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ELSEVIER Computers and Mathematics with Applications 51 (2006) 199–208
An International Journal
computers & mathematics with applications
www.elsevier.com/locate/camwa

A Fractal Poisson Process and Its Input Queue

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Abstract—An interrupted Poisson process has two states, on-state and off-state. In the on-state, Poisson arrivals occur, and on the other hand, there are no arrivals in the off-state. As the variation becomes larger, the arrival process is changed to a more complex interrupted Poisson process generated by embedding new off-states in each on-state. In such a way, recursively embedding off-states in on-states and taking a limit, a fractal structure can be found in the on-off structure. We name the arrival process fractal Poisson process and study it. The interarrival time density has a heavy tail. In addition, we study queueing models with the fractal Poisson arrivals. Even if the utilization is very low, the waiting time is very long.

Keywords—Variation, n^{th} generation interrupted Poisson process, GI/M/1 queue, Fractal Poisson process.

1. INTRODUCTION

The interrupted Poisson process (IPP) was introduced as an overflow process in circuit switch networks by Kuczura [1]. The interarrival time of the IPP is more variable [2] than the one for a Poisson process for the fixed mean interarrival time. In packet communication networks, we can also find the packet streams which have more variable one than Poisson one. The packet streams which originate from terminal equipments with some regularity become ones that have some variation due to various factors as going forward in the networks. It is well known that the regular streams are transformed into the Poisson streams and then to the IPP streams.

In the interrupted Poisson process(IPP) [1], on-state periods and off-state periods alternately occur. Being an on-state, Poisson arrivals occur with some rate. On the other hand, there are no arrivals during any off-state period. Poisson arrivals which are uniformly distributed on time axis in a Poisson process are driven into on-state periods which have finite intervals. The IPP streams have larger variation than Poisson streams. What is the transformation mechanism from the Poisson streams to the IPP streams? We can consider that this mechanism occurs because off-state periods preempt over a Poisson process with some rate. If the preemption rate is constant on the time axis, then an on-state period length follows an exponential distribution. In addition, if the off-state period length also follows an exponential one, then the on-off process is an original IPP proposed by Kuczura [1]. We call this IPP "the first generation IPP". The Poisson process considered here would be called "the 0^{th} generation IPP". Note that the longer off-state period

0898-1221/06/$ - see front matter Typeset by $\mathcal{A}_\mathcal{M}\mathcal{S}$-TEX
doi:10.1016/j.camwa.2005.11.023

implies the larger variation. As a result, the Poisson arrival rate during on-states increases. We know that, as a packet stream goes forward in the network, the variation increases. We can expect that the new generation IPP will appear.

It is natural to consider that the same mechanism, in which off-state periods preempt over a Poisson process, occurs in on-state periods themselves of the first generation IPP. That is, new off-state periods preempt over the on-state. The mean length of an off-state period newly generated, of course, is shorter than the mean off-state period length of the first generation IPP. Suppose that the latter is M times larger for some integer M than the former. We call the process "second generation IPP". In this IPP, a period which corresponds to an on-state period of the first generation IPP has an on-off structure. If we magnify the period with this on-off structure M times, then we find the first generation IPP. When the same mechanism occurs during an on-state period of the second generation IPP, the third generation IPP will appear. If this mechanism is recursively repeated n times, then we will be able to find the n^{th} generation IPP. It should be noted that the interarrival time of the $(n+1)^{\text{th}}$ generation IPP is more variable than the one of the n^{th} generation IPP. Taking a limit of n to infinity, the on-off structure has self-similarity. We call the process "*fractal Poisson process*". Under some condition, the interarrival times follow the distribution with a heavy tail. Taqqu *et al.* [3] study an on-off process in which both on-period length and off-period length have heavy tails. Our fractal Poisson process is renewable and it is convenient to deal with.

In Section 3, we study a GI/M/1 queue with the n^{th} generation IPP input. In the GI/M/1 theory, the generalized utilization [4] is the most important factor. That generalized utilization β_n of the n^{th} generation input queue becomes larger, as n increases, and finally converges to 1.

