KITTING PROCESS IN A STOCHASTIC ASSEMBLY SYSTEM

Pradip Som

W. E. Wilhelm

R. L. Disney

Department of Industrial Engineering
Texas A&M University
College Station, TX 77843-3131

Abstract

In small-lot, multi-product, multi-level assembly systems, kitting (or accumulating) components required for assembly plays a crucial role in determining system performance, especially when the system operates in a stochastic environment. This paper analyzes the kitting process of a stochastic assembly system, treating it as an assembly-like queue. If components arrive according to Poisson processes, we show that the output stream departing the kitting operation is a Markov renewal process. The distribution of time between kit completions is also derived. Under the special condition of identical component arrival streams having the same Poisson parameter, we show that the output stream of kits approximates a Poisson process with parameter equal to that of the input stream. This approximately decouples assembly from kitting, allowing the assembly operation to be analyzed separately. We also show that, in the long run, all inventory positions are equally likely and independent of the actual inventory position.

Keywords: Kitting, assembly, Markov renewal process, double-ended queue.

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1. Introduction

Traditionally, material flow analysis in assembly systems has been based on the assumption that the system operates deterministically. In recent years, attention has been directed to a more realistic analysis of assembly systems, explicitly treating the stochastic events that influence operations. An important aspect of assembly operations is kitting (or accumulating) required components and releasing the kit to initiate assembly. Due to the stochastic nature of component availability, stock-outs often occur in component inventories, thereby disrupting kitting and, consequently, assembly schedules. The goal of this paper is to better understand the kitting process in a stochastic assembly system, which we treat as an assembly-like queue.

This paper models the kitting process of an assembly system as a Markov renewal process, assuming that component arrival streams follow independent Poisson distributions. The assembly system is assumed to have a structure similar to that described in Hopp and Simon [7] and is shown in Figure 1.

Figure 1.

P_1 and P_2 are machines that process components (to prepare them for assembly) and P_3 is the assembly machine. I_1 and I_2 are the buffers for components, I_0 is the buffer for kits, and I_3 is the buffer for the end-product. P_1 and P_2 work independently, withdraw raw materials from their respective pools of unlimited supply, and deliver processed components to buffers I_1 and I_2, respectively . A component arriving at buffer I_1 (I_2) is immediately kitted with a part from buffer I_2 (I_1) if one is available, and a "kit" is said to be composed. If a kit cannot be composed, the processed part is held in buffer $I_1(I_2)$ to await the arrival of a "matching" part at buffer $I_2(I_1)$. Once composed, a kit of matching components from I_1 and I_2 is sent immediately to I_0 and the kit is considered to be one arrival at I_0. If the arriving kit finds I_0 empty and P_3 idle, it is immediately placed in the assembly machine P_3. Otherwise, the kit is held in buffer I_0.

We assume that buffers of components have limited capacity and that each component is processed according to an exponential distribution (before kitting) to prepare it for assembly . When P_3 completes an assembly, it withdraws a kit (i.e. two matched components) from I_0, whenever available, then assembles another end product and delivers it to buffer I_3. If a kit is not available in I_0 when P_3 completes an assembly, it remains idle until a completed kit arrives. Demands for end products arrive at I_3; each demand is assumed to be for a lot of unit size and is satisfied immediately if stock is available. Unsatisfied demands are backordered, causing the inventory position at I_3 to take on negative values.

Our primary result is to show that the output stream departing the kitting operation is a Markov renewal process. In the special case in which component arrival streams have the same Poisson parameter, we are able to show that the output stream approximates a Poisson process with parameter the same as that of the arrival streams.

Regarding the modus operandi of the assembly system, Harrison [6] showed that a sufficient condition for stability of operations of such systems is that component buffer sizes be finite. For a system with finite buffers, we show that, in the long run, probabilities of observing different inventory positions at I_1 (I_2) are all equally likely and are independent of any particular inventory position. Also, considering the special case of component arrival streams with the same Poisson parameter, we show that the kit completion process well approximates a Poisson process when the component buffers are large enough, permitting the kitting and assembly operation to be decoupled so that downstream operations can be analyzed separately.

Stochastic assembly systems are often studied as assembly-like queues. Harrison [6] showed that an assembly system with input streams that are independent renewal processes and with no inventory capacity limitations for any stream are unstable. He also showed that, under these conditions, the limiting distribution of the time that parts wait for assembly converges to a defective distribution.

