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A Markov Renewal Approach to $M/G/1$ Type Queues with Countably Many Background States *

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Abstract. We consider the stationary distribution of the $M/GI/1$ type queue when background states are countable. We are interested in its tail behavior. To this end, we derive a Markov renewal equation for characterizing the stationary distribution using a Markov additive process that describes the number of customers in system when the system is not empty. Variants of this Markov renewal equation are also derived. It is shown that the transition kernels of these renewal equations can be expressed by the ladder height and the associated background state of a dual Markov additive process. Usually, matrix analysis is extensively used for studying the $M/G/1$ type queue. However, this may not be convenient when the background states are countable. We here rely on stochastic arguments, which not only make computations possible but also reveal new features. Those results are applied to study the tail decay rates of the stationary distributions. This includes refinements of the existence results with extensions.

Keywords: $M/G/1$ type queue, stationary distribution, Markov additive process, duality, ladder height, Markov renewal theorem, hitting probability, decay rate

AMS subject classification: 90B22, 60K25, 60K20, 60J75

1. Introduction

The $M/GI/1$ type queue, which was termed by Neuts [12], has been extensively studied in theory as well as in applications. It is a discrete-time single server queue with a finite number of background states, and a pair of the number of customers in system and the background state constitutes a discrete-time Markov chain. It is assumed that background state transitions do not depend on the number of customers in system as long as the system is not empty, but this is not the case when the system is empty. The latter is called boundary transitions. Thus, the $M/GI/1$ type queue has a relatively simple structure. Nevertheless, it copes with various kinds of queueing models, e.g., vacation models, batch arrival queues and some others.

The finite number of the background states enables us to algorithmically compute characteristics such as the stationary joint distribution of the system and background states. This led to a great success of the $M/GI/1$ type queue, particularly, in applications. However, it also limits a class of applicable models. In general, if there are more

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than one unlimited queue, e.g., like in tandem queues, then we can not use the framework of finite background states. In other words, its applicability is much widened if this finiteness assumption is removed. So, there have been some attempts in this line. For instance, Takahashi et al. [17] recently studied a quasi-birth-and-death process with countable background states for considering a two-queue system with joining shorter queue discipline.

In this paper, we study the $M/GI/1$ type queue allowing the background state space to be countable. This model includes the quasi-birth-and-death process of [17] as a special case. Similar to the finite background case, we are mainly concerned with the stationary joint distribution of the number of customers in system and background state. The conventional $M/GI/1$ type queue is usually studied by matrix analysis. This is also the case in [17]. Unlike those matrix approach, we here mainly use stochastic analysis. This simplifies our computations, and enables us to get finer results. For instance, the recent results of [17] can be obtained under weaker conditions. Furthermore, it simultaneously reveals nice stochastic structures even for the conventional $M/GI/1$ type queue itself, i.e., the case of the finite background states.

We first observe that, when the system is not empty, the number of customers in system is described by a Markov additive process. Using this fact, we show that the stationary distribution is obtained as the solution of a Markov renewal equation, i.e., as a Markov renewal function. This leads to the following interesting facts. First, the stationary distribution can be obtained in terms of the ladder height and the associated background state of a dual Markov additive process and the boundary transitions, where the dual process is defined by changing the sign of the time reversed Markov additive process. This observation may go back to Feller [8] for the $M/G/1$ queue, but its extension is not obvious because of the different transitions at the boundary in addition to the background process. Secondly, the tail behavior of the stationary distribution can be studied through an asymptotic behavior of the dual Markov renewal process. Thirdly, the stationary distribution is also obtained through the background state distributions at the upward hitting times of the dual Markov additive process.

Technically, this paper is stimulated by recent work of Takine [18] on the stationary distribution in the $M/G/1$ type queue as well as the author's works [9,11] on the hitting probabilities in a continuous-time Markov additive process with finite background states. In those papers, Markov renewal functions are derived for studying asymptotic tail behaviors of the stationary distribution, but all of them assume the finite number of the background states and matrix analysis is extensively used. Some related discussions can be also found in [10].

This paper is composed by five sections. In section 2, we first introduce the $M/G/1$ type queue, and discuss an identity obtained by Ramaswami [14] with a simple proof. The Markov renewal equation is derived, and its variants are considered in section 3. We then discuss how the stationary distribution is obtained through the hitting probability in section 4. We finally apply the results to get the decay rate of the stationary distribution in section 5.

2. Ramaswami's identity

In this section, we introduce notation for the $M/G/1$ type queue with countable background states and for the corresponding Markov additive process. We shall use the standard notation in the literature, but sometimes slightly change them for convenience. We then give a simple proof for the identity derived by Ramaswami [14]. This identity plays a key role in our arguments. Some stability issues are also discussed.

Let S be a countable set, which serves as a state space for the background state. Let $A(n)$ and $B(n)$, $n = 0, 1, \ldots,$ be $S \times S$ nonnegative matrices such that

$$\sum_{n=0}^{\infty} A(n)\boldsymbol{e} = \boldsymbol{e} \quad \text{and} \quad \sum_{n=0}^{\infty} B(n)\boldsymbol{e} = \boldsymbol{e},$$

where $\boldsymbol{e}$ is S-column vector all of whose entries are unit. Thus, these matrix sums are stochastic. Let $\mathbb{Z}_+ = \{0, 1, \ldots\}$ and $S_1 = \mathbb{Z}_+ \times S$. Define the $S_1 \times S_1$ transition probability matrix P as

$$P = \begin{pmatrix} B(0) & B(1) & B(2) & B(3) & \ldots \\ A(0) & A(1) & A(2) & A(3) & \ldots \\ 0 & A(0) & A(1) & A(2) & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}.$$

If S is finite, the Markov chain with this transition matrix P is referred to as the $M/G/1$ *type* in the queueing literature. Since we removed the finiteness of S, we refer it as a $M/G/1$ type with countable background states. As we shall see, $A(n)$ might be better to denoted by $A(n-1)$, which directly corresponds with up and down movements of the additive component, but we keep it as it is since this notation has been widely used as well as it is convenient for queueing applications.

Usually, matrices $B(n)$ with $n \geqslant 0$ are allowed to have different row entries from S. Furthermore, the first matrix $A(0)$ in the second row blocks can be replaced by any nonnegative matrix $C(0)$ of the same column size as $A(0)$ such that $(C(0) + \sum_{n=1}^{\infty} A(n)) \times \boldsymbol{e} = \boldsymbol{e}$. However, these modifications are not essential in our arguments. So we always assume that $A(n)$ and $B(n)$ are matrices of the same sizes, and will not use $C(0)$ in the second row blocks. Let $\boldsymbol{x}(n)$, $n = 0, 1, \ldots,$ be nonnegative S-row vectors. Then, $\boldsymbol{x} \equiv \{\boldsymbol{x}(n); n \geqslant 0\}$ is said to the stationary measure (or vector) of P if $\boldsymbol{x}P = \boldsymbol{x}$, which is equivalent to

$$\boldsymbol{x}(n) = \boldsymbol{x}(0)B(n) + \sum_{\ell=1}^{n+1} \boldsymbol{x}(\ell)A(n+1-\ell), \tag{2.1}$$

and, in particular, said to the stationary distribution if $\sum_{n=0}^{\infty} \boldsymbol{x}(n)\boldsymbol{e} = 1$. To exclude a trivial case, we assume that $B(0)$ is not stochastic, i.e., there is a row of $B(0)$ such that its sum is less than unit.

We next consider the Markov chain obtained from P removing boundary states $\{0\} \times S$ and extending the state space from S_1 to $\mathbb{Z} \times S$, where $\mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\}$. That is, the transition probability from (i, m) to (j, n) of the Markov chain is $[A(n-m+1)]_{ij}$ if $n \geqslant m-1$, otherwise it is null. Let (X_n, Y_n) be the state of this Markov chain at time n, where $Y_0 = 0$. The process $\{(X_n, Y_n)\}$ is a discrete valued and discrete time Markov additive process with background process $\{X_n\}$ (see, e.g., [6]). We refer to Y_n as an additive component. From the structure of P, Y_n is skip free in the downward.

Let $Q(n) = A(n+1)$ for $n \geqslant -1$. This $\{Q(n);\ n \geqslant -1\}$ is the transition kernel of the Markov additive process (X_n, Y_n), and referred to as a Markov additive kernel. For mathematical description, $Q(n)$ is more convenient than $A(n)$, but much of the queueing literature has used $A(n)$, so we follow them except a few, and refer to $\{A(n)\}$ as a Markov additive kernel as well. Denote the generating matrix function of $\{A(n)\}$ by A^*, i.e.,

$$A^*(z) = \sum_{n=0}^{\infty} z^n A(n), \quad z > 0.$$

Similarly we let $B^*(z) = \sum_{n=0}^{\infty} z^n B(n)$. Note that the background process $\{X_n\}$ is a Markov chain with transition rate matrix $A \equiv A^*(1)$. We assume that

(i) A is irreducible and aperiodic.