2. n^{th} GENERATION IPP AND FRACTAL POISSON PROCESS AS ITS LIMIT

Consider an interrupted Poisson process (IPP). The IPP has two states, on-state and off-state, in which the two state periods alternately occur. During an on-state period, homogeneous Poisson arrivals with rate λ_1 occur and no arrivals occur during an off-state period. The on-state period length and off-state period length follow exponential distributions with means γ^{-1} and ω_1^{-1}, respectively. We call this IPP "first generation IPP". We consider that the IPP is a process in which off-states preempt with rate γ over a Poisson process with the rate λ. Since we fix the mean interarrival times for the Poisson process and IPP, the arrival rate λ_1 is greater than the rate λ. That is, when the Poisson process is transformed to the IPP, the variations of interarrival times become larger. The off-state period length density function and on-state one are, respectively, given by

$$a_1(x,\omega_1) = \omega_1 e^{-\omega_1 x} \tag{1}$$

and

$$b_1(x) = \gamma e^{-\gamma x}. \tag{2}$$

Furthermore, when the variation becomes larger, what happens? The off-state period length will become larger from ω_1^{-1} to ω_2^{-1}. At the same time, new preemptions of off-states occur in each on-state period of the first generation IPP. We suppose that the rate of preemptions is $M\gamma$ and the mean new off-state period length is $(M\omega_2)^{-1}$. Then, the off-state and on-state period length density functions are, respectively, given by

$$a_2(x,\omega_2) = k_1 a_1(x,\omega_2) + k_2 M\omega_2 e^{-M\omega_2 x}, \qquad k_1 + k_2 = 1, \tag{3}$$

and

$$b_2(x) = M\gamma e^{-M\gamma x}. \tag{4}$$

We call this IPP "second generation IPP". If we magnify, M times, an on-off state period of the second generation IPP embedded in an on-state period of the first generation IPP, we can find the first generation IPP itself there. Equation (3) shows that an off-state period of the first generation IPP appears with probability k_1 and a new off-state period with the mean length $(M\omega_2)^{-1}$ appears with probability k_2. Hereafter, we set $M = 2$ and $k_1 = k_2 = 1/2$ for simplicity.

We apply the same mechanism to the second generation IPP. Then, we have

$$a_3(x, \omega_3) = \frac{1}{2}\left\{a_2(x, \omega_3) + 2^2\omega_3 e^{-2^2\omega_3 x}\right\} \tag{5}$$

and

$$b_3(x) = 2^2\gamma e^{-2^2\gamma x}. \tag{6}$$

If we recursively repeat the same mechanism, we have the following n^{th} generation IPP:

$$a_n(x, \omega_n) = \frac{1}{2}\left\{a_{n-1}(x, \omega_n) + 2^{n-1}\omega_n e^{-2^{n-1}\omega_n x}\right\} \tag{7}$$

and

$$b_n(x) = 2^{n-1}\gamma e^{-2^{n-1}\gamma x}. \tag{8}$$

Taking a limit to ∞ for n, we have a fractal Poisson process.

THEOREM 1. *An off-state period length density function for the n^{th} generation IPP is given by*

$$a_n(x, \omega_n) = \sum_{i=1}^{n-1}\left(\frac{1}{2}\right)^{n-i} 2^i\omega_n e^{-2^i\omega_n x} + \left(\frac{1}{2}\right)^{n-1}\omega_n e^{-\omega_n x}. \tag{9}$$

PROOF. When $n = 1$, we have

$$a_1(x, \omega_1) = \omega_1 e^{-\omega_1 x}. \tag{10}$$

When $n = k$, we assume that

$$a_k(x, \omega_k) = \sum_{i=1}^{k-1}\left(\frac{1}{2}\right)^{k-i} 2^i\omega_k e^{-2^i\omega_k x} + \left(\frac{1}{2}\right)^{k-1}\omega_k e^{-\omega_k x}. \tag{11}$$

Then, we have

$$a_{k+1}(x, \omega_{k+1}) = \frac{1}{2}\left(a_k(x, \omega_{k+1}) + 2^k\omega_{k+1} e^{-2^k\omega_{k+1} x}\right). \tag{12}$$

Substituting (11), we have

$$a_{k+1}(x, \omega_{k+1}) = \sum_{i=1}^{k}\left(\frac{1}{2}\right)^{k+1-i} 2^i\omega_{k+1} e^{-2^i\omega_{k+1} x} + \left(\frac{1}{2}\right)^{k}\omega_{k+1} e^{-\omega_{k+1} x}. \tag{13}$$ ∎

Since

$$\sum_{i=1}^{n-1}\left(\frac{1}{2}\right)^{n-i} + \left(\frac{1}{2}\right)^{n-1} = 1 \tag{14}$$

in (9), $a_n(x, \omega_n)$ is a hyperexponential density function.