Since we assume that two components are required to compose a kit, the queues of components form a double-ended queue [5], [8]. A double-ended queue can be best described by the well known taxi-cab problem where taxis and passengers form two different queues. A customer waits in its queue and leaves it as soon as a taxi is available; taxis wait in queue for customers and leave when a customer is available. The two queues are interdependent and their combination is known as a double-ended queue where it is known that the related queueing process is a random walk on {.., -2, -1, 0, 1, 2, ...} and is transient or null unless the queues are bounded. The kitting process under study can be considered as a double-ended queue of the type examined by Kashyap and Chaudhury [9]. They showed that each queue length distribution is independent of occupancy when arrival rates to the double-ended queue are equal. They also derived the distribution of waiting times in double-ended queues but made no attempt to analyze its output process.

Bhat [2] incorporated limited buffer capacities in assembly like queues and derived expressions for the stationary probability vector of the queue length. Latouche [10] considered assembly systems with Poisson procurement processes and exponential processing times and derived conditions required for stability. Assembly networks that represent one-time production (for example; space-shuttle, aircraft prototype, etc.) are analyzed by Saboo and Wilhelm [11] and Wilhelm et al[13].

The output processes from queues operating according to various disciplines are reviewed by Disney and Konig [4] in detail. They describe the characteristics of the output processes resulting from GI/D/s, M/M/s, M/GI/1/L, M/E_k/1/L, M/GI/∞, GI/GI/1/L and GI/M/1/L systems. Apparently, the output process of a double-ended queue has not been studied previously. In this paper we analyze such a process as a part of our study of the kitting process.

We have organized this paper in five sections. The fundamentals and pertinent assumptions are presented in Section 2. Section 3 relates the formulation of a Markov renewal process which describes the kitting operation. The model is evaluated in Section 4 by determining the state transition matrix P, the time-stationary probability vector ?, and the distribution of time between kit completions, which is shown to be approximately Poisson under certain conditions. Practical implications of analytical results are described and conclusions are presented in Section 5.

2. Fundamentals

The structure of the assembly system under analysis is presented in Figure 1. A little thought indicates that it is not possible for both buffers I_1 and I_2 to have positive stock levels at the same time. An arrival which increases the stock level of one of the buffers to a positive value creates a "virtual backorder" at the other buffer. At any time t ($t > 0$), the inventory position `M'(defined as the number of parts on hand plus on order minus the number on back order) in one buffer is associated with inventory position `-M' in the other, and equality holds only when the inventory position is zero(0) for both buffers I_1 and I_2. The inventory positions at I_1 and I_2 may thus be viewed as "mirror images" of one another, a special structure which we exploit to analyze the kitting process.

Since the purpose of this paper is to characterize the kitting process, we study the stream of arrivals to I_0 (i.e., the output of the kitting process) in the following sections and ignore the process downstream of I_0. We present a thorough analysis of the downstream assembly system in a companion paper (Som and Wilhelm [12]).

Our model, which is based on the structure described in this section, relies upon three fundamental assumptions:

(i) processing times at the part processing machines, P_1 and P_2, are independent, identically distributed, non-negative exponential random variables with rates μ_1 and μ_2, respectively.

(ii) the capacities of buffers I_1 and I_2 are bounded from above by K_1 and K_2, respectively, representing practical limitations on buffer space, and, according to Harrison [6], allowing the system to reach a steady state. No capacity restriction is imposed on I_0.

(iii) P_1 (P_2) prepares parts exclusively for I_1 (I_2). However, when I_1 (I_2) is filled to capacity K_1 (K_2), additional arrivals are not processed in the system under analysis (e.g., they may be processed and assembled by a subcontractor).

In the following sections we formulate the model and analyze it as a Markov renewal process.

3. Formulation of a Markov renewal process

The inventory positions at I_1 and I_2 change with the arrival and departure of components to and from the respective buffers. We define the *mirror image process* (X, T) as a *marked point process*, which characterizes the inventory positions or states at the arrival and departure epochs. The sample path diagram of the *mirror image process* is presented in Figure 2.

Figure 2.

Thus, $(X, T) = \{X_m, T_m : m \in N\}$

in which, $X_m = \{{}^1X_m, {}^2X_m\}$

T_m = time of m-th state change epoch.

${}^1X_m$ = inventory position at buffer I_1 at time T_m.

${}^2X_m$ = inventory position at buffer I_2 at time T_m.