Remark 2.1. If the background state space S is finite, then (i) implies that A is positive recurrent. However, if not, the situation is greatly changed. In fact, A may be transient even if P is positive recurrent. For example, consider a Markov additive process (X_n, Y_n) with background state space $\mathbb{Z}_+$ such that $\inf_i E(X_{n+1} \mid X_n = i) > 0$. Then, A is obviously transient. However, if we put the boundary at level 0 where X_n is changed to only decrease and if $\sup_i E(Y_{n+1} - Y_n \mid X_n = i) < 0$, then P can be positive recurrent. This is actually the situation of a single server priority queue with two types of customers. Thus, it may be too restrictive to assume that A is positive recurrent.

For $n \geqslant 1$, let $\tau_n = \inf\{\ell \geqslant 1;\ Y_\ell = -n\}$, i.e., the first time when Y_n hits $-n$ from above. Since $\{Y_n\}$ is skip free and has homogenous background state transitions, it is easy to see that the embedded process $\{X_{\tau_n}\}$, where $\tau_0 = 0$, is also a Markov chain, which may terminate in a finite steps. Denote the $S \times S$ transition matrix of this Markov chain by G. Conditioning by the first step, it is obvious to see that G must satisfy

$$G = \sum_{n=0}^{\infty} A(n) G^n. \tag{2.2}$$

It is also not hard to see that G is the unique minimal nonnegative and nonzero solution of (2.2). This G is called the *fundamental matrix* in the queueing literature. Note that G may not be irreducible. For example, if $[\boldsymbol{e}A(0)]_j = 0$, then $G_{ij} = 0$ for all $i \in S$. Since any state can reach any other state under the transitions of A, so G has a single irreducible class.

Although it is too restrictive to assume for A to be positive recurrent, it will be useful to consider the positive recurrent case since we may apply this case under a suitable change of the probability measure. We now tentatively suppose that X_n has the unique stationary distribution, denoted by S-row vector $\boldsymbol{\pi}$. Furthermore, if the mean drift of Y_n is not positive, i.e.,

$$\beta_A \equiv \sum_{n=1} n\boldsymbol{\pi} A(n)\boldsymbol{e} \leqslant 1,$$

then τ_n is proper since Y_ℓ hits $-n$ with probability one. Hence, G is stochastic. We can say a little more about its recurrence by theorem 1 of Alsmeyer [2].

Lemma 2.1. Suppose that A is positive recurrent. Then, G is stochastic only if $\beta_A \leqslant 1$, and it is Harris ergodic if $\beta_A < 1$.

Here, a stochastic matrix is said to be Harris recurrent (ergodic) if the corresponding Markov chain is Harris recurrent (ergodic, respectively) (see, e.g., [4] for these Harris definitions). Since the state space is discrete, Harris recurrent (ergodic) is equivalent to that there is a single irreducible and recurrent (positive recurrent) class that can be reached from all other states.

We now back to the case that A may not be positive recurrent. Define $\Phi_A(n)$ and $\Phi_B(n)$ as

$$\Phi_A(n) = \sum_{\ell=n}^{\infty} A(\ell)G^{\ell-n} \quad \text{and} \quad \Phi_B(n) = \sum_{\ell=n}^{\infty} B(\ell)G^{\ell-n}.$$

Then, using a censored process, Ramaswami [14] shows that, if $\boldsymbol{x} \equiv \{\boldsymbol{x}(n)\}$ is the stationary distribution, then it satisfies

$$\boldsymbol{x}(n) = \boldsymbol{x}(0)\Phi_B(n) + \sum_{\ell=1}^{n} \boldsymbol{x}(\ell)\Phi_A(n+1-\ell), \quad n \geqslant 0, \tag{2.3}$$

where an empty sum is null. In [14], the background state space S is assumed to be finite, but this is clearly not essential. Equation (2.3) is rearranged for $n \geqslant 1$ to

$$\boldsymbol{x}(n) = \left[\boldsymbol{x}(0)\Phi_B(n) + \sum_{\ell=1}^{n-1} \boldsymbol{x}(\ell)\Phi_A(n+1-\ell)\right]\big(I - \Phi_A(1)\big)^{-1}, \quad n \geqslant 1, \tag{2.4}$$

where $(I - \Phi_A(1))^{-1} = \sum_{n=0}^{\infty} \Phi_A(1)^n$ is well defined since $\Phi_A(1)$ is strictly substochastic. Thus, $\boldsymbol{x}(n)$ is computed inductively if $\boldsymbol{x}(0)$ is given. This computation algorithm is known to be stable because subtractions are not needed. It should be noted that we cannot numerically compute $\boldsymbol{x}(n)$ because S may be infinite. This is a drawback of removing the finiteness assumption on S.

Assume that the stationary distribution exists and is unique and that $\boldsymbol{x}(0)$ is properly given. Then, it is easy to see that (2.3) also implies (2.1). We here prove this implication without those assumptions. This is because we shall work on (2.3) to get the stationary distribution. Furthermore, the proof is very short, so may be of independent interest.

Lemma 2.2. If (2.3) holds, then $\{\boldsymbol{x}(n)\}$ is the stationary vector of P.

Proof. Since

$$\Phi_A(n) = A(n) + \Phi_A(n+1)G \quad \text{and} \quad \Phi_B(n) = B(n) + \Phi_B(n+1)G,$$

we have, from (2.3), for $n \geqslant 0$,

$$\begin{aligned}
\boldsymbol{x}(n) &= \boldsymbol{x}(0)(B(n) + \Phi_B(n+1)G) + \sum_{\ell=1}^{n} \boldsymbol{x}(\ell)\big(A(n+1-\ell) + \Phi_A(n+2-\ell)G\big)\\
&= \boldsymbol{x}(0)B(n) + \sum_{\ell=1}^{n+1} \boldsymbol{x}(\ell)A(n+1-\ell) - \boldsymbol{x}(n+1)A(0)\\
&\qquad + \boldsymbol{x}(0)\Phi_B(n+1)G + \sum_{\ell=1}^{n+1} \boldsymbol{x}(\ell)\Phi_A(n+2-\ell)G - \boldsymbol{x}(n+1)\Phi_A(1)G\\
&= \boldsymbol{x}(0)B(n) + \sum_{\ell=1}^{n+1} \boldsymbol{x}(\ell)A(n+1-\ell) + \boldsymbol{x}(n+1)\big(G - A(0) - \Phi_A(1)G\big)\\
&= \boldsymbol{x}(0)B(n) + \sum_{\ell=1}^{n+1} \boldsymbol{x}(\ell)A(n+1-\ell),
\end{aligned}$$

where we have used (2.3) for $n+1$ to get the third equation. Thus, we get the stationary equation (2.1). □

By (2.4), $\boldsymbol{x}(n)$ is uniquely determined for a given $\boldsymbol{x}(0)$. The existence of the nonnegative and nonzero vector $\boldsymbol{x}(0)$ is assured either if there is no background sate, i.e., $|S| = 1$, or if $|S| \geqslant 2$ and the following equation has the nonnegative and nonzero solution $\boldsymbol{x}(0)$.

$$\boldsymbol{x}(0) = \boldsymbol{x}(0)\Phi_B(0). \tag{2.5}$$

Obviously, this vector uniquely exists up to a multiplicative constant if and only if $\Phi_B(0)$ is Harris recurrent. Furthermore, $\boldsymbol{x}(0)$ is a finite measure if and only if

(ii) $\Phi_B(0)$ is Harris ergodic.

Throughout the paper, we assume (ii). Let us consider the case that A is positive recurrent again. In this case, by lemma 2.1, $\Phi_B(0)$ is stochastic only if $\beta_A \leqslant 1$. So, (ii) always holds for finite S if $B^*(1)$ is aperiodic and irreducible. This may not be true for countable S. In many applications, $\{B(n)\}$ is stochastically not greater than $\{A(n)\}$, i.e., $\sum_{\ell=n}^{\infty} B(\ell) \leqslant \sum_{\ell=n}^{\infty} A(\ell)$ in componentwise for all $n \geqslant 1$. In this case, we can show that (ii) holds if $\beta_A < 1$, since the stationary distribution exists for $\{B(n)\} = \{A(n)\}$.

3. Markov renewal equations

In this section, we first derive a Markov renewal equation for the stationary measure $\{\boldsymbol{x}(n)\}$, assuming its existence. Then, we derive its variants. Throughout this and next sections, we assume that A is positive recurrent in addition to (i), i.e., we replace (i) by the stronger assumption:

(i′) A is irreducible, aperiodic, and positive recurrent with the stationary distribution $\boldsymbol{\pi}$.