Now, we fix the mean interarrival time for the n^{th} generation IPP $(n = 0, 1, \dots)$. In particular, when $n = 0$, the process is Poisson with the rate λ. If λ_n is the arrival rate in an on-state period for the n^{th} generation IPP, then we have

$$\frac{\lambda_n E[b_n]}{E[a_n] + E[b_n]} = \lambda, \tag{15}$$

where $E[a_n]$ and $E[b_n]$ are the mean off-state period length and the mean on-state length, respectively. Then, we have

$$E[a_n] = \frac{1}{\omega_n}\left(\frac{1}{2}\right)^{n-1}\frac{n+1}{2} \tag{16}$$

and

$$E[b_n] = \frac{1}{2^{n-1}\gamma}. \tag{17}$$

Equation (16) can be obtained as

$$\begin{aligned} E[a_n] &= \left(\frac{1}{2}\right)^n \sum_{i=1}^{n-1} \frac{2^i}{2^i \omega_n} + \left(\frac{1}{2}\right)^{n-1} \frac{1}{\omega_n} \\ &= \frac{1}{\omega_n}\left\{\left(\frac{1}{2}\right)^n (n-1) + \left(\frac{1}{2}\right)^{n-1}\right\} \\ &= \frac{1}{\omega_n}\left(\frac{1}{2}\right)^{n-1}\frac{n+1}{2}. \end{aligned}$$

Equation (17) is obvious.

Hence, from

$$\frac{\lambda_n/2^{n-1}\gamma}{(1/\omega_n)(1/2)^{n-1}((n+1)/2) + 1/2^{n-1}\gamma} = \frac{\lambda_n}{1+(n+1)\gamma/2\omega_n}, \tag{18}$$

we have

$$\lambda_n = \lambda\left(1 + \frac{(n+1)\gamma}{2\omega_n}\right). \tag{19}$$

Now, we will study an interarrival time probability density function $f_n(x, \omega_n)$ of the n^{th} generation IPP. The Laplace transform (LT) $f_n^*(s, \omega_n)$ of $f_n(x, \omega_n)$ is given by

$$f_n^*(s, \omega_n) = \frac{\lambda_n}{s + 2^{n-1}\gamma + \lambda_n} + \frac{2^{n-1}\gamma}{s + 2^{n-1}\gamma + \lambda_n} a_n^*(s, \omega_n) f_n^*(s, \omega_n), \tag{20}$$

where $a_n^*(s, \omega_n)$ is the LT of $a_n(x, \omega_n)$.

We can assure that the mean interarrival time of the n^{th} generation IPP is equal to λ^{-1}. That is, since

$$\begin{aligned} -\frac{d}{ds} f_n^*(s, \omega_n)\bigg|_{s=0+} &= \frac{1}{2^{n-1}\gamma + \lambda_n} \\ &+ \frac{2^{n-1}\gamma}{2^{n-1}\gamma + \lambda_n}\left(-\frac{d}{ds} a_n^*(s, \omega_n)\bigg|_{s=0+} - \frac{d}{ds} f_n^*(s, \omega_n)\bigg|_{s=0+}\right), \end{aligned} \tag{21}$$

we have from (16)

$$-\left(1 - \frac{2^{n-1}\gamma}{2^{n-1}\gamma + \lambda_n}\right)\frac{d}{ds} f_n^*(s, \omega_n)\bigg|_{s=0+} = \frac{1}{2^{n-1}\gamma + \lambda_n}\left(1 + \frac{2^{n-1}\gamma}{\omega_n}\left(\frac{1}{2}\right)^{n-1}\frac{n+1}{2}\right). \tag{22}$$

Therefore, from (19) we obtain

$$-\frac{d}{ds} f_n^*(s, \omega_n)\bigg|_{s=0+} = \frac{1}{\lambda_n}\left(1 + \frac{(n+1)\gamma}{2\omega_n}\right) = \frac{1}{\lambda}. \tag{23}$$