Due to the mirror image property of the inventory positions at I_1 and I_2, at any random time T_m, ${}^1X_m = {}^1x_m$ implies ${}^2X_m = -\,{}^1x_m$; or, equivalently, ${}^2X_m = {}^2x_m$ implies ${}^1X_m = -{}^2x_m$. Hence, it is obvious that the Mirror Image Process may be analyzed by viewing the inventory position just at I_1 (or, equivalently, just at I_2).

Whenever matching components are available at buffers I_1 and I_2, a kit is composed (instantaneously) and sent to I_0. These departure epochs (occurring simultaneously from both I_1 and I_2) and the corresponding inventory position at I_1 describe another *marked point process* which we define as the *output process*. By observing the inventory position at I_1, it is apparent that a particular subset of the epochs $\{T_m : m \in N\}$, marked by a decrease in the positive inventory position or an increase in the negative inventory position, constitutes kit completion as well as state change epochs in the *output process.*

These output epochs are a sub-sequence of the sequence $\{T_m : m \in N\}$, defined as

$t = \{t_n : n \in N\}$ with $0 = t_0 \leq t_1 \leq t_2 \leq t_3 \leq ....$ such that for $? \in O$, $t_0(?) = T_0(?) = 0$, $t_n(?) = T_k(?)$, $n \geq 1$, in which, $k = \min \{m \in N: n \leq \sum_{j=1}^{m} 1_{\{|^{1}X_{j-1}|>|^{1}X_{j}|\}}(w\ ) 1\}$ and $1_{\{X\}}(.)$ is an indicator function. Define $D_{n+1} = t_{n+1} - t_n$ as the time between successive departures, n and n+1. For $n \in N$, the random variable $D_n : O \rightarrow R^+$ represents the length of the n-th inter-departure interval. Then $t_{n+1} = t_n + D_{n+1}$, $n \in N$, defines the time of the (n+1)-th departure. The set $t = \{t_n : n \in N\}$ defines the *output time process.*

For each $n \in N$, define the random variable $Z_n : O \rightarrow E$ as the inventory position at the buffer I_1 or the system state of the *output process* immediately after the n-th departure epoch t_n. The set $Z = \{ Z_n : n \in N \}$ defines the *output state process*, and the joint random variables $\{ Z, t \} = \{ Z_n, t_n : n \in N \}$ define the *output process.* Here, D_n depends on the present state Z_n and the next state Z_{n+1}. However, given these states, D_n is independent of previous D_k and Z_k for k=1,....n-1, indicating that the *output process* $\{Z, t\}$ is a Markov renewal process on the state space E. Since a Markov renewal process is completely characterized by its semi-Markov kernel Q(i, j, t), we study this kernel in the following sub-section.

DETERMINATION OF THE SEMI-MARKOV KERNEL Q(i,j,t)

The semi-Markov kernel of the *output process* $\{Z, t\}$ may be expressed as

$$Q(i, j, t) = \Pr\{Z_{n+1} = j,\ t_{n+1} - t_n \leq t \mid Z_n = i\}.$$

For convenience, the semi-Markov kernel is expressed in the Laplace transform domain as L{Q(i, j, dt)} = Q{i, j, ds}.

The Laplace transform of $\frac{d}{dt}$ 2Pr{Z_{n+1} = j, t_{n+1} - $t_n \leq$ t | Z_n = i}, expressed as L[dP{Z_{n+1} = j, t_{n+1} - $t_n \leq$ t | Z_n = i}], can be shown to have five different forms,

depending upon inventory positions at epochs t_n and t_{n+1}. We describe the five cases below.

Case I. The starting (i.e., at t_n) inventory position is non-negative and it *does not* reach the positive boundary K_1 before the time of the next departure (i.e., at t_{n+1}).

Certain combinations of i and j define Case I:

(i) $0 < i \leq K_1$-1, $i-1 \leq j \leq K_1$-2, and

(ii) i = 0, $0 < j \leq K_1$-2.

Then,

$$dP\{Z_{n+1} = j,\ t_{n+1} - t_n \leq t \mid Z_n = i\} = \frac{e^{-m_1 t}(m_1 t)^{j-i+1}}{(j-i+1)!}\, m_2\, e^{-m_2 t} dt \ 3. \tag{1}$$

Since we are looking at two consecutive kit completion epochs, t_n and t_{n+1}, at which inventory positions at I_1 are i and j respectively, j-i+1 components must have arrived at I_1 before any arrival at I_2.