3.1. The first formulation

Let $\Psi_A(n) = \Phi_A(n+1)$ for $n \geqslant 0$. Then, (2.3) can be written as

$$\boldsymbol{x}(n) = \boldsymbol{x}(0)\big(\Phi_B(n) - \Psi_A(n)\big) + \boldsymbol{x} * \Psi_A(n), \quad n \geqslant 0, \tag{3.1}$$

or, taking its transpose,

$$\boldsymbol{x}^{\mathrm{T}}(n) = \big(\Phi_B^{\mathrm{T}}(n) - \Psi_A^{\mathrm{T}}(n)\big)\boldsymbol{x}^{\mathrm{T}}(0) + \Psi_A^{\mathrm{T}} * \boldsymbol{x}^{\mathrm{T}}(n), \quad n \geqslant 0, \tag{3.2}$$

where $*$ denotes the convolution, i.e.,

$$\boldsymbol{x} * \Psi_A(n) = \sum_{\ell=0}^{n} \boldsymbol{x}(\ell)\Psi_A(n-\ell), \qquad \Psi_A^{\mathrm{T}} * \boldsymbol{x}^{\mathrm{T}}(n) = \sum_{\ell=0}^{n} \Psi_A^{\mathrm{T}}(n-\ell)\boldsymbol{x}^{\mathrm{T}}(\ell).$$

(3.2) is a Markov renewal equation if $\sum_{n=0}^{\infty} \Psi_A^{\mathrm{T}}(n)\boldsymbol{e} \leqslant \boldsymbol{e}$, i.e, $\sum_{n=0}^{\infty} \Psi_A^{\mathrm{T}}(n)$ is substochastic. Unfortunately, this may not be true. To convert this equation to the Markov renewal equation, we use a dual process, defined below.

Assume that X_0 is subject to the stationary distribution $\boldsymbol{\pi}$. Then, $\{X_n;\ n \geqslant 0\}$ is a stationary process, so we can extend this process on the time axis of the whole integers. Since $\{Y_n\}$ is defined for the $\{X_n\}$, we can extend it also on the time axis of the whole integers, where $Y_0 = 0$ is retained. For instance, $Y_{-1} = -(n-1)$ if the transition from X_{-1} to X_0 occurs due to $A(n)$. We then define the dual process $\{(\widetilde{X}_n, \widetilde{Y}_n)\}$ by

$$\widetilde{X}_n = X_{-n} \quad \text{and} \quad \widetilde{Y}_n = -Y_{-n}, \quad n \in \mathbb{Z}.$$

As above, we put the tilde for a characteristic of the dual process unless otherwise stated. It is easy to see that $\{\widetilde{Y}_n\}$ is a Markov additive process with background process $\{\widetilde{X}_n\}$ whose Markov renewal kernel is given by

$$\widetilde{A}(n+1) = \Delta_{\boldsymbol{\pi}}^{-1} A^{\mathrm{T}}(n+1)\Delta_{\boldsymbol{\pi}}, \quad n = -1, 1, 2, \ldots,$$

where $\Delta_{\boldsymbol{\pi}}$ is the diagonal matrix whose ith entry is the corresponding one of vector $\boldsymbol{\pi}$. This convention will be used for other vectors as well. Similarly to the forward case, we put $\widetilde{A} = \sum_{n=0}^{\infty} \widetilde{A}(n)$, which is the transition matrix for $\{\widetilde{X}_n\}$. Clearly, (i′) implies that $\widetilde{A}$ is irreducible and aperiodic.

Define $\tilde{\tau}$ as

$$\tilde{\tau} = \inf\big\{n \geqslant 1 \mid \widetilde{Y}_n \geqslant 0\big\}.$$

That is, $\tilde{\tau}$ is the weak ladder epoch of $\{\widetilde{Y}_n\}$. Define $S \times S$ substochastic matrix $\widetilde{L}(\ell)$ by

$$\big[\widetilde{L}(\ell)\big]_{ij} = P\big(\widetilde{Y}_{\tilde{\tau}} = \ell,\ \widetilde{X}_{\tilde{\tau}} = j \mid \widetilde{X}_0 = i\big), \quad \ell \geqslant 0,\ i, j \in S.$$

Namely, $\{[\widetilde{L}(\ell)]_{ij};\ \ell \geqslant 0\}$ is the probability mass function of the weak ladder height for the dual process. The following result is a key for our arguments.

Lemma 3.1. Under any drift condition, we have

$$\big[\widetilde{L}(\ell)\big]_{ij} = \frac{\pi_j}{\pi_i}\big[\Psi_A(\ell)\big]_{ji}, \quad \ell \geqslant 0,\ i, j \in S. \tag{3.3}$$

Proof. For convenience, we write the probability $P(D \mid \widetilde{X}_0 = i)$ as $\widetilde{P}_i(D)$ and $P(D \mid X_0 = j)$ as $P_j(D)$ for event D. We evaluate $\pi_i \widetilde{P}_i(\tilde{\tau} = n,\ \widetilde{Y}_{\tilde{\tau}} = \ell,\ \widetilde{X}_{\tilde{\tau}} = j)$. For $n = 1$, we have

$$\begin{aligned}
\pi_i \widetilde{P}_i\big(\tilde{\tau} = 1,\ \widetilde{Y}_{\tilde{\tau}} = \ell,\ \widetilde{X}_{\tilde{\tau}} = j\big) &= P(Y_{-1} = -\ell,\ X_{-1} = j,\ Y_0 = 0,\ X_0 = i)\\
&= \pi_j P_j(Y_0 = 0,\ Y_1 = \ell,\ X_1 = i)\\
&= \pi_j\big[A(\ell+1)\big]_{ji}.
\end{aligned}$$

For $n \geqslant 2$,

$$\begin{aligned}
&\pi_i \widetilde{P}_i(\tilde{\tau} = n,\ \widetilde{Y}_{\tilde{\tau}} = \ell,\ \widetilde{X}_{\tilde{\tau}} = j)\\
&\quad = P\big(\widetilde{Y}_0 = 0,\ \widetilde{X}_0 = i,\ \widetilde{Y}_s < 0,\ s = 1, 2, \ldots, n-1,\ \widetilde{Y}_n = \ell,\ \widetilde{X}_n = j\big)\\
&\quad = P\big(\widetilde{Y}_{-n} = 0,\ \widetilde{X}_{-n} = i,\ \widetilde{Y}_s < 0,\ s = -1, -2, \ldots, -(n-1),\ \widetilde{Y}_0 = \ell,\ \widetilde{X}_0 = j\big)\\
&\quad = P\big(Y_n = 0,\ X_n = i,\ Y_s > 0,\ s = 1, 2, \ldots, (n-1),\ Y_0 = -\ell,\ X_0 = j\big)\\
&\quad = P\big(Y_n = \ell,\ X_n = i,\ Y_s > \ell,\ s = 1, 2, \ldots, (n-1),\ Y_0 = 0,\ X_0 = j\big)\\
&\quad = \pi_j P_j\big(Y_0 = 0,\ Y_s > \ell,\ s = 1, 2, \ldots, (n-1),\ Y_n = \ell,\ X_n = i\big)\\
&\quad = \pi_j \sum_{m=\ell+1}^{\infty} P_j\big(Y_0 = 0,\ Y_1 = m,\ Y_s > \ell,\ s = 2, \ldots, (n-1),\ Y_n = \ell,\ X_n = i\big)\\
&\quad = \pi_j \sum_{m=\ell+1}^{\infty} \sum_{k\in S}\big[A(m+1)\big]_{jk} P_k\big(Y_s > m-\ell,\ s = 2, \ldots, (n-1),\\
&\qquad\qquad Y_n = m-\ell,\ X_n = i\big).
\end{aligned}$$

Summing this equation over $n \geqslant 2$ yields

$$\begin{aligned}
\pi_i \widetilde{P}_i\big(\tilde{\tau} \geqslant 2,\ \widetilde{Y}_{\tilde{\tau}} = \ell,\ \widetilde{X}_{\tilde{\tau}} = j\big) &= \pi_j \sum_{m=\ell+1}^{\infty} \sum_{k\in S}\big[A(m+1)\big]_{jk} G_{ki}^{m+1-(\ell+1)}\\
&= \pi_j \sum_{m=\ell+2}^{\infty} \big[A(m)G^{m-(\ell+1)}\big]_{ji}.
\end{aligned}$$

Combining this with the case of $n = 1$, we get (3.3). $\square$

Remark 3.1. From this lemma, we have

$$\sum_{\ell=1}^{\infty}\sum_{j\in S}\frac{\pi_j}{\pi_i}\big[\Psi_A(\ell)\big]_{ji} \leqslant 1, \quad i \in S,$$

where the equality holds only if $\beta_A \geqslant 1$.

