ELSEVIER

European Journal of Operational Research 108 (1998) 74–93

EUROPEAN JOURNAL OF OPERATIONAL RESEARCH

Theory and Methodology

Analysis of a single-stage, single-product, stochastic, MRP-controlled assembly system

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Received 1 July 1995; accepted 1 December 1996

Abstract

This paper models a single-stage, single-product, stochastic assembly system, operating according to an Materials Requirements Planning controlled (MRP) ordering philosophy. It deals explicitly with the underlying stochastic process that describes the end-product inventory position, enabling production lead times to be treated as independent and generally distributed random variables. The inventory position process is identified as a Markov renewal process, and this structure is exploited to determine system performance measures such as average inventory level, average backorder level, and the probability distribution of the end-product inventory position. An example, which demonstrates the type of analysis possible, focuses on quantifying the effect of kitting on the availability of end-products. © 1998 Elsevier Science B.V.

Keywords: Manufacturing; Modeling; Material requirements planning (MRP) control; Kitting in assembly; End-product inventory position; Markov renewal process

1. Introduction

The traditional view of Materials Requirements Planning (MRP), which commonly implements the 'push' philosophy of material flow management, is based on the assumption that the production system operates deterministically. System operations are usually planned for a series of discrete time intervals known as time buckets, and order release quantities are planned, depending on forecasted demands (Orlicky [14]). The deterministic assumption seems unrealistic, since, in general, production takes place in a stochastic environment and demand for end-products is seldomly completely predictable.

The purpose of this paper is to model a single-stage, single-product, MRP-controlled (a term coined by Buzacott and Shantikumar [5]) production/inventory system in which processing time and demand are stochastic. The objectives are to analyze the underlying stochastic process that describes the inventory position and to obtain various system performance measures, lending insight into MRP-control. In particular, we demonstrate use of the model to study assembly operations, quantitatively describing the effects of the kitting

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PII S0377-2217(97)00047-7

process on end-product inventory position. MRP is most relevant to assembly systems, yet no prior models lend insight into the MRP control of such a system.

The (generic) system that we model is composed of production center P and end-product inventory I as shown in Fig. 1. The production center consists of machines $P_1, \ldots, P_n$, each of which processes a different type of part, and machine M, which assembles one part of each type into each end-product. An inventory-replenishment order for a lot of end-products requires lots of part types $1, \ldots, n$ to be produced on machines $P_1, \ldots, P_n$, respectively. Completed parts are accumulated in a kit, and, when all part types are ready, the completed kit is transferred to machine M where the parts are assembled. Customers for end-products arrive randomly at inventory I. Each demands one end-product, which is supplied immediately if stock is available. Unsatisfied demands are backordered.

In the deterministic MRP environment, inventory position (i.e., the number of end-products on hand plus on order minus the number on backorder) is checked at fixed time intervals and replenishment orders for required quantities are placed at these times. Our model is based on an analog of this practice, resulting in a specialized ordering strategy for the stochastic environment. Inventory position is checked at the instant a replenishment order arrives at I and an order for one lot of end-products is placed on P if replenishment is necessary to assure future (expected) safety stock. The rationale behind this ordering strategy is that the single-stage production facility can start processing an order only after it finishes the previous one. An order is not placed while the facility is producing, since such an order would just wait in queue. This ordering strategy allows use of the latest possible information about the end-product inventory position and the order quantity is determined accordingly, invoking the key look-ahead characteristic of MRP control.

In the deterministic MRP environment, a replenishment order is assumed to arrive after a *constant* lead time. In the stochastic case, we model replenishment (i.e., production) lead times as independent and generally distributed (G.I.) random variables.

If the criterion that assures future (expected) safety stock is satisfied when the inventory position is checked, a replenishment order is not placed. However, the inventory position is checked again after a random duration. This is the stochastic analog of the practice used in the deterministic environment in which, if no order is placed in some period, the inventory position is checked again after a deterministic duration. Whenever inventory position is checked in the stochastic case, the look-ahead feature assures future (expected) safety stock levels.

This stochastic production/inventory system can be viewed as a double-ended queue analogous to a taxi cab queue in which customers arrive at random to pick up cabs (i.e., end-products) at one end and empty cabs (i.e., replenishment orders) return at random to the other end.

Few stochastic models of MRP-controlled production/inventory systems appear in the literature because such systems present difficult challenges to analysis. A single-stage production facility serving the demand for a single product was first studied by Gavish and Graves [11], who assumed a constant production rate. In a companion paper, Gavish and Graves [12] considered an arbitrary distribution for production time and Poisson demand for the end-product, and they developed a continuous review policy to minimize expected production/inventory cost. However, their approach does not incorporate MRP control for production; rather, it

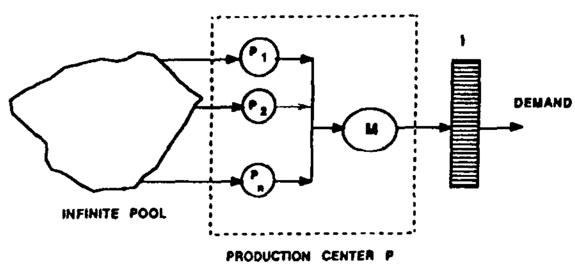

Fig. 1. Single-stage production/inventory system.

follows an (R,r) policy. In other MRP-related research, Yano [20] devised a method to optimize planned lead times in serial production systems and others (e.g., Chang [6], Wijngaard and Wortman [18]) studied the roles of safety stock and safety lead time.

A two-stage production/inventory system, operating according to a base stock policy, was studied by Zipkin [21]. By assuming that production times at both stages are exponentially distributed and that demand follows a Poisson process, he was able to analyze the system as a simple Markov process. He obtained several performance measures, including the average inventory level and the average number of backorders for the end-product.

In recent years, researchers have devoted much attention to studying pull systems (e.g., Deleersnyder et al. [8]; Zipkin [21], Di Mascolo et al. [9]) and to comparing the operating philosophies of MRP with those of kanban-controlled, pull production systems (e.g., Spearman and Zazanis [16]). Buzacott [2] modeled a kanban-controlled production system with stochastic demand and stochastic processing times as a linked queueing network, deriving various performance measures: delivery performance, inventory levels, and number of kanbans. He also showed that a conventional MRP system can be approximated by a kanban-controlled production system if the triggering of kanbans is effected by forecasted instead of actual demand.

In related work, Buzacott and Shanthikumar [4] generalized MRP and other control mechanisms to manage material flow in manufacturing systems consisting of multiple cells. Buzacott et al. [3] compared the service levels that can be attained in finished goods inventory by using either MRP or base stock control in serial production systems. Their model of system dynamics led to an approximation scheme that compared favorably with simulated performance.

Buzacott and Shantikumar [5] presented a model that is, perhaps, the most faithful representation of current MRP control. They analyzed the use of safety stock and safety lead time in a single-stage, MRP-controlled production/inventory system with stochastic demand, stochastic processing times, and limited production capacity. They modeled the effects of master scheduling, which may, in principle, be used to reduce or eliminate demand variability over the lead time, concluding that safety stock is more robust and that safety time is preferable only when lead time demand can be forecasted accurately.

A production facility with a single processor producing multiple products was studied by Altiok and Shiue [1]. Assuming Poisson demand, arbitrarily distributed processing and set-up times, and backorders, they were able to determine various performance measures including average inventory level, average backorder level, and the probability distribution of backorders. They identified that a single-product system reduces to an M/G/1 type queueing problem, which we discuss further later.

The main differences between our analysis of the MRP controlled production/inventory system and prior studies are: (1) we address assembly operations, (2) we deal explicitly with the underlying stochastic process that describes kit and end-product inventory positions, (3) we allow production lead times to be G.I., and (4) we incorporate stochastic reorder timing to correspond with the assumed stochastic environment. We identify the inventory position process to be a Markov renewal process and exploit this structure to determine system performance measures such as average inventory level, average backorder level, and the probability distribution of the end-product inventory position. We give an example to demonstrate the type of analysis possible, focusing on quantifying the effect of kitting on the availability of end-products. To our knowledge, this approach to analyzing the MRP system is novel.

