A DOUBLE-ENDED QUEUEING SYSTEM WITH LIMITED WAITING SPACE

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The double-ended queue involving taxis and customers at a taxi-stand has been considered under the assumption that there is limited waiting space both for taxis and for customers, the arrivals of taxis and customers being general and Poisson respectively. Using the supplementary variable technique (Syski 1960), an expression for the Laplace transform of the generating function of the state probabilities is obtained and specialized for the cases of k-Erlang and Poisson arrivals of taxis. In the last-mentioned case the p.g.f. corresponds to an earlier result of the author (Kashyap 1965).

1. Introduction

Kendall (1951) discussed the double-ended queueing problem in which customers and taxis arrive at a taxi-stand in Poisson streams with constant mean rates λ and μ. No limit was placed on the number of customers or taxis that can form a queue at the stand. Dobbie (1961) studied the problem taking λ and μ as functions of time, again with no limitation on the number of customers or taxis. Jain (1962) has studied the problem with limited waiting space for taxis so that not more than N taxis can wait at the stand, the arrivals of taxis and customers being general and Poisson respectively. Recently the author (Kashyap 1965) has considered the double-ended queue with limited waiting space for both customers and taxis and with Poisson arrivals for both. In the present paper the problem is studied with finite waiting space for both customers and taxis, the arrivals being Poisson and general respectively. The integrals involved are evaluated completely in terms of known functions for the case where the arrivals of taxis are according to a k-Erlang distribution, from which the case of Poisson arrivals follows as a particular case.

2. Statement of the Problem

Customers arrive at a taxi-stand in a Poisson fashion with mean rate λ and form a queue if no taxis are available. Taxis arrive at the stand, the inter-arrival time distribution being general with probability density $S(x)$, and pick up one customer each from the queue, if any, or else form a queue. Thus at any time there is either a queue of customers or one of taxis or of neither. Customers arriving in a group may be taken as one unit, assuming that a group does not consist of more persons than a taxi can accommodate.

Let there be limited waiting spaces for M customers and N taxis. If a customer arrives to find M customers already waiting, he has to leave. Similarly, a taxi arriving when N taxis are already in the queue has to go away.

Let $p_n(x, t)\, dx$ be the probability that at time t, n taxis are waiting, the time elapsed since the arrival of the last taxi lying in the interval $(x, x+dx)$. n can take integral values between, and including, $-M$ and N.

(i) If $n > 0$, it denotes the number of taxis waiting.
(ii) If $n = 0$, neither taxis nor customers are waiting.
(iii) If $n < 0$, its numerical value gives the number of customers waiting.

3. Formulation of Equations

Following Keilson and Kooharian (1960), we have the following transition equations:

$$p_{-M}(x+\Delta, t+\Delta) = p_{-M}(x, t)[1-\eta(x)\Delta]+p_{-M+1}(x, t)\cdot\lambda\Delta \quad .. \quad (1)$$

$$p_n(x+\Delta, t+\Delta) = p_n(x, t)[1-(\lambda+\eta(x))\Delta]+p_{n+1}(x, t).\, \lambda\Delta,$$
$$(-M+1 \leqslant n \leqslant N-1) \quad .. \quad (2)$$

$$p_N(x+\Delta, t+\Delta) = p_N(x, t)[1-(\lambda+\eta(x))\Delta], \quad .. \quad .. \quad .. \quad (3)$$

where $\eta(x)\,\Delta$ is the first order probability that a taxi arrives in time $(x, x+\Delta)$ conditioned that it had not arrived up to x, and is related to the probability density $S(x)$ of inter-arrival times by the relation

$$S(x) = \eta(x)\, \exp\left(-\int_0^x \eta(u)\, du\right). \quad .. \quad .. \quad .. \quad (4)$$

Equations (1) through (3) yield

$$\frac{\partial p_{-M}(x, t)}{\partial t}+\frac{\partial p_{-M}(x, t)}{\partial x}+\eta(x)p_{-M}(x, t) = \lambda p_{-M+1}(x, t) \quad .. \quad .. \quad (5)$$

$$\frac{\partial p_n(x, t)}{\partial t}+\frac{\partial p_n(x, t)}{\partial x}+[\lambda+\eta(x)]p_n(x, t) = \lambda p_{n+1}(x, t),$$
$$(-M+1 < n < N-1) \quad .. \quad (6)$$

$$\frac{\partial p_N(x, t)}{\partial t}+\frac{\partial p_N(x, t)}{\partial x}+[\lambda+\eta(x)]p_N(x, t) = 0. \quad .. \quad .. \quad .. \quad (7)$$

These equations are to be solved subject to the boundary conditions:

$$p_{-M}(0, t) = 0 \quad .. \quad .. \quad .. \quad .. \quad .. \quad .. \quad .. \quad .. \quad (8)$$

$$p_n(0, t) = \int_0^\infty p_{n-1}(x, t)\eta(x)dx, \qquad (-M+1 \leqslant n < N-1) \quad .. \quad (9)$$

$$p_N(0, t) = \int_0^\infty [p_{N-1}(x, t)+p_N(x, t)]\,\eta(x)\, dx, \quad .. \quad .. \quad .. \quad (10)$$

which follow as in Keilson and Kooharian (1960) by the considerations of limited waiting space given in para. 2 of section 2 above.