We now convert (3.2) to a Markov renewal equation. To this end, we use the following notation.

$$\begin{aligned}\tilde{\boldsymbol{x}}(n) &= \Delta_{\boldsymbol{\pi}}^{-1}\boldsymbol{x}(n)^{\mathrm{T}},\\ \widetilde{\Phi}_V(n) &= \Delta_{\boldsymbol{\pi}}^{-1}\Phi_V(n+1)^{\mathrm{T}}\Delta_{\boldsymbol{\pi}} \quad \text{for } V = A, B,\\ \widetilde{\Psi}_A(n) &= \widetilde{\Phi}_A(n+1).\end{aligned}$$

The generating functions of these vector and matrix functions are denoted by $\tilde{\boldsymbol{x}}^*(s)$, $\widetilde{\Phi}_V^*(s)$ and $\widetilde{\Psi}_A^*(s)$, respectively.

Theorem 3.1. Suppose that (i$'$) and (ii) hold. Then, the following Markov renewal equation has the solution $\{\tilde{\boldsymbol{x}}(n)\}$, and $\{\tilde{\boldsymbol{x}}(n)^{\mathrm{T}}\Delta_{\boldsymbol{\pi}}\}$ is the stationary measure.

$$\tilde{\boldsymbol{x}}(n) = \big(\widetilde{\Phi}_B(n) - \widetilde{\Psi}_A(n)\big)\tilde{\boldsymbol{x}}(0) + \widetilde{L} * \tilde{\boldsymbol{x}}(n), \quad n \geqslant 0. \tag{3.4}$$

The Markov renewal kernel $\{\widetilde{L}(n);\ n \geqslant 0\}$ is defective only if $\beta_A < 1$. In this case, $\{\boldsymbol{x}(n)\}$ is a finite measure if and only if

$$\beta_B \equiv \sum_{n=1}^{\infty} n\boldsymbol{x}(0)B(n)\boldsymbol{e} < \infty. \tag{3.5}$$

Proof. Since (3.3) can be written as $\widetilde{L}(n) = \widetilde{\Psi}_A(n)$, (3.4) is immediate from (3.2). We already know the defectiveness of $\{\widetilde{L}(n);\ n \geqslant 0\}$, so it remains to prove the last statement. For this, we take the generating functions of (3.4). Then, for $0 \leqslant s < 1$, we have

$$\tilde{\boldsymbol{x}}^*(s) = \big(I - \widetilde{\Psi}_A^*(s)\big)^{-1}\big(\widetilde{\Phi}_B^*(s) - \widetilde{\Psi}_A^*(s)\big)\tilde{\boldsymbol{x}}(0),$$

since $\widetilde{\Psi}_A^*(s)$ is strictly substochastic. Note that $\widetilde{\Psi}_A$ is still strictly substochastic. Hence, the total measure $\boldsymbol{x}^*(1)\boldsymbol{e} = \boldsymbol{\pi}\tilde{\boldsymbol{x}}^*(1)$ is finite if and only if $\boldsymbol{\pi}\widetilde{\Phi}_B^*(1)\tilde{\boldsymbol{x}}(0)$ is finite. This is equivalent to $\boldsymbol{x}(0)\Phi_B^*(1)\boldsymbol{e} < \infty$, where $\Phi_B^*(s)$ is the generating function of $\{\Phi_B(n)\}$. We now check that (3.5) is equivalent to

$$\boldsymbol{x}(0)\Phi_B^*(1)\boldsymbol{e} = \sum_{n=0}^{\infty}\boldsymbol{x}(0)\sum_{\ell=n}^{\infty}B(\ell)G^{\ell-n}\boldsymbol{e} = \sum_{\ell=0}^{\infty}(\ell+1)\boldsymbol{x}(0)B(\ell)\boldsymbol{e} < \infty.$$

This concludes the last statement. □

Remark 3.2. If S is finite and $\boldsymbol{x}(0)$ is positive, then (3.5) is equivalent to

$$\sum_{n=1}^{\infty} nB(n) < \infty.$$

This is exactly the necessary and sufficient condition for the finiteness of $\{\boldsymbol{x}(n)\}$ obtained in [13, theorem 3.2.1]. Note that complicated matrix computations are performed to prove this fact in [13], while our arguments are straightforward and valid also for countable S.

Let the overline $^{-}$ stand for the tail of a summation like $\overline{\tilde{\boldsymbol{x}}}(n) = \sum_{\ell=n}^{\infty} \tilde{\boldsymbol{x}}(\ell)$. Summing (3.4) over n yields

$$\overline{\tilde{\boldsymbol{x}}}(n) = \big(\overline{\widetilde{\Phi}}_B(n) - \overline{\widetilde{\Psi}}_A(n)\big)\tilde{\boldsymbol{x}}(0) + \overline{\widetilde{L}}(n+1)\overline{\tilde{\boldsymbol{x}}}(0) + \widetilde{L} * \overline{\tilde{\boldsymbol{x}}}(n), \quad n \geqslant 0. \tag{3.6}$$

This is a Markov renewal equation for $\{\overline{\tilde{\boldsymbol{x}}}(n)\}$.

Remark 3.3. Suppose that $A(n) = B(n)$ for $n \geqslant 0$. In this case, we have

$$\overline{\widetilde{\Phi}}_B(n) - \overline{\widetilde{\Psi}}_A(n) = \widetilde{\Phi}_A(n). \tag{3.7}$$

This is the standard setting in the $M/G/1$ type queue.

Let $\widetilde{U}$ be the mass function of the Markov renewal measure for the kernel $\{\widetilde{L}(n);\ n \geqslant 0\}$, i.e.,

$$\widetilde{U}(n) = \sum_{\ell=0}^{\infty} \widetilde{L}^{(*\ell)}(n) \quad \left(= \sum_{\ell=0}^{\infty} \widetilde{\Psi}_A^{(*\ell)}(n)\right), \quad n \geqslant 0, \tag{3.8}$$

where $\widetilde{L}^{(*\ell)}(n) = \widetilde{L}^{(*\ell-1)} * \widetilde{L}(n)$ by matrix convolution $A * B(n)$ defined as

$$A * B(n) = \sum_{\ell=0}^{n} A(\ell)B(n-\ell),$$

and $\widetilde{L}^{(0)}(n) = 1(n=0)I$. Then theorem 3.1 implies:

Corollary 3.1. Under the same assumption of theorem 3.1,

$$\tilde{\boldsymbol{x}}(n) = \big(\widetilde{U} * \big(\widetilde{\Phi}_B - \widetilde{\Psi}_A\big)\big)(n)\tilde{\boldsymbol{x}}(0), \quad n \geqslant 0. \tag{3.9}$$

Here, if $\beta_A < 1$, the stationary distribution $\{\boldsymbol{x}(n)\}$ is obtained as

$$\boldsymbol{x}(n) = \boldsymbol{x}(0)\big((\Phi_B - \Psi_A) * U\big)(n), \quad n \geqslant 0, \tag{3.10}$$

where $U(n) = \Delta_{\boldsymbol{\pi}} \widetilde{U}(n)^{\mathrm{T}} \Delta_{\boldsymbol{\pi}}^{-1}$, so $U(n)$ is given by

$$U(n) = \sum_{\ell=0}^{\infty} \Psi_A^{(*\ell)}(n), \quad n \geqslant 0.$$

Note that $U(0) = I + \Psi_A(0)U(0)$, so (3.10) with $n = 0$ reduces to (2.5). It should be noticed that all entries of $U(n)$'s are finite for all $n \geqslant 0$ if $\beta_A \leqslant 1$, and $U(\infty)$ is finite only if $\beta_A < 1$. In section 4, (3.9) will be used to consider an asymptotic behavior of $\boldsymbol{x}(n)$ as n goes to infinity.

3.2. Alternative formulations

There are some variants of the renewal equation (3.4) and the corresponding renewal function (3.8). Takine's [18] formulation is one of them. Generally speaking, those variants come from different choices of the Markov renewal kernel and the initial term of the sequence $\{\tilde{\boldsymbol{x}}(n)\}$.