The body of this paper is organized in five sections. Section 2 describes the model of a single-stage MRP-controlled assembly system and the replenishment order placement strategy. The stochastic process underlying the *inventory position process* is identified as a Markov renewal process in Section 3. The model is analyzed in Section 4 by obtaining measures of the *inventory position process*: the Palm probability vector ν, observed at replenishment order arrival times, and the stationary probability vector, η, observed at arbitrary times. The stationary probability vector of the end-product inventory position, f, is also determined. Using the performance measures of the assembly system obtained in Section 4, the effect of kitting on MRP-controlled assembly is described through an example in Section 5. Finally, the conclusion is presented in Section 6.

2. Model description

Fig. 1 depicts a system in which production facility P produces end-products and sends them to end-product inventory I. We assume that demands occur for one end-product at a time according to the Poisson process. We note that, in practice, it is possible for an MRP system to 'freeze' requirements over a certain future duration with the goal of reducing uncertainty in demand. However, customers might still cancel orders or request later due dates, and the profit incentive would discourage a company from refusing a new order with a short lead time if it could be produced with available capacity. Thus, uncertainties exist in any case, so we follow the lead of most related studies (see the literature review) and invoke the Poisson assumption.

Within the production facility, machines $P_1, P_2, \ldots, P_n$ each withdraw raw materials from an infinite pool and prepare parts required for assembly. As soon as a kit of parts required to produce one batch of end-products is composed, it is sent immediately to machine M for assembly. After assembly, the batch of end-products is sent immediately to inventory I.

The inventory position is observed at an instant when a replenishment order arrives at I. Depending on the inventory position immediately after the arrival of a replenishment order, the quantity of the next order to be placed on P is determined. We agree, a priori, that an order will be placed, either for quantity Q or for nothing at all, depending on the inventory position at that time. This ordering strategy has been used for deterministic assembly systems (Vollman et al. [22]) in which replenishment order lead time is assumed to be constant. In deterministic MRP, when an order is released for quantity Q, it arrives after a *constant* lead time L. In our case, we consider this lead time to be a random variable. Again, in deterministic MRP, when no order is placed in some period (i.e., the order release quantity is zero), the projected available balance is checked at the end of a time bucket of deterministic duration, and a new order release quantity is determined. In modeling stochastic MRP, when the order release quantity is zero, we observe the system again after a *random* lead time.

Various MRP lot-sizing techniques are available in the literature; for example, lot-for-lot, Economic Ordering Quantity (EOQ), Part Period Balancing (PPB) rule, and Least Unit Cost (LUC) rule (Nahmias [13]). In this paper we assume that order quantity Q is the Economic Ordering Quantity. However, no attempt has been made to obtain the optimum batch size, since stochastic optimization is beyond the scope of this study.

To describe the ordering philosophy, we first define the following notation:

s = safety stock;
Q = Economic Ordering Quantity;
τ_m = instant of the mth replenishment order arrival;
X_m = inventory position at τ_m (prior to a replenishment order arrival);
R_m = replenishment order quantity arriving at time τ_m (either 0 or Q);
λ = mean rate of the Poisson demand process.

After an order for Q items is placed, the production center P takes random time L_Q to produce the lot, and the replenishment order arrives at time τ_{m+1}, a time duration of L_Q after τ_m. If no order is placed at time τ_m (i.e., replenishment quantity $R_{m+1} = 0$), the inventory position is checked again at time τ_{m+1}, after random time duration L_R following τ_m, to determine if an order should be placed at that time. The random variables L_Q and L_R are G.I. distributed with distribution functions $\Phi(\cdot)$ and $\Psi(\cdot)$, respectively.

At time τ_m an order is placed, either for Q or 0 end-products, assuring that the expected safety stock that will exist when the replenishment order (of lot size R_{m+1}) arrives at time τ_{m+1} will be at least s. For practical purposes, we assume that $E[L_Q] \geq E[L_R]$. If the order quantity is Q, the following inequality must be satisfied to assure an expected safety stock of s end-products at time τ_{m+1}:

$$X_m + R_m - \lambda E\left[L_Q\right] \geq s,$$

in which $E[L_Q]$ = the expected duration of the random lead time L_Q. Rearranging, the criterion becomes

$$X_m + R_m \geq K_1,$$

in which $K_1 = s + \lambda E[L_Q]$. If the order quantity is zero, the following inequality must be satisfied to assure an expected safety stock of s end-products at time τ_{m+1}:

$$X_m + R_m - \lambda E[L_R] - \lambda E[L_Q] \geq s,$$

in which $E[L_R]$ = the expected duration of the random lead time L_R. Note that $\lambda E[L_Q]$ must be included, since, if $R_m = 0$, the earliest expected time that a replenishment order of quantity Q might arrive is $\tau_m + L_R + L_Q$. Rearranging, the criterion becomes

$$X_m + R_m \geq K,$$

in which $K = s + \lambda E[L_Q] + \lambda E[L_R]$. Since $K \geq K_1$, the event of placing an order when inventory position reaches K contains the event of placing an order when inventory position reaches K_1. The ordering strategy which we propose to represent MRP control can be summarized by:

$$\text{if } K - X_m - R_m > 0, \text{ order } Q, \tag{1}$$

$$\text{if } K - X_m - R_m \leq 0, \text{ order } 0. \tag{2}$$

When the inventory position at time τ_m is less than the pre-specified value K, the quantity Q is ordered. Otherwise, no order is placed at time τ_m.

In a deterministic MRP environment, the system is observed at fixed time intervals and replenishment orders for required quantities are placed at these times. In the stochastic environment, we propose to observe the system at random times $\tau_m : m \in \mathbb{N}$. The decision setting the order quantity R_{m+1} at τ_m is made considering the expected inventory level at time τ_{m+1}, and, thus, represents the look ahead feature of a deterministic MRP system. In the next section, we identify the underlying stochastic structure of this production/inventory system.

3. The inventory position process

In this MRP-controlled system, the inventory positions at I change at the epochs at which replenishment orders arrive and also at the epochs at which end-products are demanded. To identify the underlying stochastic structure, we observe the system only at the epochs at which replenishment orders arrive at I, and we model the system state as a *marked point process*.

A realization, ω, of the *marked point process* is described by the sequence $\omega = \{\bar{x}_m, t_m : m \in \mathbb{N}\}$ in which $\bar{x}_m \in E$ and $t_m \in \mathbb{R}^+$. t_m is the time at which the mth replenishment order arrives at I and indicates the mth state change epoch. The vector $\bar{x}_m$ is the two-tuple (x_m, r_m) with state space $(-\infty, K+Q-1] \times [Q,0]$, in which x_m indicates the inventory position at t_m^- and r_m is the replenishment order quantity that arrives at time t_m.

Let the set of all such realizations ω be denoted by Ω. The system state *marked point process* can be represented by the probability space $(\Omega, \sigma(\Omega), P)$ where $\sigma(\Omega)$ is a σ-algebra on Ω, and P is a probability measure on $\sigma(\Omega)$. We follow the treatment by Disney and Kiessler [10] of *marked point processes* and assume that for each realization, ω, the number of ordered pairs $(\bar{x}_m, t_m)$ in $E \times B$ is finite where B is a bounded set in $\mathbb{R}^+$. Define mappings $T_m : \Omega \to \mathbb{R}^+$ and $\bar{x}_m : \Omega \to E$ (where E is a countable set), as $\tau_m(\omega) = t_m$, $X_m(\omega) = x_m$. We describe $\bar{X}_m$ as a two-tuple (X_m, R_m) where $X_m(\omega) = x_m$, $R_m(\omega) = r_m$. The stochastic process $(\bar{X}, \tau) = \{X_m, R_m, \tau_m : m \in \mathbb{N}\}$ is defined as the *Inventory Position Process* and the set $(\bar{X}, \tau) = \{X_m, R_m : m \in \mathbb{N}\}$ is defined as the *Inventory Position State Process*.

In what follows, we denote x_m and x_{m+1}, the inventory positions at two consecutive instances τ_m and τ_{m+1}, as i_X and j_X respectively, since it is conventional to represent the semi-Markov kernel as $Q(i,j,t)$, in which i

and j are two consecutive states. Similarly, we denote r_m and r_{m+1} as i_R and j_R, respectively. Next, we establish that $(\bar{X},\tau)$ is a Markov renewal process.

Theorem 1. *The inventory position process $(\bar{X},\tau)=\{X_m,R_m,\tau_m : m \in \mathbb{N}\}$ is a Markov renewal process on the state space E.*

It is straightforward to show that the Markov renewal properties are satisfied, and the proof is, therefore, omitted. Corollary 1, an important implication of Theorem 1, identifies the underlying Markov chain of the *inventory position process.*

Corollary 1. *The inventory position state process $\bar{X}$ forms a Markov chain on the state space E with one-step transition probability matrix P. The elements of the matrix P are given by*

$$P(i,j)=\lim_{t\to\infty} Q(i,j,t), \quad \forall i,j \in E.$$

The *inventory position process* $(\bar{X},\tau)$ is analyzed in the next section.