In Laplace transform form,

L[dP{Z_{n+1} = j, t_{n+1} - $t_n \leq$ t | Z_n = i}] =

$$\left(\frac{m_2}{m_1}\right)\left(\frac{m_1}{m_1+m_2}\right)^{j-i+2} 4 \left(\frac{m_1+m_2}{m_1+m_2+s}\right)^{j-i+2} 5 \tag{2}$$

The other four cases follow similarly.

Case II. (i) $-K_2+1 \leq i < 0$, $-K_2+2 \leq j \leq i+1$, and

(ii) i = 0, $-K_2+2 \leq j < 0$.

L[dP{Z_{n+1} = j, t_{n+1} - $t_n \leq$ t | Z_n = i}] =

$$\left(\frac{m_1}{m_2}\right)\left(\frac{m_2}{m_1+m_2}\right)^{|j|-|i|+2} 6 \left[\frac{m_1+m_2}{m_1+m_2+s}\right]^{|j|-|i|+2} 7. \tag{3}$$

Case III. $0 \le i \le K_1\text{-}1$, $j = K_1\text{-}1$

L[dP{$Z_{n+1} = K_1\text{-}1$, $t_{n+1} - t_n \le t \mid Z_n = i$}] =

$$\left(\frac{m_1}{m_1+m_2}\right)^{K_1-i} 8\left[\frac{m_2}{m_2+s}\right]\left[\frac{m_1+m_2}{m_1+m_2+s}\right]^{K_1-i} 9. \tag{4}$$

Case IV. $-K_2+1 \le i \le 0$, $j = -K_2 + 1$.

L[dP{$Z_{n+1} = -K_2+1$, $t_{n+1} - t_n \le t \mid Z_n = i$}]=

$$\left(\frac{m_2}{m_1+m_2}\right)^{K_2+i} 10\left[\frac{m_1}{m_1+s}\right]\left[\frac{m_1+m_2}{m_1+m_2+s}\right]^{K_2+i} 11. \tag{5}$$

Case V. $i = 0$, $j = 0$.

L[dP{$Z_{n+1} = 0$, $t_{n+1} - t_n \le t \mid Z_n = 0$}] =

$$\frac{2m_1 m_2}{\left(m_1+m_2\right)^2} 12\left[\frac{m_1+m_2}{m_1+m_2+s}\right]^2 13. \tag{6}$$

The interval $t_{n+1} - t_n$ includes an initial period which has an exponential distribution with rate $\mu_1 + \mu_2$. Using Bernoulli probabilities $\mu_1/(\mu_1 + \mu_2)$ and $\mu_2/(\mu_1 + \mu_2)$ and convolving with the distribution of the remainder of the interval, we get the above result.

Combining equations (2) through (6), we obtain the semi-Markov kernel Q(i, j, t), which is expressed in Laplace transform form and is presented as equation (7) in Table 1. The state transition matrix P of the underlying Markov chain Z embedded at time t_n is obtained by setting s = 0 in equation (7) and is presented as equation (8) in Table 2. An analysis of the output process {Z,t} is presented in the following section.

Table 1.

Table 2.
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4. Analysis

In this section, we analyze the *output process* {Z,t}, deriving the following:

(i) the stationary probability vector ? of the underlying Markov chain Z, and

(ii) the distribution of time between kit completions.

The vector ? indicates the time-stationary probability distribution of the inventory position at I_1, observed at a randomly selected kit completion epoch.

DETERMINATION OF STATIONARY PROBABILITY VECTOR ?

Clearly, the output process {Z,t} = {Z_n, t_n : $n \in N$} is an irreducible, nonnull, recurrent, and persistent Markov renewal process for K_1, $K_2 < \infty$; under these conditions, it possesses a stationary distribution defined as ? [3]. Note that the process {Z,t} will be recurrent null, if K_1 and K_2 are infinite. The stationary probability vector ? of the underlying Markov chain Z is obtained from the set of equations expressed in the matrix form

$$? = ?\,P.$$

Using equation (14) for P, the balance equations can be expressed for specific states