(a) *Modifying the starting term.* In the Markov renewal equation (3.2), the sequence starts with $n = 0$. Instead of this, let it start with $n = 1$, then we have

$$\tilde{\boldsymbol{x}}(n) = \widetilde{\Psi}_B(n-1)\tilde{\boldsymbol{x}}(0) + \sum_{\ell=1}^{n} \widetilde{L}(n-\ell)\tilde{\boldsymbol{x}}(\ell), \quad n \geqslant 1. \tag{3.11}$$

Since $\boldsymbol{x}(0)$ is determined by (2.5), this is reasonable, and makes the solution simplify. Namely, we have

$$\boldsymbol{x}(n) = \boldsymbol{x}(0)(\Psi_B * U)(n-1), \quad n \geqslant 1, \tag{3.12}$$

where $\Psi_B(n) = \Phi_B(n+1)$. Let us check whether (3.12) is identical with (3.10). The right-hand side of (3.12) becomes

$$\begin{aligned}\boldsymbol{x}(0)(\Psi_B * U)(n-1) &= \boldsymbol{x}(0)\sum_{\ell=0}^{n-1} \Phi_B(n-\ell)U(\ell) \\ &= \boldsymbol{x}(0)\left(\sum_{\ell=0}^{n} \Phi_B(n-\ell)U(\ell) - \Phi_B(0)U(n)\right) \\ &= \boldsymbol{x}(0)(\Phi_B * U)(n) - \boldsymbol{x}(0)U(n),\end{aligned}$$

where (2.5) is used to get the last equality. Since $U(n) = \Psi_A * U(n)$ for $n \geqslant 1$, (3.12) is indeed identical with (3.10).

(b) *Modifying the Markov renewal kernel.* We next consider to change the Markov renewal kernel corresponding with the alternative equation (2.4) as well as starting with $n = 1$. Since $\widetilde{\Phi}_A(1) = \widetilde{\Psi}_A(0) = \widetilde{L}(0)$, we have

$$\tilde{\boldsymbol{x}}(n) = \bigl(I - \widetilde{L}(0)\bigr)^{-1}\widetilde{\Phi}_B(n)\tilde{\boldsymbol{x}}(0) + \sum_{\ell=1}^{n-1}\bigl(I - \widetilde{L}(0)\bigr)^{-1}\widetilde{L}(n-\ell)\tilde{\boldsymbol{x}}(\ell), \quad n \geqslant 1. \tag{3.13}$$

In this case, the Markov renewal kernel $\{\widetilde{\Gamma}(n);\ n \geqslant 1\}$ is given by

$$\widetilde{\Gamma}(n) = \bigl(I - \widetilde{L}(0)\bigr)^{-1}\widetilde{L}(n), \quad n \geqslant 1. \tag{3.14}$$

This $\{\widetilde{\Gamma}(n)\}$ is a right kernel since

$$\sum_{n=1}^{\infty}\widetilde{\Gamma}(n)\boldsymbol{e} = \bigl(I - \widetilde{L}(0)\bigr)^{-1}\sum_{n=1}^{\infty}\widetilde{L}(n)\boldsymbol{e} \leqslant \boldsymbol{e},$$

due to $\sum_{n=0}^{\infty}\widetilde{L}(n)\boldsymbol{e} \leqslant \boldsymbol{e}$. Note that this inequality is obtained by lengthy matrix computations in [18] for finite S, while our verification is probabilistic and does not require for S to be finite. Furthermore, (3.14) is interpreted as a certain conditional probability mass function.

Summing (3.13) over n, we have, similar to (3.6),

$$\overline{\tilde{\boldsymbol{x}}}(n) = \bigl(I - \widetilde{L}(0)\bigr)^{-1}\overline{\widetilde{\Phi}}_B(n)\tilde{\boldsymbol{x}}(0) + \widetilde{\Gamma} * \overline{\tilde{\boldsymbol{x}}}(n), \quad n \geqslant 1. \tag{3.15}$$

This is the Markov renewal equation that is obtained by Takine [18]. This equation is convenient to directly see the effect of the boundary transition $\{B(n)\}$, while the renewal kernel is more complicated.

4. Hitting probabilities

In this section, we assume that $\{\boldsymbol{x}(n)\}$ is the stationary distribution in addition to the assumptions of section 3. For each $n \geqslant 1$, let

$$\widetilde{T}_n = \inf\bigl\{\ell \geqslant 1 \mid \widetilde{Y}_\ell \geqslant n\bigr\}.$$

That is, $\widetilde{T}_n$ is the hitting time of the dual process $\{\widetilde{Y}_\ell\}$ at upward level $n \geqslant 1$. It is well known that, if $A(n) = B(n)$ for all $n \geqslant 0$, then the stationary distribution $\{\boldsymbol{x}(n)\}$ is obtained as

$$\bigl[\overline{\boldsymbol{x}}(n)\bigr]_i = \begin{cases} \pi_i\widetilde{P}_i(\tilde{\tau} < \infty), & n = 1,\\ \pi_i\widetilde{P}_i\bigl(\widetilde{T}_{n-1} < \infty\bigr), & n \geqslant 2, \end{cases} \tag{4.1}$$

where $\overline{\boldsymbol{x}}(0) = \boldsymbol{e}$. In this section, we show that this can be generalized using $S \times S$ matrix $\widetilde{H}(n)$ for $n \geqslant 0$, defined as

$$\bigl[\widetilde{H}(n)\bigr]_{ij} = \widetilde{P}_i\bigl(\widetilde{T}_n < \infty,\ \widetilde{X}_{\widetilde{T}_n} = j\bigr), \quad i, j \in S,$$

where $\widetilde{T}_0 = 0$, so $\widetilde{H}(0) = I$. We refer these probabilities as *hitting probabilities* at upper level n.

We aim to see how the hitting probabilities are involved in the stationary distribution $\{\boldsymbol{x}(n)\}$. We first note relationship between the hitting probabilities and the renewal function $\{\widetilde{U}(n)\}$.

Lemma 4.1. Let $\Delta\widetilde{H}(n) = \widetilde{H}(n+1) - \widetilde{H}(n)$ for $n \geqslant 0$, then

$$\widetilde{U}(n) = -\Delta\widetilde{H}(n)\big(I - \overline{\widetilde{L}}(0)\big)^{-1}, \quad n \geqslant 0. \tag{4.2}$$

Proof. We derive a renewal equation for this $H(n)$. For $n \geqslant 1$,

$$\begin{aligned}
\big[H(n)\big]_{ij} &= \widetilde{P}_i\big(\widetilde{Y}_{\tilde{\tau}} \geqslant n,\ \widetilde{X}_{\tilde{\tau}} = j\big) + \widetilde{P}_i\big(\widetilde{Y}_{\tilde{\tau}} \leqslant n-1,\ \widetilde{X}_{\widetilde{T}_n} = j\big) \\
&= \big[\overline{\widetilde{L}}(n)\big]_{ij} + \sum_{\ell=0}^{n-1} \big[\widetilde{L}(s)H(n-s)\big]_{ij} \\
&= \big[\overline{\widetilde{L}}(n+1) + \widetilde{L} * H(n)\big]_{ij}.
\end{aligned}$$

Since $H(0) = \overline{\widetilde{L}}(1) + \widetilde{L} * H(0) + I - \overline{\widetilde{L}}(0)$, we have

$$H(n) = \widetilde{U}(n)\big(I - \overline{\widetilde{L}}(0)\big) + \sum_{\ell=0}^{n} \widetilde{U}(\ell)\overline{\widetilde{L}}(n+1-\ell), \quad n \geqslant 0. \tag{4.3}$$

Hence, we have, for $n \geqslant 0$,

$$\begin{aligned}
H(n) - H(n+1) &= \big(\widetilde{U}(n) - \widetilde{U}(n+1)\big)\big(I - \overline{\widetilde{L}}(0)\big) \\
&\quad + \sum_{\ell=0}^{n} \widetilde{U}(\ell)\widetilde{L}(n+1-\ell) - \widetilde{U}(n+1)\overline{\widetilde{L}}(0) \\
&= \widetilde{U}(n)\big(I - \overline{\widetilde{L}}(0)\big) + \sum_{\ell=0}^{n+1} \widetilde{U}(\ell)\widetilde{L}(n+1-\ell) - \widetilde{U}(n+1) \\
&= \widetilde{U}(n)\big(I - \overline{\widetilde{L}}(0)\big).
\end{aligned}$$

Thus we get (4.2). □

From (4.3) and (4.2), we have, for $n \geqslant 1$,

$$\begin{aligned}
\widetilde{H}(n) &= \widetilde{H}(n-1) - \widetilde{U}(n-1)\big(I - \overline{\widetilde{L}}(0)\big) \\
&= \sum_{\ell=0}^{n} \widetilde{U}(\ell)\overline{\widetilde{L}}(n-\ell).
\end{aligned} \tag{4.4}$$

Using this equation together with lemma 4.1, we prove the following result.