The variance of interarrival times

$$V_n = \frac{d^2}{ds^2} f_n^*(s,\omega_n)\bigg|_{s=0+} - \left(\frac{1}{\lambda}\right)^2 \tag{24}$$

can be obtained by the following manner: differentiating (20) twice and substituting $s = 0+$, we have

$$\begin{aligned} \frac{d^2}{ds^2} f_n^*(s,\omega_n)\bigg|_{s=0+} &= \frac{2}{(2^{n-1}\gamma+\lambda_n)^2} + 2\frac{2^{n-1}\gamma}{(2^{n-1}\gamma+\lambda_n)^2}\left(E(a_n)+\frac{1}{\lambda}\right) \\ &+ \frac{2^{n-1}\gamma}{2^{n-1}\gamma+\lambda_n}\left(\frac{d^2}{ds^2}a_n(s)\bigg|_{s=0+} + \frac{2}{\lambda}E(a_n) + \frac{d^2}{ds^2} f_n^*(s,\omega_n)\bigg|_{s=0+}\right). \end{aligned} \tag{25}$$

Since

$$\begin{aligned} \frac{d^2}{ds^2}a_n(s)\bigg|_{s=0+} &= \sum_{i=1}^{n-1}\left(\frac{1}{2}\right)^{n-i}\frac{1}{(2^i\omega_n)^2} + \left(\frac{1}{2}\right)^{n-1}\frac{1}{\omega_n^2} \\ &= \left(\frac{1}{2}\right)^n \frac{1}{\omega_n^2}, \end{aligned} \tag{26}$$

it is satisfied that

$$\begin{aligned} \left(1-\frac{2^{n-1}\gamma}{2^{n-1}\gamma+\lambda_n}\right)\frac{d^2}{ds^2} f_n^*(s,\omega_n)\bigg|_{s=0+} &= \frac{2}{(2^{n-1}\gamma+\lambda_n)^2}\left[1+2^{n-1}\gamma\left(E(a_n)+\frac{1}{\lambda}\right)\right] \\ &+ \frac{2^{n-1}\gamma}{2^{n-1}\gamma+\lambda_n}\left[\left(\frac{1}{2}\right)^n\frac{1}{\omega_n^2}+\frac{2}{\lambda}E(a_n)\right], \end{aligned} \tag{27}$$

where $E(a_n)$ is given in (16). Now, we have

$$\begin{aligned} \frac{d^2}{ds^2} f_n^*(s,\omega_n)\bigg|_{s=0+} &= \frac{2}{\lambda_n(2^{n-1}\gamma+\lambda_n)}\left[1+2^{n-1}\gamma\left(E(a_n)+\frac{1}{\lambda}\right)\right] \\ &+ \frac{2^{n-1}\gamma}{\lambda_n}\left[\left(\frac{1}{2}\right)^n\frac{1}{\omega_n^2}+\frac{2}{\lambda}E(a_n)\right]. \end{aligned} \tag{28}$$

In (28), we have for some constant c

$$\lim_{n\to\infty}\frac{2^{n-1}E(a_n)}{\lambda_n} = c < \infty. \tag{29}$$

In the second term of the right-hand-side of (28), we have

$$\frac{2^{n-1}\gamma}{\lambda_n}\left(\frac{1}{2}\right)^n\frac{1}{\omega_n^2} = \frac{\gamma}{2\lambda_n\omega_n^2}. \tag{30}$$

Since, in (19),

$$\lambda_n = \lambda\left(1+\frac{(n+1)\gamma}{2\omega_n}\right)$$

is satisfied, we obtain

$$\lambda_n\omega_n^2 = \lambda\left(\omega_n^2+\frac{n+1}{2}\gamma\omega_n\right). \tag{31}$$

Now, we get an important result. That is, if the sequence $\{\omega_n,\, n=0,1,\dots\}$ satisfies that

$$\lim_{n\to\infty}(n+1)\omega_n = 0,$$

for example, if, for $\varepsilon > 0$ and $\omega > 0$,

$$\omega_n = \frac{\omega}{(1+\varepsilon)^n},$$

then

$$\lim_{n\to\infty} \lambda_n \omega_n^2 = 0. \tag{32}$$

That means that

$$\lim_{n\to\infty} \frac{d^2}{ds^2} f_n^*(s,\omega_n)\bigg|_{s=0+} = \infty \tag{33}$$

or the variance V_∞ is infinite. The interarrival time distribution has a heavy tail.