4. Analysis

We first derive the semi-Markov kernel, $Q(i,j,t)$, of the Markov renewal process $(\bar{X},\tau)$. Subsequently, using $Q(i,j,t)$, the ergodic distribution of the underlying Markov chain of the $(\bar{X},\tau)$ process is obtained. Finally, we derive a semi-regenerative process describing the end-product inventory position and obtain the stationary distribution of the end-product inventory position observed at arbitrary times. The reader who is not interested in this mathematical detail can observe the resulting models at the end of the section and their application in Section 5.

4.1. Semi-Markov kernel Q(i,j,t)

The semi-Markov kernel of the inventory position process $(\bar{X},\tau)$ is expressed as

$$\begin{aligned} Q(i,j,t) &= P\{X_{m+1}=j_X,\ R_{m+1}=j_R,\ \tau_{m+1}-\tau_m \le t \mid X_m,R_m\} \\ &= \int_0^t P\{R_{m+1}=j_R \mid X_m=i_X,\ R_m=i_R\}P\{X_{m+1}=j_X \mid X_m=i_X,\ R_m=i_R, \\ &\qquad \tau_{m+1}-\tau_m=u\}\, \mathrm{d}P\{\tau_{m+1}-\tau_m \le u \mid R_{m+1}=j_R\}. \end{aligned} \tag{3}$$

For all $i,j \in E$, it follows from the structure of the problem that

$$P\{R_{m+1}=j_R \mid X_m=i_X,\ R_m=i_R\} \begin{array}{lllll} =1 & \text{for} & j_R=Q, & i_X<K-Q, & i_R=Q, \\ & \text{or} & j_R=0, & K-Q \le i_X \le K-1, & i_R=Q, \\ & \text{or} & j_R=Q, & i_X<K, & i_R=0, \\ & \text{or} & j_R=0, & K \le i_X \le K+Q-1, & i_R=0, \\ =0, & & & \text{otherwise}. & \end{array} \tag{4}$$

For a given time interval u, considering Poisson demand for the end-product, it follows that

$$\begin{aligned} P\{X_{m+1}=j_X \mid X_m=i_X,\ R_m=i_R,\ \tau_{m+1}-\tau_m=u\} &= \frac{e^{-\lambda u}(\lambda u)^{i_x+Q-j_x}}{(i_x+Q-j_x)!} \quad \text{for } \begin{matrix} j_X \le i_X+Q, & i_R=Q \\ i_X \in (-\infty, K-1] & \end{matrix} \\ &= \frac{e^{-\lambda u}(\lambda u)^{i_x-j_x}}{(i_x-j_x)!} \quad \text{for } \begin{matrix} j_X \le i_X, & i_R=0 \\ i_X \in (-\infty, K+Q-1] & \end{matrix} \\ &= 0, \quad \text{otherwise.} \end{aligned} \tag{5}$$

The time between the arrivals of two successive replenishment orders $(\tau_{m+1}-\tau_m)$, depends on the time required by P to process lot size j_R. It follows that for all $i,j \le E$, $t \in \mathbb{R}^+$ and $m \in \mathbb{N}$

$$\begin{aligned} P\{\tau_{m+1}-\tau_m \le u \mid R_{m+1}=j_R\} &= \Phi(u) \quad \text{if} \quad j_R=Q, \\ &= \Psi(u) \quad \text{if} \quad j_R=0. \end{aligned} \tag{6}$$

Substituting (4), (5), and (6) in (3), we obtain the semi-Markov kernel

$$Q(i,j,t)=\int_0^t \frac{e^{-\lambda u}(\lambda u)^{i_x+Q-j_x}}{(i_x+Q-j_x)!}\,d\phi(u), \quad \begin{matrix} j_R=Q, & i_R=Q, \\ i_X<K-Q, & j_X \le i_X+Q, \end{matrix}$$

$$Q(i,j,t)=\int_0^t \frac{e^{-\lambda u}(\lambda u)^{i_x-j_x}}{(i_x-j_x)!}\,d\phi(u), \quad \begin{matrix} j_R=Q, & i_R=0, \\ j_X \le i_X, & i_X<K, \end{matrix}$$

$$Q(i,j,t)=\int_0^t \frac{e^{-\lambda u}(\lambda u)^{i_x+Q-j_x}}{(i_x+Q-j_x)!}\,d\psi(u), \quad \begin{matrix} j_R=0, & i_R=Q, \\ j_X \le i_X+Q, & K-Q \le i_X \le K-1, \end{matrix}$$

$$Q(i,j,t)=\int_0^t \frac{e^{-\lambda u}(\lambda u)^{i_x-j_x}}{(i_x-j_x)!}\,d\psi(u) \quad \begin{matrix} j_R=0, & i_R=0, \\ j_X \le i_X, & K \le i_X \le K+Q-1, \end{matrix}$$

$Q(i,j,t)=0$, otherwise.

4.2. Stationary distribution of the underlying Markov chain

The ergodic distribution of the inventory position at order placement epochs can be obtained from the semi-Markov kernel $Q(i,j,t)$ of the $(\bar{X},\tau)$ process. To accomplish this, we partition the state space into

$$\alpha=\{i \in E: i_R=Q\} \text{ and } \beta=\{i \in E: i_R=0\}, \tag{7}$$

in which α represents the set of all states that triggers an order quantity of Q, and β represents the set of all states that trigger an order quantity of 0. Thus, α and β partition the state space E into mutually exclusive, collectively exhaustive subsets: $\alpha \cup \beta = E$ and $\alpha \cap \beta = \varnothing$. Moreover, since E is countable, we are able to partition the semi-Markov kernel $Q(i,j,t)$:

$$Q(t)=\begin{bmatrix} Q_{\alpha\alpha}(t) & Q_{\alpha\beta}(t) \\ Q_{\beta\alpha}(t) & Q_{\beta\beta}(t) \end{bmatrix}.$$

The one-step transition probability matrix of the inventory position process at replenishment order arrival epochs can be determined by taking the limit as $t \to \infty$, that is,

$$P(i,j) = \lim_{t \to \infty} Q(i,j,t),$$

in which P is the one-step transition probability matrix of the underlying Markov chain. We also define

$$Q_{\alpha\alpha} = \lim_{t \to \infty} Q_{\alpha\alpha}(t),$$

$$Q_{\alpha\beta} = \lim_{t \to \infty} Q_{\alpha\beta}(t),$$

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$$Q_{\beta\beta} = \lim_{t \to \infty} Q_{\beta\beta}(t).$$

Then, P can be written as

$$P = \begin{bmatrix} Q_{\alpha\alpha} & Q_{\alpha\beta} \\ Q_{\beta\alpha} & Q_{\beta\beta} \end{bmatrix}.$$

Let

$$\nu(j) = \lim_{m \to \infty} P[X_m = j_X,\ R_m = j_R].$$

Using the identity $\nu = \nu P$, and partitioning ν as $\nu = [\pi_\alpha\ \ \pi_\beta]$, the equations for the stationary distribution of the underlying Markov chain can be written as

$$\begin{bmatrix} \pi_\alpha & \pi_\beta \end{bmatrix} = \begin{bmatrix} \pi_\alpha & \pi_\beta \end{bmatrix} \begin{bmatrix} Q_{\alpha\alpha} & Q_{\alpha\beta} \\ Q_{\beta\alpha} & Q_{\beta\beta} \end{bmatrix}.$$