$-K_2+1 \leq j \leq K_1-1$ as

$$?(0) = ?(-1)?(1-v) + ?(0)2v(1-v) + ?(1)(1/?)v \tag{9}$$

$$?(K_1-1) = ?(K_1-2) \tag{10}$$

$$?(-K_2+1) = ?(-K_2+2) \tag{11}$$

$$?(j) = ?(0)(1/?)v^{(j+2)} + ?(1)(1/?)v^{(j+1)} + ?(2)(1/?)v^{j} + \ldots + (1/?)?(j+1)$$

$$j = 1,2,\ldots,K_1-2. \tag{12}$$

$$?(j) = ?(0)?(1-v)^{(-j+2)} + ?(1)?(1-v)^{(-j+1)} + ?(2)?(1-v)^{(-j)} + \ldots + ?(1-v)?(-j+1)$$

$j = -1, -2, \ldots, -K_2+2.$ (13)

in which, $? = \mu_1/\mu_2$, $v = \mu_1/(\mu_1 + \mu_2)$.

In addition, we have the normalizing expression

$\sum_j$ 14 ?(j) = 1. (14)

The solution to equations(9)-(14) can be expressed as

$$\Pi(0) = \frac{m_1 + m_2}{m_1 K_2 + m_2 K_1} \quad 15 \tag{15}$$

$$\Pi(j) = \frac{m_2}{m_1 K_2 + m_2 K_1} \quad 16 \qquad j = 1,2,\ldots, K_1-1 \tag{16}$$

$$\Pi(j) = \frac{m_1}{m_1 K_2 + m_2 K_1} \quad 17 \qquad j = -1,-2,\ldots, -K_2+1. \tag{17}$$

It may be observed that ?(j), the stationary probability of positive(negative) stock in buffer I_1 observed at a kit completion time, is a constant independent of j, the stock position.

DISTRIBUTION OF TIME BETWEEN KIT COMPLETIONS, D_n

To determine the distribution of time between kit completions, we concentrate on analyzing the *output time process* $t = \{t_n : n \in N\}$, which specifies the arrival stream (of kits) to buffer I_0.

Considering the stationary distribution ? of the underlying Markov chain Z and for $t \in R^+$, the distribution of time between two consecutive kit completions is given by

$P\{t_{n+1} - t_n \leq t\}$ = ? Q(i, j, t) U (18)

in which U is a column vector with all elements equal to 1.

Expressing equation (18) in Laplace transform form we obtain:

$L[dP(t_{n+1} - t_n \leq t)]$ = ? Q(i, j, ds) U. (19)

Substituting the values of ? and Q(i, j, ds) from equations(15) to (17) and (7) into equation (19),

$$L[dP(t_{n+1} - t_n \le t)] = \left(\frac{m_1}{m_1 + s}\right)\left[\frac{K_2 m_1 + m_2}{m_1 K_2 + m_2 K_1}\right]18 \qquad + \qquad \left(\frac{m_2}{m_2 + s}\right)\left[\frac{K_1 m_2 + m_1}{m_1 K_2 + m_2 K_1}\right]19$$

$$- \qquad ?(0)\left(\frac{m_1 + m_2}{m_1 + m_2 + s}\right)20 \qquad (20)$$

It is apparent that if equation (20) is inverted (i.e., to the time domain), the distribution of the time between kit completions, D_{n+1}, would be the weighted sum of three exponential distributions with rates μ_1, μ_2 and $\mu_1+\mu_2$.

A SPECIAL CASE WITH INFINITE BUFFER CAPACITY

We consider a special case in which the capacity of one buffer (either I_1 or I_2) approaches infinity while the other buffer capacity is finite.

Without loss of generality, consider $K_1 \to \infty$, $K_2 < \infty$ and $\frac{K_2}{K_1}21 \to 0$; it can be easily seen from equation (20) that

$$L[dP(t_{n+1} - t_n \le t)] = \frac{m_2}{m_2 + s}22.$$

Similarly, with $K_2 \to \infty$, $K_1 < \infty$ and $\frac{K_1}{K_2}23 \to 0$,

$$L[dP(t_{n+1} - t_n \le t)] = \frac{m_1}{m_1 + s}24.$$

In this case we observe that the distribution of the time between kit completions is asymptotically exponential and is identically the same as the distribution of the time between component arrivals at the finite buffer.

A SPECIAL CASE WITH $\mu_1 = \mu_2 = \mu$

This section specializes the case in which component processing times at machines P_1 and P_2 are independent exponential random variables with the same rates (i.e., $\mu_1 = \mu_2 = \mu$). In practice, this situation may occur when components are obtained from independent suppliers with identical (and independent) lead time distributions. Also, the same situation may occur during "in-house" production where the machines employed, P_1 and P_2, are identical (and independent). In the following sub-sections we show that the distribution of time between kit completions, D_n, can be approximated by independent and identically distributed exponential random variables.