We will next consider the interarrival time density functions themselves. From (20), it is satisfied that

$$\begin{aligned} f_n^*(s,\omega_n) &= \frac{\lambda_n/(s+2^{n-1}\gamma+\lambda_n)}{1-(2^{n-1}\gamma/(s+2^{n-1}\gamma+\lambda_n))\, a_n^*(s,\omega_n)} \\ &= \frac{\lambda_n(s+\omega_n)x_n(s)}{(s+2^{n-1}\gamma+\lambda_n)(s+\omega_n)x_n(s)-2^{n-1}\gamma y_n(s)}. \end{aligned} \tag{34}$$

Here,

$$x_n(s) = \prod_{i=1}^{n-1}\left(s+2^i\omega_n\right) \tag{35}$$

and

$$y_n(s) = (s+\omega_n)\sum_{i=1}^{n-1}\left(\frac{1}{2}\right)^{n-i} 2^i\omega_n \frac{x_n(s)}{s+2^i\omega_n} + \left(\frac{1}{2}\right)^{n-1}\omega_n x_n(s). \tag{36}$$

THEOREM 2. *Spectra $-\alpha_i^{(n)}$, $i=1,\dots,n+1$ of $f_n^*(s,\omega_n)$, that is, roots of the $(n+1)^{th}$ polynomial equation*

$$z_n(s) = \left(s+2^{n-1}\gamma+\lambda_n\right)(s+\omega_n)x_n(s) - 2^{n-1}\gamma y_n(s) = 0$$

are real and simple, and each of them is in the following interval:

$$\begin{aligned} &-\alpha_1^{(n)} \in (-\omega_n, 0), \\ &-\alpha_i^{(n)} \in \left(-2^{i-1}\omega_n, -2^{i-2}\omega_n\right), \qquad i=2,\dots,n, \\ &-\alpha_{n+1}^{(n)} \in \left(-\infty, -2^{n-1}\omega_n\right). \end{aligned}$$

PROOF. The $(n+1)^{\text{th}}$ polynomial $z_n(s)$ is continuous and the following inequalities are satisfied:

$$z_n(0) > 0$$

and

$$(-1)^i z_n\left(-2^{i-1}\omega_n\right) > 0, \qquad i=1,\dots,n.$$

Then, $z_n(s)=0$ has one root at least in each of $n+1$ intervals $(-\infty,-2^{n-1}\omega_n), (-2^{n-1}\omega_n, -2^{n-2}\omega_n), (-2^{n-2}\omega_n, -2^{n-3}\omega_n), \dots, (-4\omega_n, -2\omega_n), (-2\omega_n, -\omega_n)$, and $(-\omega_n, 0)$. Since $z_n(s)=0$ has $n+1$ roots, so there exists just one root in each interval. ∎

Theorem 2 mentions that $f_n^*(s,\omega_n)$ can be written as

$$f_n^*(s,\omega_n) = \frac{\lambda_n(s+\omega_n)x_n(s)}{\prod_{i=1}^{n+1}\left(s+\alpha_i^{(n)}\right)}. \tag{37}$$

The Laplace inversion of (37) gives

$$f_n(x,\omega_n) = \sum_{j=1}^{n+1} \frac{\lambda_n\left(-\alpha_j^{(n)}+\omega_n\right) x_n\left(-\alpha_j^{(n)}\right)}{\prod_{\substack{i=1\\ i\neq j}}^{n+1}\left(-\alpha_j^{(n)}+\alpha_i^{(n)}\right)} e^{-\alpha_j^{(n)}x}. \tag{38}$$

Here, all the coefficients of $e^{-\alpha_j^{(n)}x}$ for $j = 1,\dots,n+1$ are positive. The probability density function $f_n(x,\omega_n)$ is hyperexponential and completely monotone.