Theorem 4.1. Suppose that (i′), (ii), the stability conditions $\beta_A < 1$ and (3.5) hold. Then, we have

$$\overline{\boldsymbol{x}}^{\mathrm{T}}(1) = \Delta_{\boldsymbol{\pi}}\big[\big(\overline{\widetilde{\Phi}}_B(1) - \overline{\widetilde{\Phi}}_A(1)\big)\tilde{\boldsymbol{x}}(0) + \widetilde{H}(0)\bar{\tilde{\boldsymbol{x}}}(0)\big]. \tag{4.5}$$

and, for $n \geqslant 2$,

$$\overline{\boldsymbol{x}}^{\mathrm{T}}(n) = \Delta_{\boldsymbol{\pi}}\left[-\sum_{\ell=0}^{n-2} \Delta\widetilde{H}(\ell)\big(\overline{\widetilde{\Phi}}_B - \overline{\widetilde{\Phi}}_A\big)(n-\ell)\tilde{\boldsymbol{x}}(0) + \widetilde{H}(n-1)\bar{\tilde{\boldsymbol{x}}}(0)\right]. \tag{4.6}$$

In particular, if $A(n) = B(n)$, then (4.1) is certainly obtained.

Proof. From (3.6) and (4.4), we have, for $n \geqslant 1$

$$\begin{aligned}
\bar{\tilde{\boldsymbol{x}}}(n) &= \widetilde{U} * \big(\overline{\widetilde{\Phi}}_B - \overline{\widetilde{\Psi}}_A\big)(n)\tilde{\boldsymbol{x}}(0) + \sum_{\ell=0}^{n} \widetilde{U}(\ell)\overline{\widetilde{L}}(n+1-\ell)\bar{\tilde{\boldsymbol{x}}}(0) \\
&= \widetilde{U} * \big(\overline{\widetilde{\Phi}}_B - \overline{\widetilde{\Phi}}_A\big)(n)\tilde{\boldsymbol{x}}(0) + \widetilde{U} * \widetilde{\Phi}_A(n)\tilde{\boldsymbol{x}}(0) + \widetilde{H}(n+1)\bar{\tilde{\boldsymbol{x}}}(0).
\end{aligned}$$

We compute the second term in the following way.

$$\begin{aligned}
&\widetilde{U} * \widetilde{\Phi}_A(n)\tilde{\boldsymbol{x}}(0) \\
&\quad = \left(\sum_{\ell=0}^{n-1} \widetilde{U}(\ell)\widetilde{L}(n-1-\ell) + \widetilde{U}(n)\big(\widetilde{\Phi}_A(0) - \widetilde{\Phi}_B(0)\big) + \widetilde{U}(n)\widetilde{\Phi}_B(0)\right)\tilde{\boldsymbol{x}}(0) \\
&\quad = \big(\widetilde{U}(n-1) + \widetilde{U}(n)\big)\tilde{\boldsymbol{x}}(0) - 1(n=1)\tilde{\boldsymbol{x}}(0) + \widetilde{U}(n)\big(\widetilde{\Phi}_A(0) - \widetilde{\Phi}_B(0)\big)\tilde{\boldsymbol{x}}(0)
\end{aligned}$$

From (3.6) with $n = 0$, we have

$$\bar{\tilde{\boldsymbol{x}}}(0) = \big(\overline{\widetilde{\Phi}}_B(0) - \overline{\widetilde{\Psi}}_A(0)\big)\tilde{\boldsymbol{x}}(0) + \overline{\widetilde{L}}(1)\bar{\tilde{\boldsymbol{x}}}(0) + \widetilde{L}(0)\bar{\tilde{\boldsymbol{x}}}(0).$$

This yields

$$\begin{aligned}
\big(I - \overline{\widetilde{L}}(0)\big)\bar{\tilde{\boldsymbol{x}}}(0) &= \big(\overline{\widetilde{\Phi}}_B(0) - \overline{\widetilde{\Psi}}_A(0)\big)\tilde{\boldsymbol{x}}(0) \\
&= \big(\overline{\widetilde{\Psi}}_B(0) - \overline{\widetilde{\Psi}}_A(0)\big)\tilde{\boldsymbol{x}}(0) + \widetilde{\Phi}_B(0)\tilde{\boldsymbol{x}}(0).
\end{aligned}$$

Hence,

$$\tilde{\boldsymbol{x}}(0) = \widetilde{\Phi}_B(0)\tilde{\boldsymbol{x}}(0) = \big(I - \overline{\widetilde{L}}(0)\big)\bar{\tilde{\boldsymbol{x}}}(0) - \big(\overline{\widetilde{\Psi}}_B(0) - \overline{\widetilde{\Psi}}_A(0)\big)\tilde{\boldsymbol{x}}(0). \tag{4.7}$$

Substituting this into the above computed term, we have

$$\begin{aligned}
\widetilde{U} * \widetilde{\Phi}_A(n)\tilde{\boldsymbol{x}}(0) = &\big(\widetilde{U}(n-1) + \widetilde{U}(n)\big)\big(I - \overline{\widetilde{L}}(0)\big)\bar{\tilde{\boldsymbol{x}}}(0) \\
&- \widetilde{U}(n-1)\big(\overline{\widetilde{\Phi}}_B(1) - \overline{\widetilde{\Phi}}_A(1)\big)\tilde{\boldsymbol{x}}(0) \\
&- \widetilde{U}(n)\big(\overline{\widetilde{\Phi}}_B(0) - \overline{\widetilde{\Phi}}_A(0)\big)\tilde{\boldsymbol{x}}(0) - 1(n=1)\tilde{\boldsymbol{x}}(0).
\end{aligned}$$

Thus, using lemma 4.1, we get (4.6) for $n \geqslant 2$. For $n = 1$, we note the following equation obtained from (4.7).

$$\overline{\tilde{x}}(0) - \tilde{x}(0) = \overline{\widetilde{L}}(0)\overline{\tilde{x}}(0) - \left(\overline{\widetilde{\Phi}}_B(1) - \overline{\widetilde{\Phi}}_A(1)\right)\tilde{x}(0).$$

Then, we can get (4.5). □

5. Application to the tail decay rate

We finally consider asymptotic tail behaviors of the stationary measure or distribution. In this section, we assume (i) and (ii), from which P has the unique stationary measure $\{\boldsymbol{x}(n)\colon n \geqslant 0\}$. However, neither A nor P is assumed to be positive recurrent. Similar to [9,18], we apply the Markov renewal theorem (e.g., see [4,7]) to the renewal equation of theorem 3.1. To this end, we first require to change the Markov renewal kernel $\{Q(n)\}$ so that it has a stationary distribution, then we need an aperiodicity for a stochastic kernel corresponding to $\{\widetilde{L}^*(n)\}$. Following Alsmeyer [1] and Shurenkov [16], we introduce the period of Markov renewal kernel $\{Q(n)\}$, where $Q(n) = A(n+1)$, as follows: a positive integer d is said to be the period of $\{Q(n)\}$ if it is the *greatest* positive integer such that, if $[Q(n)]_{ij} > 0$, then, $n = \gamma(i) - \gamma(j) + d\ell$ for some integer ℓ, where γ is a function from S to $\{0, 1, \ldots, d-1\}$. This γ is called a shift function. We will use the following condition.

(iii) Markov additive kernel $\{Q(n);\ n \geqslant -1\}$ is aperiodic, i.e., has unit period. In this case $\gamma \equiv 0$.

Remark 5.1. A sufficient condition for (iii) is that $\{Q(n);\ n \geqslant 0\}$, i.e., $\{A(n);\ n \geqslant 1\}$, is irreducible and aperiodic. This is little stronger, but enough for many applications.

We now closely look at the arguments in section 3.1. Then, we can see that $\boldsymbol{x}^{\mathrm{T}}(n)$, $\Phi_B^{\mathrm{T}}(n)$ and $\Psi_A^{\mathrm{T}}(n)$ can be replaced by $z^n\boldsymbol{x}^{\mathrm{T}}(n)$, $z^n\Phi_B^{\mathrm{T}}(n)$ and $z^n\Psi_A^{\mathrm{T}}(n)$, respectively, in the Markov renewal equation (3.2). Hence, if we can find z such that $\Psi_A^*(z) \equiv \sum_{n=0}^{\infty} z^n\Psi(n)$ has the left and right invariant positive vectors, we can obtain a stochastic kernel for the modified Markov renewal equation. Then, we can apply the Markov renewal theorem, which concludes under appropriate conditions that $z^n\boldsymbol{x}^{\mathrm{T}}(n)$ converges as n goes to infinity.