APPROXIMATION OF D_n BY THE EXPONENTIAL DISTRIBUTION

Making appropriate changes in equations (7) and (8) to accommodate the special case, the semi-Markov kernel, Q(i, j, t), in Laplace transform form and the transition probability matrix P of the underlying Markov chain Z may be expressed by equations (21) and (22) which are presented in Tables 3 and 4, respectively.

Table 3.

Table 4.

The stationary probabilities of this Markov chain are given by

$$?(0) = \frac{2}{K_1 + K_2} 25 \tag{23}$$

$$?(j) = \frac{1}{K_1 + K_2} 26 \qquad \forall\ j \neq 0. \tag{24}$$

These results have striking similarities - but at the same time, important differences - with those obtained by Bhat[1] for the limiting distribution of the population in the finite buffer of a double-ended queue.

The distribution of time between kit completions, D_n, can be expressed in Laplace transform form as

$$L[dP(t_n - t_{n-1} \leq t)] = ?\ Q(i, j, ds)\ U \tag{25}$$

in which U is a column vector with each element equal to 1. Substituting equations (21), (22), (23) and (24), equation (25) specializes to

$$L[dP(t_n - t_{n-1} \leq t)] = 27 \tag{26}$$

Clearly, for large values of $K_1 + K_2$, the distribution of time between kit completions, D_n, is approximately exponential with rate μ. The value of $K_1 + K_2$ necessary to allow this approximation can be determined as a function of the degree of approximation desired. The e-approximate distribution of D_n is

$$? \, Q(i, j, ds) \, U = L \, [dP(t_n - t_{n-1} \leq t)] = \frac{m}{m+s} 28, \tag{27}$$

which is the Laplace transform of an exponential distribution with rate μ.

APPROXIMATE INDEPENDENCE OF D_n

In this section, we discuss the independence of m consecutive random variables D_n, n=1,2,.....m. We show that for sufficiently large $K_1 + K_2$, the m consecutive random variables D_n, n=1,2.....m, become independent to within an error of e.

This independence holds if the joint distribution of the m consecutive random variables D_n equals the product of the m marginal distributions of the random variables D_n. Statistical independence should hold for m $\rightarrow \infty$, but this limiting case is not easily evaluated.

To establish the approximation, we must show (writing Q(i, j, ds) = Q(ds)),

$?Q(ds_1)Q(ds_2)Q(ds_3)....Q(ds_m)U$

$$= \{?Q(ds_1)U\}\{?Q(ds_2)U\}\{?Q(ds_3)U\}...\{?Q(ds_m)U\}. \tag{28}$$

The left hand side of equation (28) is

$$?Q(ds_1)Q(ds_2)Q(ds_3)....Q(ds_m)U \quad = \prod_{i=1}^{m}\left(\frac{m}{m+s_i}\right)\left[1 - \frac{2}{K_1+K_2}\right] 29 + \frac{2}{K_1+K_2}\prod_{i=2}^{m}\left(\frac{m}{m+s_i}\right)$$

$$\mathrm{x}\left[b(\frac{1}{2}a)^{K_2} + (\frac{1}{2}a)^{K_2} + ...+(\frac{1}{2}a)^3 + (\frac{1}{2})a\prod_{i=2}^{m}\left(\frac{m}{m+s_i}\right) + (\frac{1}{2}a)^3 + ...+(\frac{1}{2}a)^{K_1} + b(\frac{1}{2}a)^{K_1}\right]$$

30

$$= \prod_{i=1}^{m}\left(\frac{m}{m+s_i}\right) - \frac{2}{K_1+K_2}\prod_{i=1}^{m}\left(\frac{m}{m+s_i}\right) - \frac{2}{K_1+K_2}\prod_{i=2}^{m}\left(\frac{m}{m+s_i}\right)$$

$$\text{x}\left[b(\frac{1}{2}a)^{K_2} + ... + (\frac{1}{2}a)^3 + (\frac{1}{2})a\prod_{i=2}^{m}\left(\frac{m}{m+s_i}\right) + (\frac{1}{2}a)^3 + ... + (\frac{1}{2}a)^{K_1} + b(\frac{1}{2}a)^{K_1}\right] 31$$

(29)

By making $K_1 + K_2$ sufficiently large, the right hand side of equation (29) can be approximated by $\prod_{i=1}^{m}\left(\frac{m}{m+s_i}\right)$32, the product of the Laplace transform of the m marginal distributions of the random variables D_n for $n \in N$. Hence, equation (28) holds for sufficiently large $K_1 + K_2$, indicating that the random variables D_n, n = 1,...,m, are independent. The required value of $K_1 + K_2$ depends upon the degree of approximation desired.