3. A GI/M/1 QUEUE WITH n^{th} GENERATION IPP INPUTS

In this section, we study a GI/M/1 queue with n^{th} generation IPP input. Suppose that the interarrival time probability density function is given by $f_n(x,\omega_n)$ and further that the service time distribution is exponential with rate μ. For example, in packet communication networks, a terminal equipment which receives packets can be modelled as a GI/M/1 queue with n^{th} generation IPP input. Here, customers correspond to packets. Letting X_n denote the number of customers in the system as seen by the n^{th} arrival, it is easy to see that the process $\{X_n;\ n = 1,2,\dots\}$ is an imbedded Markov chain. The transition probability p_{ij} for this Markov chain is given by

$$p_{i,i+1-j} = \int_0^\infty e^{-\mu x}\frac{(\mu x)^j}{j!} f_n(x,\omega_n)\,dx, \qquad j=0,1,\dots,i, \tag{39}$$

and

$$p_{i,0} = \int_0^\infty \sum_{k=i+1}^{\infty} e^{-\mu x}\frac{(\mu x)^k}{k!} f_n(x,\omega_n)\,dx, \qquad i=0,1,2,\dots. \tag{40}$$

The stationary probabilities π_k, $k=0,1,\dots$ can be as the unique solution of

$$\pi_k = \sum_{i=0}^{\infty} \pi_i p_{ik}, \qquad k=0,1,\dots. \tag{41}$$

Equations (41) reduce to

$$\pi_k = \sum_{i=k-1}^{\infty} \pi_i \int_0^\infty e^{-\mu x}\frac{(\mu x)^{i+1-k}}{(i+1-k)!} f_n(x,\omega_n)\,dx, \qquad k=1,2,\dots, \tag{42}$$

and

$$\sum_{k=0}^{\infty} \pi_k = 1. \tag{43}$$

When the mean interarrival time is greater than the mean service time μ^{-1}, that is, $\rho < 1$, the stationary probability π_k is geometrically distributed and can be written as

$$\pi_k = (1-\beta_n)\beta_n^k \tag{44}$$

for some constant β_n. Substitution into (42) leads to

$$\begin{aligned}
\beta_n^k &= \sum_{i=k-1}^{\infty} \beta_n^i \int_0^\infty e^{-\mu x}\frac{(\mu x)^{i+1-k}}{(i+1-k)!} f_n(x,\omega_n) \\
&= \int_0^\infty e^{-\mu x}\beta_n^{k-1}\sum_{i=k-1}^{\infty}\frac{(\beta_n\mu x)^{i+1-k}}{(i+1-k)!} f_n(x,\omega_n)\,dx \\
&= \int_0^\infty e^{-\mu x}\beta_n^{k-1} e^{\beta_n \mu x} f_n(x,\omega_n)\,dx
\end{aligned} \tag{45}$$

or

$$\beta_n = \int_0^\infty e^{-\mu(1-\beta_n)x} f_n(x, \omega_n)\, dx = f_n^*(\mu(1-\beta_n), \omega_n). \tag{46}$$

If the utilization $\rho < 1$, then (46) has a unique solution in $(0, 1)$. That is because that we have

$$1 = f_n^*(0+, \omega_n),$$

$$\frac{d}{ds} f_n^*(\mu(1-s), \omega_n)|_{s=1-} = \mu\lambda^{-1} > 1,$$

$$f_n^*(\mu, \omega_n) > 0,$$

and

$$\frac{d^2}{ds^2} f_n^*(\mu(1-s), \omega_n) > 0, \qquad \text{for every } s \in (0,1).$$

The β_n is a generalized utilization. In particular, the input process is Poisson; it is satisfied that $\beta_n = \rho$. We have the following theorem for the n^{th} generation IPP input queue.

THEOREM 3. *When we let β_n denote the generalized utilization for the n^{th} generation input queue,* $\mathrm{GI}(f_n(x, \omega_n))/\mathrm{M}(\mu)/1$, $n = 0, 1, \dots$, *it is satisfied that*

$$0 < \beta_0 (= \rho) < \beta_1 < \cdots < 1. \tag{47}$$

In addition, if it can be written for $\varepsilon > 0$ and ω that

$$\omega_n = (1 + \varepsilon)^{-n} \omega,$$

then

$$\lim_{n \to \infty} \beta_n = 1. \tag{48}$$

PROOF. Let X_n $(n = 1, 2, \dots)$ denote nonnegative random variables for the distribution

