\alpha_{011} & \pi_{1011}(z) &= A(z)/z \left( \pi_{0110}(z) + \nu(z)\pi_{0111}(z) - \alpha_{011} \right) \\
\pi_{0100}(z) &= A(z)\alpha_{100} & \pi_{1100}(z) &= A(z)/z \left( \pi_{1000}(z) + \nu(z)\pi_{1001}(z) - \alpha_{100} \right) \\
\pi_{0101}(z) &= A(z)\alpha_{101} & \pi_{1101}(z) &= A(z)/z \left( \pi_{1010}(z) + \nu(z)\pi_{1011}(z) - \alpha_{101} \right) \\
\pi_{0110}(z) &= A(z)\alpha_{110} & \pi_{1110}(z) &= A(z)/z \left( \pi_{1100}(z) + \nu(z)\pi_{1101}(z) - \alpha_{110} \right) \\
\pi_{0111}(z) &= A(z)\alpha_{111} & \pi_{1111}(z) &= A(z)/z \left( \pi_{1110}(z) + \nu(z)\pi_{1111}(z) - \alpha_{111} \right)
\end{align*}
\]

\[2^{N-1}\text{ unknowns}\ : \ \alpha_{\sigma} \triangleq \pi_{\sigma 0}(0) + \delta \pi_{\sigma 1}(0)\]
Calculate the unknowns

- Probability mass function of arrival distribution:

\[
a_i, \ i = 0, 1, \ldots \text{ such that } A(z) = \sum_{i=0}^{\infty} a_i z^i
\]

- Define \( \tilde{a}_i = \delta a_i + (1 - \delta) a_{i-1} \) (\( i > 0 \)) and \( \tilde{a}_0 = \delta a_0 \)

Construct \((j \times (j + 1))\)-matrices \( P_j \) and \((j \times 1)\)-matrices \( Q_j \) (\( j = 1, \ldots, N - 2 \))

\[
P_0^1 = \begin{bmatrix} a_1 & a_0 & 0 & \cdots & 0 \\ a_2 & a_1 & a_0 & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ a_j & \cdots & a_0 & \end{bmatrix}, \quad P_1^1 = \begin{bmatrix} \tilde{a}_1 & \tilde{a}_0 & 0 & \cdots & 0 \\ \tilde{a}_2 & \tilde{a}_1 & \tilde{a}_0 & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \tilde{a}_j & \cdots & \tilde{a}_0 & \end{bmatrix}
\]

\[
Q_0^1 = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_j \end{bmatrix}, \quad Q_1^1 = \begin{bmatrix} \tilde{a}_1 \\ \tilde{a}_2 \\ \vdots \\ \tilde{a}_j \end{bmatrix}
\]
Calculate the unknowns

- Define \( \gamma_\sigma \triangleq a_0 P_1^{\sigma_1} P_2^{\sigma_2} \cdots P_{k-1}^{\sigma_{k-1}} Q_k^{\sigma_k} \)

Then for ‘stage \( k \)’:

\[
\pi(1)_k(0) = \sum_{\sigma} \gamma_\sigma \alpha_{\sigma}
\]

\( \Rightarrow 2^{N-1} - 1 \) relations

Last relation: normalisation condition \( \sum_{\sigma} \alpha_\sigma = 1 - \frac{\lambda}{\delta} \)

- Solve linear system of \( 2^{N-1} \) equations
  computationally complex!!
Distribution of the sender queue content

- **Sender queue content** \( u = \text{transmit queue } h + \text{transit queue } m_1 + \ldots + m_N \)

\[
U(z) \triangleq \mathbb{E}[z^{h+m_1+m_2+\ldots+m_N}]
\]

\[
= \sum_{\sigma} \pi_{\sigma}(z)
\]

\[
= \ldots
\]

\[
= \frac{\delta(z - 1)A^{N+1}(z)}{z - \nu(z)A(z)} \sum_{\sigma} \alpha_{\sigma}(\nu(z)||\sigma||}
\]

where \( ||\sigma|| = \text{number of 1's in the bit sequence } \sigma \)

- **Mean sender queue content:**

\[
U'(1) = (N + 1)\lambda + \frac{2(1 - \delta)\lambda + A''(1)}{2(\delta - \lambda)} + \frac{\delta(1 - \delta)}{\delta - \lambda} \sum_{\sigma} \alpha_{\sigma}||\sigma||
\]

- **Higher-order moments** . . .