The implications of equations (27) and (29) lead to the following theorem.

THEOREM 1

The arrival process of kits at buffer I_0 can be approximated by a Poisson process with rate μ, the degree of approximation depending on the value of K_1+K_2.■

DEGREE OF APPROXIMATION: AN EXAMPLE

To illustrate the relationship between the degree of approximation of the arrival rate at I_0 and the buffer capacities K_1 (K_2), we consider the following example with equal buffer capacities $K_1 = K_2 = K$ and equal Poisson arrival rates $\mu_1 = \mu_2 = \mu$ at buffers I_1 and I_2, respectively.

Using equation (26), the density function of the time between kit completions, D_n, may be expressed in Laplace transform form as:

$$\text{f(s)} = \left(\frac{m}{m+s}\right) - \frac{1}{K}\left(\frac{m}{m+s}\right)\left(\frac{s}{2m+s}\right)33. \qquad (30)$$

Inverting to the time domain, the density function of D_n is obtained as

$$f(t) = m e^{-mt} + \frac{m}{K} e^{-mt} - \frac{2m}{K} e^{-2mt} \; 34, \; t \geq 0. \tag{31}$$

We define an error term e(t), expressed as the absolute difference between the exponential density and the actual density of D_n:

$$e(t) = \frac{m}{K} | 2 e^{-2mt} - e^{-mt} | 35. \tag{32}$$

Using equation (31), graphs of f(t) are plotted for μ=1 and K=2, 5 and 10 against time t ≥ 0 in Figure 3. It is observed from Figure 3 that the density of D_n rapidly approaches an exponential density as K increases. The graph of e(t) against K, plotted for μ=1 and t=0.2, is presented in Figure 4, which also indicates that the error term e(t) approaches zero(0) rapidly as K increases.

Figure 3.

Figure 4.

Using equation (32), it is easily seen that for ∀t, t>0, $| 2 e^{-2mt} - e^{-mt} | 36 \leq 1$. Hence, for a given e>0 and for any arrival rate μ, we can find a K such that

$$\frac{m}{K} 37 \leq e.$$

Therefore, the inventory capacity required to effect the desired approximation can be easily determined knowing the component arrival rate.

5. Discussion and conclusion

We have proven conditions for which the inter-arrival times of kits arriving to assembly are approximately independent and identically distributed exponential random variables. If components arrive at I_1 and I_2 according to independent and identical Poisson arrival streams and if $K_1 + K_2$ is sufficiently large, the output stream from kitting approximates a Poisson process. The practical importance of this result is that the assembly process downstream of the kitting operation can be decoupled from kitting for further analysis. The required conditions (for decoupling) are not restrictive and may, in fact, hold in actual applications.

It is also interesting to note that the long-term probability distribution of inventory positions at I_1 (I_2), observed at kit completion epochs, is independent of the actual inventory positions (positive or negative). Our result also indicates that all the positive (negative) inventory positions are equally likely with probability that is proportional to the rate μ_1 (μ_2), of Poisson arrivals. If arrival rates to I_1 and I_2 are equal (i.e., $\mu_1 = \mu_2 = \mu$) , all the inventory positions except zero become equally likely with probability that is inversely proportional to the total inventory capacity ($K_1 + K_2$). The incidence of observing both buffers empty is twice as likely as observing a positive (negative) stock position at either of the buffers.

Harrison [6] showed that a sufficient condition for an assembly-like queue to reach steady state is that buffer capacities must be bounded from above. We have shown that the total buffer capacity, $K_1 + K_2$, must be "sufficiently large" to obtain a Poisson approximation of the output stream of kits. However, from the example in section 4, we find that $K_1 + K_2$ need not be impractically large to achieve an approximate Poisson output stream; the value of $K_1 + K_2$ being dependent upon the degree of approximation desired. Since the arrival process at assembly machine P_3 may be approximated by a Poisson distribution, the downstream assembly system can be approximated by the much studied M/G/1 queue.

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