We next observe that, for $z > 1$,

$$\begin{aligned}\Psi_A^*(z) &= \sum_{n=0}^{\infty} z^n\Phi_A(n+1) = \sum_{n=0}^{\infty}\sum_{\ell=n+1}^{\infty} z^n A(\ell)G^{\ell-(n+1)}\\ &= \sum_{\ell=1}^{\infty}\sum_{n=0}^{\ell-1} z^n A(\ell)G^{\ell-(n+1)}\\ &= \left(A^*(z) - zI\right)(zI - G)^{-1} + I. \end{aligned} \tag{5.1}$$

This equation can read as the Wiener–Hopf factorization of the Markov additive process (see, e.g., [3]). In the view of (3.2), it suggests the following conditions for the geometric decay of $\boldsymbol{x}(n)$. For some $\alpha > 1$, there exist positive vectors $\boldsymbol{h}$ and $\boldsymbol{\mu}$ such that

$$A^*(\alpha)\boldsymbol{h} = \alpha\boldsymbol{h} \quad \text{and} \quad \boldsymbol{\mu}A^*(\alpha) = \alpha\boldsymbol{\mu}. \tag{5.2}$$

In addition to this condition, let us assume

$$\boldsymbol{\mu}\boldsymbol{h} < \infty. \tag{5.3}$$

By (5.1), $\boldsymbol{\mu}$ is also a left invariant positive vector of $\Psi_A^*(\alpha)$, and $\boldsymbol{\ell} \equiv (\alpha I - G)\boldsymbol{h}$ is its right invariant vector. In the following arguments, this $\boldsymbol{\ell}$ is shown to be nonnegative.

We first introduce the Markov additive process $\{(\widehat{X}_n, \widehat{Y}_n)\}$ that has the following Markov additive kernel $\{\widehat{Q}(n)\}$. Define $\widehat{Q}(n)$ and $\widehat{A}(n)$ as

$$\widehat{Q}(n-1) = \widehat{A}(n) = \alpha^{n-1}\Delta_{\boldsymbol{h}}^{-1}A(n)\Delta_{\boldsymbol{h}}, \quad n \geqslant 0.$$

Let $\widehat{A} = \sum_{n=0}^{\infty}\widehat{A}(n)$. Clearly, $\widehat{A}$ is irreducible and stochastic by (i) and (5.2), and $\{\widehat{Q}(n)\}$ is aperiodic by (iii). Furthermore, $\widehat{A}$ has a finite and positive left invariant vector $\boldsymbol{\nu} \equiv \boldsymbol{\mu}\Delta_{\boldsymbol{h}}$ by (5.3). Hence, $\widehat{A}$, i.e., $\{\widehat{X}_n\}$, is positive recurrent, and we can apply lemma 3.1 for $\{\widehat{A}(n)\}$ and $\boldsymbol{\nu}$ instead of $\{A(n)\}$ and $\boldsymbol{\pi}$. Thus, the ladder height distribution of the dual process for $\{(\widehat{X}_n, \widehat{Y}_n)\}$ is given by

$$\widetilde{L}_{\widehat{A}}(n) \equiv \Delta_{\boldsymbol{\nu}}^{-1}\Psi_{\widehat{A}}^{\mathrm{T}}(n)\Delta_{\boldsymbol{\nu}}, \quad n \geqslant 0. \tag{5.4}$$

Let $\widetilde{L}_{\widehat{A}}^*(z) = \sum_{\ell=0}^{\infty} z^{\ell}\widetilde{L}_{\widehat{A}}(\ell)$. We can also see that $\widehat{G}$ for $\{(\widehat{X}_n, \widehat{Y}_n)\}$ corresponding with G is given by

$$\widehat{G} = \alpha^{-1}\Delta_{\boldsymbol{h}}^{-1}G\Delta_{\boldsymbol{h}}. \tag{5.5}$$

This implies that $\widehat{G}$ is strictly substochastic. Indeed, if $\widehat{G}$ is stochastic, then (5.5) leads to the contradiction that G has eingenvalue $\alpha > 1$ with eigenvector $\boldsymbol{h}$. Hence, applying lemma 2.1 for $\{(\widehat{X}_n, \widehat{Y}_n)\}$, we must have

$$\beta_{\widehat{A}} \equiv \sum_{n=1}^{\infty} n\boldsymbol{\nu}\widehat{A}(n)\mathbf{e} > 1. \tag{5.6}$$

Since $\{\widehat{A}(n)\}$ is aperiodic, irreducible and positive recurrent, this implies that $\{\widetilde{L}_{\widehat{A}}(n)\}$ is aperiodic and Harris ergodic by [2, theorem 1]. Thus, $\widetilde{L}_{\widehat{A}}^*(1)$ has the unique left invariant nonnegative vector. From (5.5), we also have

$$\Psi_{\widehat{A}}(n) = \alpha^n \sum_{\ell=n+1}^{\infty} \Delta_{\boldsymbol{h}}^{-1}A(\ell)G^{\ell-n-1}\Delta_{\boldsymbol{h}} = \alpha^n\Delta_{\boldsymbol{h}}^{-1}\Psi_A(n)\Delta_{\boldsymbol{h}}.$$

Hence, (5.4) leads to

$$\widetilde{L}_{\widehat{A}}^*(z) = \Delta_{\boldsymbol{\mu}}^{-1}\big(\Psi_A^*(\alpha z)\big)^{\mathrm{T}}\Delta_{\boldsymbol{\mu}}, \tag{5.7}$$

and $\widetilde{L}_{\widehat{A}}^*(1)$ has the left invariant vector $\boldsymbol{\eta} \equiv \boldsymbol{\ell}^{\mathrm{T}}\Delta_{\boldsymbol{\mu}}$, so $\boldsymbol{\ell}$ must be nonnegative.

Remark 5.2. We have used [2, theorem 1] to prove that $\{\widetilde{L}_{\widehat{A}}(\ell)\}$ is Harris ergodic. We can also directly proved it using the following facts. First, note that

$$\widetilde{L}^*_{\widehat{A}}(1)\mathbf{e} = \Delta_{\boldsymbol{\mu}}^{-1}\big(\Psi^*_A(\alpha)\big)^{\mathrm{T}}\boldsymbol{\mu}^{\mathrm{T}} = \mathbf{e},$$

which implies that $\widetilde{L}^*_{\widehat{A}}(1)$ is stochastic. Since $\widehat{G}$ is substochastic and $\boldsymbol{\ell} \neq \mathbf{0}$, we have

$$0 < |\boldsymbol{\eta}|\mathbf{e} \leqslant \alpha\boldsymbol{\mu}\boldsymbol{h} + \boldsymbol{\mu}G\Delta_{\boldsymbol{h}}\mathbf{e} \leqslant 2\alpha\boldsymbol{\mu}\boldsymbol{h} < \infty,$$

where $|\boldsymbol{\eta}|$ stands for the vector whose ith entry is the absolute value of the i-entry of $\boldsymbol{\eta}$, so $|\boldsymbol{\eta}|$ represents a finite measure. Then, using the dominated convergence theorem,

$$|\boldsymbol{\eta}| \leqslant |\boldsymbol{\eta}|\widetilde{L}^*_{\widehat{A}}(1) \leqslant \cdots \leqslant |\boldsymbol{\eta}|\big(\widetilde{L}^*_{\widehat{A}}(1)\big)^n, \quad n \geqslant 1,$$

implies that $\widetilde{L}^*_{\widehat{A}}(1)$ must have a stationary distribution. Since $\widehat{A}$ is irreducible, it is not hard to see that $\widetilde{L}^*_{\widehat{A}}(1)$ has a single positive recurrent class, so it is Harris ergodic.

Let $\Phi^*_B(z) = \sum_{n=0}^{\infty}\Phi_B(n)z^n$ for $z > 1$, then, similar to (5.1), we have

$$\Phi^*_B(z) = z\big(B^*(z) - I\big)(zI - G)^{-1} + I. \tag{5.8}$$

We arrive at a main result of this section.

Theorem 5.1. Suppose conditions (i), (ii) and (iii) and that there exist $\alpha > 1$ and positive vectors $\boldsymbol{h}$ and $\boldsymbol{\mu}$ satisfying (5.2) and (5.3). Let $\{\boldsymbol{x}(n)\}$ be the stationary measure of P. If

$$\boldsymbol{x}(0)B^*(\alpha)\boldsymbol{h} < \infty, \tag{5.9}$$

then we have

$$\lim_{n\to\infty}\alpha^n\boldsymbol{x}(n) = \frac{\boldsymbol{x}(0)(B^*(\alpha) - I)\boldsymbol{h}}{\boldsymbol{\mu}(A^*)'(\alpha)\boldsymbol{h}}\boldsymbol{\mu}, \tag{5.10}$$

so $\{\boldsymbol{x}(n)\}$ can be normalized as a probability measure, where $(A^*)'(\alpha) = \mathrm{d}/\mathrm{d}z A^*(z)|_{z=\alpha}$, and $\boldsymbol{\mu}(A^*)'(\alpha)\boldsymbol{h}$ is positive but may be infinite. In the latter case, the right-hand side of (5.10) is considered as zero. Otherwise, if $\boldsymbol{x}(0)B^*(\alpha) = \infty$, then

$$\boldsymbol{x}(n) = \boldsymbol{x}(0)\Phi_B * U(n) + \mathrm{o}\big(\boldsymbol{x}(0)\Phi_B * U(n)\big), \quad n \to \infty, \tag{5.11}$$

where $\mathrm{o}(\boldsymbol{f}(n))$ is the componentwise small order for vector $\boldsymbol{f}(n)$.