$$F_n(x, \omega_n) = \int_0^x f_n(t, \omega_n)\, dt.$$

From the definition of the n^{th} generation IPP, X_{n+1} is more variable than X_n, that is,

$$\int_a^\infty (1 - F_n(x, \omega_n))\, dx < \int_a^\infty (1 - F_{n+1}(x, \omega_{n+1}))\, dx,$$

for all $a > 0$. In addition, the means are fixed, that is,

$$\lambda^{-1} = E(X_n) = \int_0^\infty (1 - F_n(x, \omega_n))\, dx = \int_0^\infty (1 - F_{n+1}(x, \omega_{n+1}))\, dx = E(X_{n+1}).$$

Then, for all convex function h, we have [2]

$$E[h(X_n)] < E[h(X_{n+1})].$$

Since e^{-sx} $(0 \le s < \infty)$ is a convex function, we have

$$E\left(e^{-sX_n}\right) < E\left(e^{-sX_{n+1}}\right).$$

That means that

$$f_n^*(\mu(1-s), \omega_n) < f_{n+1}^*(\mu(1-s), \omega_{n+1}),$$

for all $s \in (0,1)$. Thus, we have

$$\beta_n < \beta_{n+1}, \qquad n = 0, 1, \dots.$$

Now, we assume that

$$\lim_{n\to\infty} \beta_n = \beta < 1.$$

Since in (19) we have

$$\lambda_n = \lambda\left(1 + \frac{(n+1)\gamma}{2\omega_n}\right),$$

it follows that when $\varepsilon < 1$,

$$\lim_{n\to\infty} \frac{\lambda_n}{2^{n-1}\gamma} = \lim_{n\to\infty} \lambda\left(\frac{1}{2^{n-1}\gamma} + \frac{n+1}{2^{n-1}\gamma}\frac{(1+\varepsilon)^n}{\omega}\right) = 0.$$

From (20), we have

$$f_n^*(s,\omega_n) = \frac{\lambda_n/2^{n-1}\gamma}{s/2^{n-1}\gamma + 1 + \lambda_n/2^{n-1}\gamma} + \frac{1}{s/2^{n-1}\gamma + 1 + \lambda_n/2^{n-1}\gamma} a_n^*(s,\omega_n) f_n^*(s,\omega_n).$$

Taking a limit to ∞ for n, we have

$$f_\infty^*(s,\omega_\infty) = a_\infty^*(s,\omega_\infty) f_\infty^*(s,\omega_\infty).$$

Setting $s = \mu(1-\beta)$, we have

$$\beta = a_\infty^*(\mu(1-\beta),\omega_\infty)\beta.$$

This means that

$$a_\infty^*(\mu(1-\beta),\omega_\infty) = 1.$$

This is contradictory, because the interarrival time is stochastically larger than the off-period length, that is,

$$1 \geq f_\infty^*(s,\omega_\infty) > a_\infty^*(s,\omega_\infty),$$

for any $s \geq 0$.

On the other hand, when $\varepsilon \geq 1$, we have in (20)

$$f_n^*(s,\omega_n) = \frac{1}{s/\lambda_n + 2^{n-1}\gamma/\lambda_n + 1} + \frac{2^{n-1}\gamma/\lambda_n}{s/\lambda_n + 2^{n-1}\gamma/\lambda_n + 1} a_n^*(s,\omega_n) f_n^*(s,\omega_n).$$

Taking a limit to ∞ for n,

$$f_\infty^*(s,\omega_\infty) = 1.$$

This contradicts $\beta < 1$. ∎

Theorem 3 mentions that, as the generation n changes, the waiting time becomes larger stochastically [2]. Even if the utilization ρ is low enough, the waiting time is tremendously large for sufficiently large n of the generation. In this situation, suppose that we get a new machine which can deal with sent packets at the speed twice compared with an older one. However, the waiting times still remain long. It is meaningless to replace the old machine with the new one. Our fractal Poisson process offers a serious problem on performance.

4. CONCLUSIONS

We have studied n^{th} generation interrupted Poisson process. The larger n implies the larger variation of interarrival times. Taking a limit of n to ∞, the interarrival time distribution has a heavy tail. We have named a fractal Poisson process for it. We have studied a GI/M/1 queue with the fractal Poisson process input and have shown that the waiting time diverges.

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