in which,

$$\pi_\alpha = \{\pi_\alpha(j) : j \in [K-1, -\infty)\},$$

$$\pi_\beta = \{\pi_\beta(j) : j \in [K-Q+1, -\infty)\}$$

and

$$\pi_\alpha(j) = \lim_{m \to \infty} P\{X_m = j,\ R_m = Q\},$$

$$\pi_\beta(j) = \lim_{m \to \infty} P\{X_m = j,\ R_m = 0\}.$$

Also,

$$Q_{\alpha\alpha} = P[X_{m+1} = j_x, R_{m+1} = Q \mid X_m = i_x, R_m = Q]$$

$$= \begin{array}{rc} & \begin{array}{ccccccccccc} K-1 & . & . & 2 & 1 & 0 & -1 & -2 & . & . & . \ -\infty \end{array} \\ \begin{array}{c} K-Q-1 \\ K-Q-2 \\ . \\ . \\ -Q \\ -Q-1 \\ . \\ -\infty \\ \\ \end{array} & \begin{bmatrix} a_K & a_{K-1} & . & a_3 & a_2 & a_1 & a_0 & a_{-1} & . & . & . \\ & a_K & . & a_4 & a_3 & a_2 & a_1 & a_0 & . & . & . \\ & & a_K & . & a_4 & a_3 & a_2 & a_1 & . & . & . \\ & & & & . & . & . & . & . & . & . \\ & & & & & a_K & a_{K-1} & . & . & . & . \\ & & & & & & a_K & a_{K-1} & . & . & . \\ & & & & & & & . & . & . & . \\ & & & & & & & . & . & . & . \\ & & & & & & & & . & . & . \end{bmatrix} \end{array}, \tag{8}$$

in which

$$a_l = \int_0^\infty \frac{e^{-\lambda t}(\lambda t)^{K-l}}{(K-l)!}\,d\phi(t), \quad l = -\infty,\dots,-2,-1,0,\dots,K.$$

$$Q_{\alpha\beta} = P[X_{m+1} = j_x,\ R_{m+1} = Q \mid X_m = i_x,\ R_m = 0]$$

$$
= \begin{array}{rc}
 & \begin{array}{ccccccccccccccc} K+Q-1 & \cdot & \cdot & K+1 & K & K-1 & \cdot & \cdot & 2 & 1 & 0 & -1 & -2 & \cdot & -\infty \end{array} \\
\begin{array}{c} K-Q \\ K-Q+1 \\ K-Q+2 \\ \cdot \\ \cdot \\ K-1 \end{array} &
\left[\begin{array}{ccccccccccccccc}
 & & & & b_K & b_{K-1} & \cdot & \cdot & b_2 & b_1 & b_0 & b_{-1} & b_{-2} & \cdot & \cdot \\
 & & & b_K & b_{K-1} & \cdot & \cdot & \cdot & b_1 & b_0 & b_{-1} & b_{-2} & b_{-3} & \cdot & \cdot \\
 & & b_K & b_{K-1} & b_{K-2} & \cdot & \cdot & \cdot & b_0 & b_{-1} & b_{-2} & b_{-3} & b_{-4} & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
b_K & b_{K-1} & \cdot & \cdot & b_{K-Q+1} & b_{K-Q} & \cdot & \cdot & \cdot & \cdot & b_{-Q+1} & b_{-Q} & b_{-Q-1} & \cdot & \cdot
\end{array}\right]
\end{array}, \tag{9}
$$

in which,

$$b_l = \int_0^\infty \frac{e^{-\lambda t}(\lambda t)^{K-l}}{(K-l)!}\,d\psi(t), \quad l = -\infty,\dots,-2,-1,0,\dots,K.$$

$$Q_{\beta\alpha} = P[X_{m+1} = j_x,\ R_{m+1} = 0 \mid X_m = i_x,\ R_m = Q]$$

$$
= \begin{array}{rc}
 & \begin{array}{cccccccccccc} K-1 & \cdot & \cdot & 2 & 1 & 0 & -1 & -2 & \cdot & \cdot & \cdot & -\infty \end{array} \\
\begin{array}{c} K-1 \\ K-2 \\ \cdot \\ \cdot \\ 0 \\ -1 \\ \cdot \\ \cdot \\ -\infty \end{array} &
\left[\begin{array}{ccccccccccc}
a_K & a_{K-1} & \cdot & a_3 & a_2 & a_1 & a_0 & a_{-1} & \cdot & \cdot & \cdot \\
 & a_K & \cdot & a_4 & a_3 & a_2 & a_1 & a_0 & \cdot & \cdot & \cdot \\
 & & a_K & \cdot & a_4 & a_3 & a_2 & a_1 & \cdot & \cdot & \cdot \\
 & & & & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
 & & & & & a_K & a_{K-1} & \cdot & \cdot & \cdot & \cdot \\
 & & & & & & a_K & a_{K-1} & \cdot & \cdot & \cdot \\
 & & & & & & & \cdot & \cdot & \cdot & \cdot \\
 & & & & & & & \cdot & \cdot & \cdot & \cdot \\
 & & & & & & & & \cdot & \cdot & \cdot
\end{array}\right]
\end{array}, \tag{10}
$$

in which

$$a_l = \int_0^\infty \frac{e^{-\lambda t}(\lambda t)^{K-l}}{(K-l)!}\,d\phi(t), \quad l = -\infty,\dots,-2,-1,0,..,K.$$

$$Q_{\beta\beta} = P[X_{m+1} = j_x,\ R_{m+1} = 0 \mid X_m = i_x,\ R_m = 0]$$

$$
= \begin{array}{rc}
 & \begin{array}{ccccccccccccccc} K+Q-1 & \cdot & \cdot & K+1 & K & K-1 & \cdot & \cdot & 2 & 1 & 0 & -1 & -2 & \cdot & -\infty \end{array} \\
\begin{array}{c} K \\ K+1 \\ K+2 \\ \cdot \\ \cdot \\ K+Q-1 \end{array} &
\left[\begin{array}{ccccccccccccccc}
 & & & & b_K & b_{K-1} & \cdot & \cdot & b_2 & b_1 & b_0 & b_{-1} & b_{-2} & \cdot & \cdot \\
 & & & b_K & b_{K-1} & \cdot & \cdot & \cdot & b_1 & b_0 & b_{-1} & b_{-2} & b_{-3} & \cdot & \cdot \\
 & & b_K & b_{K-1} & b_{K-2} & \cdot & \cdot & \cdot & b_0 & b_{-1} & b_{-2} & b_{-3} & b_{-4} & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
b_K & b_{K-1} & \cdot & \cdot & b_{K-Q+1} & b_{K-Q} & \cdot & \cdot & \cdot & \cdot & b_{-Q+1} & b_{-Q} & b_{-Q-1} & \cdot & \cdot
\end{array}\right]
\end{array}, \tag{11}
$$

in which

$$b_l = \int_0^\infty \frac{e^{-\lambda t}(\lambda t)^{K-l}}{(K-l)!}\,d\psi(t), \quad l = -\infty,\dots,-2,-1,0,\dots,K.$$

Now, let

$$\pi_{\alpha.1} = \pi_\alpha Q_{\alpha\alpha}, \tag{12}$$

$$\pi_{\beta.1} = \pi_\beta Q_{\beta\alpha}, \tag{13}$$

$$\pi_{\alpha.2} = \pi_\alpha Q_{\alpha\beta}, \tag{14}$$

$$\pi_{\beta.2} = \pi_\beta Q_{\beta\beta}, \tag{15}$$

and note that

$$\pi_\alpha = \pi_{\alpha.1} + \pi_{\beta.1}, \tag{16}$$

$$\pi_\beta = \pi_{\alpha.2} + \pi_{\beta.2}. \tag{17}$$

Eqs. (12)–(17) are evaluated using Eqs. (8)–(11); results are presented in Table 1.

Now, define geometric transforms

$$\Pi_\alpha(z) = \sum_{j=-\infty}^{K-1} \pi_\alpha(j) z^j, \tag{18}$$

$$\Pi_\beta(z) = \sum_{j=-\infty}^{K+Q-1} \pi_\beta(j) z^j. \tag{19}$$

Substituting the values of $\pi_\alpha(j)$ and $\pi_\beta(j)$ from Table 1 into (18), we get

$$\Pi_\alpha(z) = \sum_{j=-\infty}^{K-1} \sum_{i=1}^{K-j} a_{i+j}\{\pi_\alpha(K-Q-i) + \pi_\beta(K-i)\} z^j = [A(z)/z^{K-Q}][\Pi_\alpha(z) - \Pi_\alpha^*(z)] + [A(z)/z^K][\boldsymbol{\Pi}_\beta(z) - \boldsymbol{\Pi}_\beta^*(z)]$$

and, substituting into (19), we obtain

$$\Pi_\beta(z) = [B(z)/z^{K-Q}]\Pi_\alpha^*(z) + [B(z)/z^K]\Pi_\beta^*(z), \tag{20}$$