Proof. Let $\widehat{\boldsymbol{x}}_{\boldsymbol{\mu}}(n) = \alpha^n\Delta_{\boldsymbol{\mu}}^{-1}\boldsymbol{x}(n)^{\mathrm{T}}$. Then, premultiplying (3.2) by $\alpha^n\Delta_{\boldsymbol{\mu}}^{-1}$ yields

$$\hat{\boldsymbol{x}}_{\boldsymbol{\mu}}(n) = \alpha^n\Delta_{\boldsymbol{\mu}}^{-1}\big(\Phi_B(n) - \Psi_A(n)\big)^{\mathrm{T}}\boldsymbol{x}(0)^{\mathrm{T}} + \widetilde{L}_{\widehat{A}} * \hat{\boldsymbol{x}}_{\boldsymbol{\mu}}(n), \quad n \geqslant 0.$$

Since the Markov renewal kernel $\{\widetilde{L}_{\widehat{A}}(n)\}$ is aperiodic and Harris recurrent, we can apply the Markov renewal theorem (e.g., see [7, theorem 4.17]), and we get

$$\lim_{n\to\infty}\hat{\boldsymbol{x}}_{\boldsymbol{\mu}}(n) = \frac{1}{\boldsymbol{\eta}(\widetilde{L}^*_{\widehat{A}})'(1)\mathbf{e}}\mathbf{e}\boldsymbol{\ell}^{\mathrm{T}}\big(\Phi^*_B(\alpha) - \Psi^*_A(\alpha)\big)^{\mathrm{T}}\boldsymbol{x}(0)^{\mathrm{T}},$$

if $\boldsymbol{x}(0)(\Phi_B^*(\alpha)-\Psi_A^*(\alpha))\boldsymbol{\ell}$ is finite. We show that (5.9) implies this. From (5.1) and (5.8), we have

$$\begin{aligned}\big(\Phi_B^*(\alpha) - \Psi_A^*(\alpha)\big)\boldsymbol{\ell} &= \big(\alpha B^*(\alpha) - A^*(\alpha)\big)(\alpha I - G)^{-1}\boldsymbol{\ell}\\ &= \big(\alpha B^*(\alpha) - A^*(\alpha)\big)\boldsymbol{h}\\ &= \alpha\big(B^*(\alpha) - I\big)\boldsymbol{h},\end{aligned}$$

where the second equality is obtained by $\boldsymbol{\ell} = (\alpha I - G)\boldsymbol{h}$. Since $\boldsymbol{x}(0)(\Phi_B^*(\alpha) - \Psi_A^*(\alpha))\boldsymbol{\ell}$ must be nonnegative, we get its finiteness by (5.9). From (5.1), we have

$$\big(\Psi_A^*(z) - I\big)(zI - G) = A^*(z) - zI.$$

Differentiate this by $z < \alpha$, then premultiply the resulted formula by $\boldsymbol{\mu}$ and postmultiplying it by $\boldsymbol{h}$. Letting $z \uparrow \alpha$ in this formula yields

$$\boldsymbol{\mu}\big(\Psi_A^*\big)'(\alpha)\boldsymbol{\ell} = \boldsymbol{\mu}\big(A^*\big)'(\alpha)\boldsymbol{h}.$$

Since (5.7) implies

$$\boldsymbol{\eta}\big(\widetilde{L}_{\widehat{A}}^*)'(1)\mathbf{e} = \alpha\boldsymbol{\mu}\big(\Psi_A^*\big)'(\alpha)\boldsymbol{\ell},$$

we arrive at (5.10). Finally, (5.11) is immediate from (3.10). □

Remark 5.3. Theorem 5.1 generalizes [17, theorem 1], which considers the case that $A(n) = 0$ for $n \geqslant 3$ and $B(n) = 0$ for $n \geqslant 2$. Theorem 5.1 also weakens the conditions of theorem 1 in such a way that the first part of condition (9) and condition (10) there can be removed.

Remark 5.4. If S is finite, the first part of theorem 5.1 is obtained for the tail probabilities in [18], in which kernel $\{\widetilde{\Gamma}(n)\}$ is used. In [18], the aperiodic conditions are given for $\{\widetilde{\Gamma}(n)\}$, and the periodic case is also considered (see [18, theorem 2]). It is not hard to see that theorem 5.1 can be similarly extended for the periodic case using [1, theorem 2.1].

For the completeness of our discussions, we consider other possibilities that condition (5.2) and (5.3) may not be satisfied. In what follows, we assume (5.9), and refer to the surprimum of $z > 1$ such that $\limsup_{n\to\infty} z^n f(n) < \infty$ as the weak geometric decay rate of a function $f \geqslant 0$.

We first closely look at (3.2) and our arguments before theorem 5.1. Then, we can see that the convergence radii of

$$g_{ij}(s,z) = \sum_{n=0}^{\infty} s^n\big[\big(\Psi_A^*(z)\big)^n\big]_{ij}$$

with respect to s play a key role. For simplicity, suppose that $\Psi_A^*(1)$ is irreducible. Then, $g_{ij}(s,z)$ have a common convergence radius for each fixed $z > 0$ (see [15, chapter 6]). Let $\alpha = \sup\{z; g_{ij}(1,z) < \infty\}$, and consider the following three cases. Note that $g(1,\alpha) = +\infty$ implies $\alpha > 1$.

Case 1. $g(1, \alpha) = \infty$ and $[\Psi_A^*(\alpha)]_{ij} < \infty$ for all $i, j \in S$. This case is called 1-recurrence, and $\Psi_A^*(\alpha)$ always have positive left and right invariant vectors (see, e.g., [15]). Hence, (5.10) is obtained. Note that this case is implied by (5.2) and (5.3). On the other hand, these conditions hold in this case if $\alpha^n \boldsymbol{x}(n)$ converges to a positive constant as n goes infinity, since the convergence implies $\alpha \boldsymbol{\mu}(A^*)'(\alpha)\boldsymbol{h} < \infty$.

Case 2. $g(1, \alpha) = \infty$ and $[\Psi_A^*(\alpha)]_{ij} = \infty$ for some $i, j \in S$. In this case, $(\Psi_A^*(\alpha))^k = \infty$ for sufficiently large k by the irreducibility. Note that (3.14) and (3.15) hold if $\widetilde{L}(n)$ and $\tilde{\boldsymbol{x}}(n)$ are replaced by $\Psi_A(n)^{\mathrm{T}}$ and $\boldsymbol{x}^{\mathrm{T}}(n)$, respectively, where $\widetilde{\Gamma}(n)$ is replaced by $(I - \Psi_A(0)^{\mathrm{T}})^{-1}\Psi_A(n)^{\mathrm{T}}$. Hence, from (3.15), it can be seen that

$$\sum_{n=1}^{\infty} \alpha^n \overline{\boldsymbol{x}}(n)^{\mathrm{T}} = \infty.$$

On the other hand, for any positive $\beta < \alpha$, $g(1, \beta) < \infty$. So, $(I - \widetilde{\Psi}_A^*(\beta))^{-1}$ exists. From this and the corresponding formula to (3.15), we can see

$$\sum_{n=1}^{\infty} \beta^n \overline{\boldsymbol{x}}(n)^{\mathrm{T}} < \infty.$$

Hence, the α is the weak geometric decay rate.

Case 3. $g(1, \alpha) < \infty$. In this case, the similar arguments as above show that $\alpha^n \boldsymbol{x}(n)$ converges to zero vector as n goes to infinity. On the other hand, for any $\beta > \alpha$, $[\Psi_A^*(\beta)]_{ij} = \infty$ for some $i, j \in S$. Hence, the arguments in case 2 shows that $\sum_{n=1}^{\infty} \beta^n \overline{\boldsymbol{x}}(n)^{\mathrm{T}} = \infty$. Thus, if $\alpha > 1$, then the α is again the weak geometric decay rate. However, if $\alpha = 1$, we do not have any geometric decay rate.

As we noted, conditions (5.2) and (5.3) characterize the geometric decay of $\boldsymbol{x}(n)$ in the strict sense, i.e., in the sense that $\alpha^n \boldsymbol{x}(n)$ converges to a positive constant for some α. However, we have assumed that A is irreducible and aperiodic. This may not be true in general even if P is positive recurrent. So, conditions (5.2) and (5.3) may not be required for the strictly geometric decay if A is not irreducible or periodic.

It may be also interesting to consider the cases when the background process has a more general state space and when the additive component is continuous. Similar results can be expected, but there would be many technical issues to overcome to get them. Another important issue is how to find the α and the corresponding eigenvectors of $A^*(\alpha)$ in each application. At present, we only have a few examples (e.g., see [17]). Those issues would be challenging problems.

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