- **Tail distribution**

Geometric decay rate determined by smallest positive real root of denominator \( z - \nu(z)A(z) \)
Konheim’s challenge (1980)

- For large $N$, there is a **state space explosion**

For block ARQ there are no special numerical problems encountered in tabulating the probabilities. In select ARQ the matter is quite different; there appears no way of obtaining a solution without a consideration of a number of states (corresponding to the history) which grows exponentially with $N$. This limits severely the applicability of the solution for example in the evaluation of a satellite system in which $N$ will be considerably larger than 4. Hopefully, this paper will stimulate others to consider approximate methods to analyze the model.

- We need to find a solution for large $N$ that is **asymptotically correct**
Approximate analysis: ‘Ideal SR-ARQ’
Approximate analysis: ‘Ideal SR-ARQ’
Approximate analysis: ‘Ideal SR-ARQ’


Ideal SR-ARQ assumptions

- ACK/NACK message arrives immediately after transmission
- Leads to GI-Geo-1 queue at the sender
  queueing model with geometric service times!

\[
U_{\text{GI-Geo-1}}(z) = \frac{\delta(z - 1)A(z)}{z - \nu(z)A(z)} \left(1 - \frac{\lambda}{\delta}\right)
\]

- Ideal ARQ ‘ADJUSTMENTS’ for mean packet delay and mean queue content:

\[
E[d_{\text{Ideal SR}}] = \frac{E[u_{\text{GI-Geo-1}}]}{\lambda} + (N - 1) \frac{1}{1 - e}
\]

\[
E[u_{\text{Ideal SR}}] = E[u_{\text{GI-Geo-1}}] + \lambda(N - 1) \frac{1}{1 - e}
\]

To account for the fact that each transmissions actually takes \(N\) slots, (only ok for first-order moments)
Comparison Ideal SR-ARQ with exact analysis

- Poisson arrival distribution
- $e = 0.5$
- $N = 1, 3, 5, 7, 8$
Extension to Markov-modulated errors

Two-state channel model

- Two possible channel states: GOOD (0) or BAD (1)
  - channel state \( r_k \) in slot \( k \) is 0 or 1

Errors depend on channel state!

In state \( i = 0, 1 \):
- probability of NACK is \( e_i \)
- probability of ACK is \( 1 - e_i \)

Exact analysis

Increase the size of the state space even further:
- additional system variable \( r_k \)
- Complexity doubles!

\[ \langle r_k; m_{1,k}, m_{2,k}, \ldots, m_{N,k}; h_k \rangle \]
Extension to Markov-modulated errors

Ideal SR-ARQ

- The Ideal SR-ARQ approximation is ‘ideal’ to account for correlation: correlated arrivals, correlated errors

- The small state space of Ideal SR-ARQ is easily extended to account for correlation

  E.g.

    Two-state Markov-model for the channel errors

    General Markov-modulation of both arrivals and errors (independent)
Further work

SR-ARQ model with independent channel errors

- Find a solution that uses smaller state space \( (< 2^{N-1}) \) is asymptotically correct for \( N \rightarrow \infty \)
- Pgf of the sender queue content and delay distribution
- Resequencing at the receiver side

SR-ARQ model with Markov-modulated channel errors

- Find appropriate system equations and determine unknowns
- Pgf of the sender queue content and delay distribution
- Find suitable asymptotic \( (N \rightarrow \infty) \) solution
- How good is Ideal SR-ARQ in this case?

Numerical algorithms
Minimise computational complexity as much as possible