in which

$$\Pi_\alpha^*(z) = \sum_{j=K-Q}^{K-1} \pi_\alpha(j) z^j, \tag{21}$$

$$\Pi_\beta^*(z) = \sum_{j=K}^{K+Q-1} \pi_\beta(j) z^j.$$

Substituting Eq. (20) in Eq. (19), it follows that

$$\Pi_\alpha(z) = [A(z)/(z^K - z^Q A(z))][(B(z)/z^K) - 1][z^Q \Pi_\alpha^*(z) + \Pi_\beta^*(z)],$$

in which

$$A(z) = \sum_{j=-\infty}^{K} a_j z^j,$$

$$B(z) = \sum_{j=-\infty}^{K} b_j z^j.$$

The values of the constants $\pi_\alpha(K-Q)$, $\pi_\alpha(K-Q+1), \ldots, \pi_\alpha(K-1)$ and $\pi_\beta(K)$, $\pi_\beta(K+1), \ldots, \pi_\beta(K+Q-1)$ can be found such that $\Pi_\alpha(z) + \Pi_\beta(z)$ becomes a p.g.f. using Takagi's [17] approach. Define

$$\pi(j) = \lim_{m\to\infty} P\{X_m = j\}$$

Table 1

Form	Equation	Condition 1	Result 1	Condition 2	Result 2
$\pi_{\alpha,1} = \pi_\alpha Q_{\alpha\alpha}$	(18)	$-\infty < j \leq K-1$	$\pi_{\alpha,1}(j) = \sum_{i=1}^{i=K-j} \pi_\alpha(K-Q-i) a_{i+j}$	$K \leq j < \infty$	$\pi_{\alpha,1}(j) = 0$
$\pi_{\beta,1} = \pi_\beta Q_{\beta\alpha}$	(19)	$-\infty < j \leq K-1$	$\pi_{\beta,1}(j) = \sum_{i=1}^{i=K-j} \pi_\beta(K-i) a_{i+j}$	$K \leq j < \infty$	$\pi_{\beta,1}(j) = 0$
$\pi_{\alpha,2} = \pi_\alpha Q_{\alpha\beta}$	(20)	$-\infty < j \leq K$	$\pi_{\alpha,2}(j) = \sum_{i=0}^{i=Q-1} \pi_\alpha(K-Q+i) b_{j-i}$	$K \leq j \leq K+Q-1$	$\pi_{\alpha,2}(j) = \sum_{i=j-K}^{i=Q-1} \pi_\alpha(K-Q+i) b_{j-i}$
$\pi_{\beta,2} = \pi_\beta Q_{\beta\beta}$	(21)	$-\infty < j \leq K$	$\pi_{\beta,2}(j) = \sum_{i=0}^{i=Q-1} \pi_\beta(K+i) b_{j-i}$	$K \leq j \leq K+Q-1$	$\pi_{\beta,2}(j) = \sum_{i=j-K}^{i=Q-1} \pi_\beta(K+i) b_{j-i}$
$\pi_\alpha(j) = \pi_{\alpha,1}(j) + \pi_{\beta,1}(j)$	(22)	$-\infty < j \leq K-1$	$\pi_\alpha(j) = \sum_{i=1}^{K-j} \{\pi_\alpha(K-Q-i) + \pi_\beta(K-i)\} a_{i+j}$	$K \leq j < \infty$	$\pi_\alpha(j) = 0$
$\pi_\beta(j) = \pi_{\alpha,2}(j) + \pi_{\beta,2}(j)$	(23)	$-\infty < j \leq K$	$\pi_\beta(j) = \sum_{i=0}^{i=Q-1} \{\pi_\alpha(K-Q-i) + \pi_\beta(K+i)\} b_{j-i}$	$K \leq j \leq K+Q-1$	$\pi_\beta(j) = \sum_{i=j-K}^{i=Q-1} \{\pi_\alpha(K-Q+i) + \pi_\beta(K+i)\} b_{j-i}$

and

$$\Pi(z) = \sum_{j=-\infty}^{j=K+Q-1} \pi(j) z^j.$$

Then, note that

$$\Pi(z) = \Pi_\alpha(z) + \Pi_\beta(z). \tag{22}$$

We substitute Eqs. (20) and (21) in Eq. (22) and obtain the p.g.f. of the stationary distribution of the end-product inventory position observed at the order placement points.

4.3. Stationary distribution at arbitrary times

To obtain the stationary distribution of the end-product inventory position at arbitrary times, we formulate a *semi-regenerative process* to describe the inventory position. The next proposition establishes the fact that the inventory position process is a semi-regenerative process.

Proposition 1. *The end-product inventory position, Z_t, observed at any time t, $t \in \mathbb{R}^+$, forms a stochastic process $Z = \{Z_t : t \in \mathbb{R}^+\}$ and is semi-regenerative.*

Proof. We use the definition of semi-regenerative process by Cinlar [7] to establish the result. The function $t \to z_t(\omega)$ is right continuous and for almost all ω has left hand limits. There exists an embedded Markov renewal process $(\overline{X}, \tau)$ which satisfies the following:

(i) $\tau = \{\tau_m : m \in \mathbb{N}\}$ is a set of stopping times for Z since, for each $m \in \mathbb{N}$, $\overline{X_{\tau_m}} = X_m$,
(ii) for each $m \in \mathbb{N}$, $\overline{X}_m$ is determined by $\{Z_u ; u \le \tau_m\}$,
(iii) from Markov renewal property of $(\overline{X},\tau)$, for each $m \in \mathbb{N}$, $n \ge t_1$, $0 \le t_1 \le t_2 \le \ldots \le t_n$, and positive function f defined on E^n,

$$E_i\left[f(Z_{\tau_m+1},\ldots,Z_{\tau_m+t_n} \mid Zu : u \le \tau_m\right] = E_j\left[f(Z_{t_1},\ldots,Z_{t_n})\right] \text{ on } X_m = j.$$

Since the process $(Z,\tau\}$ satisfies the semi-regenerative property, it is a semi-regenerative process. □

Assuming that the underlying Markov chain of the $(\overline{X},\tau)$ process is irreducible, aperiodic, recurrent with invariant measure π and $m(\bar{k}) = \mathrm{E}[T_1 \mid \overline{X} = \bar{k}]$ and suppose that $\pi m = \sum_{\bar{k} \in E} \pi(\bar{k}) m(\bar{k}) < \infty$, (Cinlar [7]),

$$\lim_{t\to\infty} P_t(\bar{i}, A) = \lim_{t\to\infty} P\left\{\overline{X}_t \in A \mid \overline{X}_0 = \bar{i}\right\}$$

or

$$\lim_{t\to\infty} P_t\left(X_t = \bar{j}\right) = \frac{1}{\pi m} \sum_{\bar{i}} \pi(\bar{i}) \int_0^\infty P(\overline{X}_t = \bar{j} \mid \tau_1 > t, \overline{X}_0 = \bar{i}) P\left\{\tau_1 > t \mid \overline{X}_0 = \bar{i}\right\} \mathrm{d}t,$$

in which $\lim_{t\to\infty} P_t(\overline{X}_t = \bar{j})$, for all $\bar{i},\bar{j} \in E$, is the joint stationary distribution of inventory position X and reordering quantity R, observed at arbitrary times.

Partitioning the state space in α and β as in (7), and defining

$$\lim_{t\to\infty} P(\overline{X}_t = \bar{j}) = \eta(\bar{j})$$

and

$$G(\bar{i},\bar{j}) = \int_0^\infty P\{\overline{X}_t = \bar{j} \mid \tau_1 > t, \overline{X}_0 = \bar{i}\} P\{\tau_1 > t \mid \overline{X}_0 = \bar{i}\} \mathrm{d}t,$$

we can write

$$\begin{bmatrix} \eta_\alpha & \eta_\beta \end{bmatrix} = \frac{1}{\pi m} \begin{bmatrix} \pi_\alpha & \pi_\beta \end{bmatrix} \begin{bmatrix} G_{\alpha\alpha} & G_{\alpha\beta} \\ G_{\beta\alpha} & G_{\beta\beta} \end{bmatrix}, \tag{23}$$

in which

$$\begin{aligned} \eta_\alpha(j) &= \lim_{t\to\infty} P(X_t = j,\ R_t = Q), \\ \eta_\beta(j) &= \lim_{t\to\infty} P(X_t = j,\ R_t = 0). \end{aligned} \tag{24}$$

Note that

$$G_{\alpha\alpha}(i,j) = \int_0^\infty P\{X_t = j,\ R_t = Q \mid \tau_1 > t,\ X_0 = i,\ R_0 = Q\} P\{\tau_1 > t \mid X_0 = i,\ R_0 = Q\}\,\mathrm{d}t,$$

$$G_{\beta\beta}(i,j) = \int_0^\infty P\{X_t = j,\ R_t = 0 \mid \tau_1 > t,\ X_0 = i,\ R_0 = 0\} P\{\tau_1 > t \mid X_0 = i,\ R_0 = 0\}\,\mathrm{d}t,$$

$$G_{\alpha\beta}(i,j) = \int_0^\infty P\{X_t = j,\ R_t = Q \mid \tau_1 > t,\ X_0 = i,\ R_0 = 0\} P\{\tau_1 > t \mid X_0 = i,\ R_0 = 0\}\,\mathrm{d}t,$$

$$G_{\beta\alpha}(i,j) = \int_0^\infty P\{X_t = j,\ R_t = 0 \mid \tau_1 > t,\ X_0 = i,\ R_0 = Q\} P\{\tau_1 > t \mid X_0 = i,\ R_0 = Q\}\,\mathrm{d}t.$$

At the first regeneration time $\tau_1 > t$, R_t equals R_0 due to the semi-regenerative structure of $Z = \{Z_t : t \in \mathbb{R}^+\}$. Hence, the conditional probabilities $P\{R_t = Q \mid R_0 = 0\}$ and $P\{R_t = 0 \mid R_0 = Q\}$, which appear in $G_{\alpha\beta}$ and $G_{\beta\alpha}$, are both 0, and it follows that

$$G_{\alpha\beta} = G_{\beta\alpha} = 0.$$

$G_{\alpha\alpha}(i,j)$ and $G_{\beta\beta}(i,j)$ can be represented as follows:

$$\begin{aligned} G_{\alpha\alpha}(i,j) &= \int_0^\infty \frac{\mathrm{e}^{-\lambda t}(\lambda t)^{i+Q-j}}{(i+Q-j)!}[1-\phi(t)]\,\mathrm{d}t \quad \text{for } i < K-Q,\ j \le i+Q \\ &= \int_0^\infty \frac{\mathrm{e}^{-\lambda t}(\lambda t)^{i+Q-j}}{(i+Q-j)!}[1-\psi(t)]\,\mathrm{d}t \quad \text{for } K-Q \le i \le K-1,\ j \le i+Q, \end{aligned}$$

$$G_{\alpha\alpha} = \begin{array}{cc} & \begin{array}{ccccccccccccc} K+Q-1 & . & . & K+1 & K & K-1 & . & . & 1 & 0 & -1 & . & . \end{array} \\ \begin{array}{c} . \\ . \\ -Q-1 \\ -Q \\ -Q+1 \\ . \\ . \\ K-Q-1 \\ K-Q \\ K-Q+1 \\ . \\ . \\ K-1 \end{array} & \left[\begin{array}{ccccccccccccc} & & & & & & & & & & & & . \\ & & & & & & & & & & & . & . \\ & & & & & & & & & & c_K & c_{K-1} & . \\ & & & & & & & & & c_K & c_{K-1} & . & . \\ & & & & & & & & c_K & c_{K-1} & c_{K-2} & . & . \\ & & & & & & . & . & . & . & . & . & . \\ & & & & & . & . & . & . & . & . & . & . \\ & & & & c_K & c_{K-1} & . & . & c_2 & c_1 & c_0 & c_{-1} & . \\ & & & d_K & d_{K-1} & . & . & . & d_1 & d_0 & d_{-1} & d_{-2} & . \\ & & d_K & d_{K-1} & . & . & . & . & d_0 & d_{-1} & d_{-2} & d_{-3} & . \\ . & . & . & . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . & . & . & . \\ d_K & & & & d_{K-Q}\cdot & . & . & . & . & d_{-Q+1} & . & . & . \end{array} \right] \end{array}, \tag{25}$$

in which

$$c_l = \int_0^\infty \frac{e^{-\lambda t}(\lambda t)^{K-l}}{(K-l)!}[1-\phi(t)]\,dt, \quad l = -\infty,\dots,-2,-1,0,\dots,K,$$

$$d_l = \int_0^\infty \frac{e^{-\lambda t}(\lambda t)^{K-l}}{(K-l)!}[1-\psi(t)]\,dt, \quad l = -\infty,\dots,-2,-1,0,\dots,K,$$

$$\begin{aligned} G_{\beta\beta}(i,j) &= \int_0^\infty \frac{e^{-\lambda t}(\lambda t)^{i-j}}{(i-j)!}[1-\phi(t)]\,dt \quad \text{for } i<K,\ j\le i \\ &= \int_0^\infty \frac{e^{-\lambda t}(\lambda t)^{i-j}}{(i-j)!}[1-\psi(t)]\,dt \quad \text{for } K\le i\le K+Q-1,\ j\le i, \end{aligned}$$

$$G_{\beta\beta} = \begin{array}{cc} & \begin{array}{ccccccccccccc} K+Q-1 & \cdot & \cdot & K+1 & K & K-1 & \cdot & \cdot & 1 & 0 & -1 & \cdot & \cdot \end{array} \\ \begin{array}{c} \cdot \\ \cdot \\ -1 \\ 0 \\ 1 \\ \cdot \\ \cdot \\ K-1 \\ K \\ K+1 \\ \cdot \\ \cdot \\ K-1 \end{array} & \left[\begin{array}{ccccccccccccc} & & & & & & & & & & & & \cdot \\ & & & & & & & & & & & \cdot & \cdot \\ & & & & & & & & & & c_K & c_{K-1} & \cdot \\ & & & & & & & & & c_K & c_{K-1} & \cdot & \cdot \\ & & & & & & & & c_K & c_{K-1} & c_{K-2} & \cdot & \cdot \\ & & & & & & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ & & & & & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ & & & & c_K & c_{K-1} & \cdot & \cdot & c_2 & c_1 & c_0 & c_{-1} & \cdot \\ & & & d_K & d_{K-1} & \cdot & \cdot & \cdot & d_1 & d_0 & d_{-1} & d_{-2} & \cdot \\ & & d_K & d_{K-1} & \cdot & \cdot & \cdot & \cdot & d_0 & d_{-1} & d_{-2} & d_{-3} & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ d_K & & & & d_{K-Q}\cdot & \cdot & \cdot & \cdot & \cdot & d_{-Q+1} & \cdot & \cdot & \cdot \end{array}\right] \end{array}, \tag{26}$$

in which

$$c_l = \int_0^\infty \frac{e^{-\lambda t}(\lambda t)^{K-l}}{(K-l)!}[1-\phi(t)]\,dt, \quad l = -\infty,\dots,-2,-1,0,\dots,K,$$

$$d_l = \int_0^\infty \frac{e^{-\lambda t}(\lambda t)^{K-l}}{(K-l)!}[1-\psi(t)]\,dt, \quad l = -\infty,\dots,-2,-1,0,\dots,K.$$

The vector of the stationary probabilities of the end-product inventory position, observed at arbitrary times, is represented by

$$f = \{f(j) : j \in \mathbb{N}\}.$$

From Eqs. (23) and (24), it follows that

$$\begin{aligned} f(j) &= \lim_{t\to\infty} P(X_t = j) \\ &= \eta_\alpha(j) + \eta_\beta(j) \quad \text{for each } j \in \mathbb{N}. \end{aligned} \tag{27}$$

Using Eqs. (23), (25) and (26) in Eq. (27), we can show that

$$f(j) = \frac{1}{\pi m}\left\{\sum_{i=j}^{K-1} c_{K-i+j}\left[\pi_\alpha(-Q+i)+\pi_\beta(i)\right] + \sum_{i=0}^{i=Q-1} d_{j-i}\left[\pi_\alpha(K-Q+i)+\pi_\beta(K+i)\right]\right\}$$
$$\text{for } -\infty < j \le K-1$$
$$= \frac{1}{\pi m}\left\{\sum_{i=j-K}^{i=Q-1} d_{j-i}\left[\pi_\alpha(K-Q+i)+\pi_\beta(K+i)\right]\right\} \quad \text{for } K \le j \le K+Q-1. \tag{28}$$

Defining the p.g.f of the end-product inventory distribution as

$$F(z) = \sum_{j=-\infty}^{K+Q-1} f(j) z_j \tag{29}$$

and substituting Eq. (28) in (29), in can be shown that

$$F(z) = \frac{1}{\pi m}\left[\left[C(z)/A(z)\right]\Pi_\alpha(z) + \left[D(z)/B(z)\right]\Pi_\beta(z)\right], \tag{30}$$

in which

$$A(z) = \sum_{j=-\infty}^{K} a_j z^j = \sum_{j=-\infty}^{K} z^j \int_0^\infty \frac{e^{-\lambda t}(\lambda t)^{K-j}}{(K-j)!}\,d\phi(t) = \int_0^\infty e^{-\lambda(1-z)t}\,d\phi(t),$$

$$B(z) = \sum_{j=-\infty}^{K} b_j z^j = \sum_{j=-\infty}^{K} z^j \int_0^\infty \frac{e^{-\lambda t}(\lambda t)^{K-j}}{(K-j)!}\,d\psi(t) = \int_0^\infty e^{-\lambda(1-z)t}\,d\psi(t),$$

$$C(z) = \sum_{j=-\infty}^{K} c_j z^j = \sum_{j=-\infty}^{K} z^j \int_0^\infty \frac{e^{-\lambda t}(\lambda t)^{K-j}}{(K-j)!}\left[1-\phi(t)\right]dt = \int_0^\infty e^{-\lambda(1-z)t}\left[1-\phi(t)\right]dt,$$

$$D(z) = \sum_{j=-\infty}^{K} d_j z^j = \sum_{j=-\infty}^{K} z^j \int_0^\infty \frac{e^{-\lambda t}(\lambda t)^{K-j}}{(K-j)!}\left[1-\psi(t)\right]dt = \int_0^\infty e^{-\lambda(1-z)t}\left[1-\psi(t)\right]dt. \tag{31}$$

To evaluate $F(z)$, we must find $\pi m = \sum_{k \in E} \pi(k) m(k)$ and proceed as follows. Dividing the state spaces ascribed to α and β, we get

$$\pi m = \begin{bmatrix} \pi_\alpha & \pi_\beta \end{bmatrix}\begin{bmatrix} m_\alpha \\ m_\beta \end{bmatrix}, \tag{32}$$

in which

$$\pi_\alpha(j) = \lim_{m\to\infty} P\{X_m = j,\ R_m = Q\},$$

$$\pi_\beta(j) = \lim_{m\to\infty} P\{X_m = j,\ R_m = 0\},$$

$$m_\alpha(j) = \mathrm{E}\left[\tau_1 \mid X_0 = j,\ R_0 = Q\right],$$

$$m_\beta(j) = \mathrm{E}\left[\tau_1 \mid X_0 = j,\ R_0 = 0\right].$$

Substituting the values of $\pi_\alpha(j)$ and $\pi_\beta(j)$ from Table 1 in (32), it can be shown that

$$\pi m = L_Q + (L_R - L_Q)\left[\sum_{j=K-Q}^{K-1} \pi_\alpha(j) + \sum_{j=K}^{K+Q-1} \pi_\beta(j)\right],$$

in which we used the relation

$$\lim_{z \uparrow 1} \Pi(z) = \lim_{z \uparrow 1} \{\Pi_\alpha(z) + \Pi_\beta(z)\} = 1.$$

Substituting πm in Eq. (30), we obtain the expression for $F(z)$. All the moments of $f(j)$ may be obtained by successive differentiation of $F(z)$ with respect to z and evaluating the derivative as $z \to 1$.

5. Assembly operations: The effect of kitting on end-product inventory position

In this section, we demonstrate the application of our model to investigate the effect of kitting in an assembly/inventory system with MRP-control. Kitting is a crucial process by which part flows are coordinated to initiate assembly; its effect on system performance has not been well described but the process has been studied under several conditions.

Som et al. [15] modeled a system in which part types are fed by an infinite pool, processed independently by different machines according to exponential distributions, then moved to buffers. When a matched set of parts is available in the buffers, a kit is composed, and it transits to a buffer ahead of an assembly machine, which processes according to an exponential distribution. They showed that the output process of this system can, under certain conditions, be approximated by an exponential distribution. These results allow the assembly portion of a system to be decomposed from downstream operations, for example, a finished goods inventory. However, this decomposition requires rather restrictive assumptions. In this paper, we allow processing times to be G.I., and it does not appear possible to decompose the system in a similar manner to determine the distribution of time between completions of assembly operations (i.e., the distribution of arrivals to finished goods inventory).

In an earlier study, Wilhelm and Wang [19] modeled a system that produces to order, kitting parts delivered by vendors. Assuming that part delivery times are jointly normally distributed, they gave empirical evidence that kit completion times can be approximated by a normal distribution with parameters determined by a procedure they described. Such a 'recursion' model is able to deal with time-dependent operations, but not the steady state operations addressed in this paper. Furthermore, this paper gives exact results rather than approximations.

The objective of this section is to identify how the end-product inventory position is related quantitatively to the number of part types required to compose a kit for producing each batch of end-products. The assembly system operates exactly as described in Section 1. We assume that n different types of components are needed to assemble one end-product. Raw materials for each part type are withdrawn from an infinite pool, and a random amount of time is required to process each part. When the MRP control system places a production order for quantity Q, raw materials are withdrawn, and the part production processes start, each producing a lot large enough to produce Q end-products (an end-product may require more than 1 part of each type). When all the parts required for assembly have been processed, a kit is said to be composed and transits immediately to the assembly station. Let

X_i = time to produce a lot of the ith part to assemble Q end-products.

V = time required to assemble end-products of lot size Q at machine M.

The lead time to compose a kit

$$Z = \max[X_i : i = 1, \ldots, n],$$

and the total lead time to produce a lot of Q end-products, defined previously as L_Q, can be defined in more detail to reflect kitting as

$$L_Q = Z + V = \max_i\{X_i\} + V.$$

For the period when no order is placed (i.e., the order quantity is 0), the lead time is L_R, since there is no kitting associated with such an order.

We present a numerical example based on the following assumptions:

- the distribution function of the random variable X_i is $\Theta_i(t) = 1 - e^{-\mu_i t}$ that the corresponding density function is $d\Theta_i(t) = \mu_i e^{-\mu_i t}$;
- the distribution function of the random variable L_Q, the time to assemble a lot size of Q end-products from a kit, is $\rho(t) = 1 - e^{-\mu t}$, so that the corresponding density function is $d\rho(t) = \mu e^{-\mu t}$;
- the distribution function of the random variable L_R, the time to produce 0 end-products, is $\Psi(t) = 1 - e^{-\nu t}$, so that the corresponding density function is $d\Psi(t) = \nu e^{-\nu t}$;
- the distribution function of the time between demands for end-products is $\xi(t) = 1 - e^{-\lambda t}$, so that the corresponding density function is $d\xi(t) = \lambda e^{-\lambda t}$.

The distribution function of the kitting time, Z, described as $\chi(t)$, is obtained as follows:

$$\begin{aligned}\chi(t) &= \prod_{i=1}^{n} \Theta_i(t) \\ &= \prod_{i=1}^{n} (1 - e^{-\mu_i t}).\end{aligned}$$

The distribution function $\Phi(t)$ of the lead time required to produce Q end-products from raw materials, including the kitting operation, is obtained, using $*$ as the convolution operator, as follows:

$$\Phi(t) = \chi * \rho(t).$$

Part processing times and the assembly time on machine M are assumed to be exponentially distributed, since it is convenient to work with both the maximum and convolution operators when the related distributions are exponential. We emphasize, however, that our procedure is not restricted to dealing only with exponential distributions.

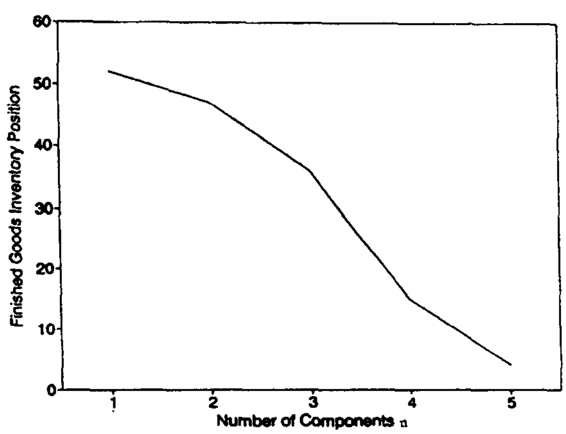

Fig. 2. Average inventory position vs. number of components.

Table 2
Long-run average end-product inventory position

Number of part types in a kit n	1	2	3	4	5
Average inventory level I	52	47	36	15	4

For the following specific cases, we considered $\mu = 8$, $\lambda = 10$, $\nu = 4$, $Q = 5$, $K = 6$, five different values of n (1, 2, 3, 4, and 5) and corresponding $\mu_1 = 5$, $\mu_2 = 4$, $\mu_3 = 6$, $\mu_4 = 5.5$, and $\mu_5 = 4.5$. Considering a particular case with $n = 2$, we elaborate some key steps to derive an expression for the average inventory level. For $n = 2$, the density function of the time to compose a kit is

$$\begin{aligned} \chi(t) &= \prod_{i=1}^{2} \Theta_i(t) \\ &= \prod_{i=1}^{2} (1 - \mathrm{e}^{-\mu_i t}) \\ &= 1 - \mathrm{e}^{-\mu_1 t} - \mathrm{e}^{-\mu_2 t} + \mathrm{e}^{-(\mu_1+\mu_2)t}, \end{aligned}$$

the distribution function of kitting plus assembly time is

$$\begin{aligned} \Phi(t) &= \chi * \rho(t) \\ &= \int_0^y (1 - \mathrm{e}^{\mu_1(y-t)} - \mathrm{e}^{-\mu_2(y-t)} + \mathrm{e}^{-(\mu_1+\mu_2)(y-t)}) \mu \mathrm{e}^{-\mu t} \mathrm{d}t \\ &= 1 - \mathrm{e}^{-\mu t} - [\mu/\mu - \mu_1)][\mathrm{e}^{-\mu_1 t} - \mathrm{e}^{-\mu t}] \\ &\quad - [\mu/(\mu - \mu_2)][\mathrm{e}^{-\mu_2 t} - \mathrm{e}^{-\mu t}] - [\mu/(\mu - \mu_1 - \mu_2)][\mathrm{e}^{-(\mu_1+\mu_2)t} - \mathrm{e}^{-\mu t}]. \end{aligned}$$

Substituting $\Phi(t)$, $\mathrm{d}\Phi(t)$, $\Psi(t)$ and $\mathrm{d}\Psi(t)$ in Eq. (31), we obtain the values for $A(z)$, $B(z)$, $C(z)$, and $D(z)$. Then, substituting the values of $A(z)$, $B(z)$, $C(z)$ and $D(z)$ into Eq. (30), we get the value of $F(z)$. The long-term average inventory position is obtained by taking the first derivative of $F(z)$ with respect to z and evaluating it as $z \to 1$. Similar derivations were used to determine the long-term average inventory position for $n = 1$, 3, 4, and 5 (Fig. 2), but the values of $A(z)$, $B(z)$, $C(z)$, and $D(z)$ are rather intricate and are, therefore, not presented in this paper.

Table 2 quantifies the average end-product inventory position in the long run as a function of the number of different part types required in a kit. As expected, as the number of part types required to assemble an end-product increases, the average end-product inventory position decreases (assuming that the same order size Q is used for each value of n). The difference in average inventory position for the cases $n = 1$ and 2 is relatively minor, indicating that when the end-product is assembled from a kit consisting of either one or two part types, the effect of kitting is not significant. However, the inventory position decreases sharply for $n = 4$ and 5, indicating that the kitting of several types of parts does affect the average inventory position significantly. It may be noted that optimal batch size, Q^*, may be different for each value of n (1, 2, 3, 4, and 5) and that optimal values of the average inventory levels may, thus, be different from those shown. However, no attempt has been made to obtain the optimal lot size Q in this analysis, since doing so is beyond the scope of this paper.

6. Conclusion

This paper analyzes underlying stochastic processes, describing operation of a single-stage, single-product assembly system that operates in an MRP-controlled environment. In particular, a model is developed, allowing production times to be treated as general, independent random variables.

The Palm probability distribution of the end-product inventory position observed at replenishment order arrival times, π, is different from the ergodic distribution of the end-product inventory position observed at arbitrary times, f. This result is as anticipated, since the lead times for replenishment order arrivals are assumed to be G.I. distributed random variables. It is also observed that the capacity of the buffer I must be bounded from above by (finite) K for any ergodic distribution of the end-product inventory position to exist. In the limit as $K \to \infty$, the long-term probability reduces to the trivial solution, zero. The probability generating function (p.g.f), $F(z)$, of the distribution of the end-product inventory position, $f(j)$, must equal unity in the limit as $z \to \infty$.

The effect of kitting on the average end-product inventory position is quantified as a means of describing some operating characteristics of stochastic assembly systems operating under MRP control. As the number of part types in a kit increases, the average end-product inventory position decreases rapidly, indicating poor coordination of material flow due to the longer lead time required to compose kits.

This paper considers a single-stage, single-product MRP-controlled assembly system. Logical extensions that could be undertaken by future research would be to address multi-stage and/or multi-product MRP-control.

Acknowledgements

This material is based on work supported by the National Science Foundation on Grant numbers DMC-8896303 and DMI 9500211.

References

[1] T. Altiok, G.A. Shiue, Single-stage, multi-product production/inventory systems with backorders, IIE Transactions 26 (2) (1994) 52–61.
[2] J.A. Buzacott, Queueing models of kanban and MRP controlled production systems, Engineering Costs and Production Economics 17 (1989) 3–20.
[3] J.A. Buzacott, S.M. Price, J.G. Shantikumar, Service Level in multistage MRP and base stock controlled production systems, in: G. Fandel, T. Gulledge, A. Jones (Eds.), New Directions for Operations Research in Manufacturing Systems, Springer, Berlin, 1992, pp. 445–463.
[4] J.A. Buzacott, J.G. Shantikumar, A general approach for coordinating production in multiple-cell manufacturing systems, Production and Operations Management. 1 (1) (1992) 34–52.
[5] J.A. Buzacott, J.G. Shantikumar, Safety stock versus safety lead time in MRP controlled production systems, Management Science 40 (12) (1994) 1678–1689.
[6] C.A. Chang, The interchangeability of safety stocks and safety lead time, Journal of Operations Management 6 (1) (1985) 35–42.
[7] Cinlar, E, Introduction to Stochastic Processes, Prentice-Hall, Englewood Cliffs, New Jersey, 1975.
[8] J.L. Deleersnyder, T.L. Hodgson, H. Muller, P.J. O'Grady, Kanban controlled pull systems: An analytical approach, Management Science 35 (1989) 1079–1091.
[9] M. Di Mascolo, Y. Frein, Y. Dallery, An analytical method for performance evaluation of kanban controlled production systems, Operations Research 44 (1) (1996) 50–64.
[10] R.L. Disney, P.C. Kiessler, Traffic Processes in Queueing Networks: A Markov Renewal Approach, Johns Hopkins University Press, 1987.
[11] B. Gavish, S.C. Graves, A one-product production/inventory problem under continuous review policy, Operations Research 28 (5) (1980) 1228–1236.
[12] B. Gavish, S.C. Graves, Production/inventory systems with a stochastic production rate under a continuous review policy, Computers and Operations Research 8 (3) (1981) 169–183.
[13] S. Nahmias, Production and Operations Analysis, R.D. Irwin Inc., Homewood, IL, 1989.
[14] J.A. Orlicky, Materials Requirements Planning, McGraw-Hill, New York, 1975.
[15] P. Som, W.E. Wilhelm, R.L. Disney, Kitting process in a stochastic assembly, System, Queueing Systems: Theory and Applications 17 (1994) 471–490.

[16] M.L. Spearman, M.A. Zazanis, Push and pull production systems: Issues and comparisons, Operations Research 40 (3) (1992) 521–532.
[17] H. Takagi, Queueing analysis of vacation models, IBM TRL Research Report, 1987.
[18] J. Wijngaard, J.C. Wortman, MRP and inventories, European Journal of Operational Research 20 (1985) 281–293.
[19] W.E. Wilhelm, L. Wang, Management of component accumulation in small-lot assembly systems, Journal of Manufacturing Systems 5 (1) (1986) 27–39.
[20] C.A. Yano, Setting planned lead times in serial production systems with tardiness costs, Management Science 33 (1) (1987) 95–106.
[21] P. Zipkin, Models for design and control of stochastic multi-item batch production systems, Operations Research 34 (1) (1986) 91–104.
[22] T.E. Vollman, W.L. Berry and D.C. Whybark, Manufacturing Planning and Control Systems, Irwin, Homewood, IL, 1984.