4. Solution of the Problem

Let us define the following generating functions:

$$f(x, \alpha, t) = \sum_{n=-M}^{N} \alpha^n p_n(x, t) \qquad \ldots \quad (11)$$

and

$$\pi(\alpha, t) = \sum_{n=-M}^{N} \alpha^n p_n(t), \qquad \ldots \quad (11a)$$

where $p_n(t)$ is the probability that n taxis are waiting at time t, irrespective of the value of x, so that

$$\pi(\alpha, t) = \int_0^\infty f(x, \alpha, t)\, dx.$$

Multiplying the equations (5) through (7) by appropriate powers of α and adding, we have

$$\frac{\partial f(x, \alpha, t)}{\partial t} + \frac{\partial f(x, \alpha, t)}{\partial x} + [\lambda + \eta(x) - \lambda/\alpha] f(x, \alpha, t) = \lambda \alpha^{-M}(1 - 1/\alpha) p_{-M}(x, t). \quad (12)$$

The boundary conditions (8) through (10) similarly yield

$$f(0, \alpha, t) = \alpha \int_0^\infty f(x, \alpha, t) \eta(x)\, dx + \alpha^N (1-\alpha) \int_0^\infty p_N(x, t) \eta(x)\, dx. \quad \ldots \quad (13)$$

Let the system start from the arrival of a taxi which makes the number of units in the system equal to i, where $-M < i < N$; then

$$p_n(x, 0) = \delta_{in} \delta(x), \qquad \ldots \quad (14)$$

where δ_{in} is the Kronecker delta and $\delta(x)$ is the Dirac delta function.

Therefore,

$$f(x, \alpha, 0) = \alpha^i \delta(x). \qquad \ldots \quad (15)$$

Let $\bar{f}(x, \alpha, s)$ denote the Laplace transform (L.T.) of $f(x, \alpha, t)$, defined by

$$\bar{f}(x, \alpha, s) = \int_0^\infty \exp(-st) f(x, \alpha, t)\, dt.$$

Similarly, let $\bar{p}_n(x, s)$ denote the L.T. of $p_n(x, t)$, $(n = -M, -M+1, \ldots, N)$.

On taking L.T.s and using (15), equation (12) gives

$$\frac{\partial \bar{f}}{\partial x} + [\lambda + s + \eta(x) - \lambda/\alpha] \bar{f} = \alpha^i \delta(x) + \lambda \alpha^{-M}(1 - 1/\alpha) \bar{p}_{-M}(x, s). \qquad \ldots \quad (16)$$

Equations (5) through (7) yield

$$\frac{\partial \bar{p}_{-M}(x, s)}{\partial x}+[s+\eta(x)]\bar{p}_{-M}(x, s) = \lambda \bar{p}_{-M+1}(x, s) \qquad \text{(17)}$$

$$\frac{\partial \bar{p}_n(x, s)}{\partial x}+[\lambda+s+\eta(x)]\bar{p}_n(x, s) = \lambda \bar{p}_{n+1}(x, s)+\delta_{in}\delta(x), \qquad \text{(18)}$$
$$(-M+1 \leqslant n \leqslant N-1)$$

$$\frac{\partial \bar{p}_N(x, s)}{\partial x}+[\lambda+s+\eta(x)]\bar{p}_N(x, s) = 0. \qquad \text{(19)}$$

And (8) through (10), and (13) give

$$\bar{p}_{-M}(0, s) = 0 \qquad \text{(20)}$$

$$\bar{p}_n(0, s) = \int_0^\infty \bar{p}_{n-1}(x, s)\eta(x)\, dx, \qquad (-M+1 \leqslant n \leqslant N-1) \qquad \text{(21)}$$

$$\bar{p}_N(0, s) = \int_0^\infty \bar{p}_{N-1}(x, s)\eta(x)\, dx+\int_0^\infty \bar{p}_N(x, s)\eta(x)\, dx \qquad \text{(22)}$$

$$\bar{f}(0, \alpha, s) = \alpha \int_0^\infty \bar{f}(x, \alpha, s)\eta(x)\, dx-\alpha^N(\alpha-1)\int_0^\infty \bar{p}_N(x, s)\eta(x)\, dx. \qquad \text{(23)}$$

Solution of (19) is

$$\bar{p}_N(x, s) = \bar{p}_N(0, s) \exp[-(\lambda+s)x] \exp\left[-\int_0^x \eta(u)\, du\right]. \qquad \text{(24)}$$

Taking $n = N-1$ in (18) and using (24), we have

$$\frac{\partial \bar{p}_{N-1}(x, s)}{\partial x}+[\lambda+s+\eta(x)]\bar{p}_{N-1}(x, s) = \exp[-(\lambda+s)x] \exp\left[-\int_0^x \eta(u)\, du\right]\bar{p}_N(0, s)\,.\,\lambda,$$

whence

$$\bar{p}_{N-1}(x, s) = [\lambda x\bar{p}_n(0, s)+\bar{p}_{N-1}(0, s)] \exp[-(\lambda+s)x] \exp\left[-\int_0^x \eta(u)\, du\right]. \qquad \text{(25)}$$

Similarly, taking $n = N-2$ in (18) and using (25), and solving the resulting differential equation, we have

$$\bar{p}_{N-2}(x, s) = \left[\frac{\lambda^2x^2}{2}\bar{p}_N(0, s)+\lambda x\bar{p}_{N-1}(0, s)+\bar{p}_{N-2}(0, s)\right] \exp\left[-(\lambda+s)x-\int_0^x \eta(u)\, du\right], \qquad \text{(26)}$$

and so on.

The differential equation for $\bar{p}_i(x, s)$ will be

$$\frac{\partial \bar{p}_i(x, s)}{\partial x}+[\lambda+s+\eta(x)]\bar{p}_i(x, s)$$
$$= \lambda\left[\frac{(\lambda x)^{N-i-1}}{(N-i-1)!}\bar{p}_N(0, s)+\frac{(\lambda x)^{N-i-2}}{(N-i-2)!}\bar{p}_{N-1}(0, s)+\ldots+\bar{p}_{i+1}(0, s)\right]$$
$$\times \exp[-(\lambda+s)x] \exp\left[-\int_0^x \eta(u)\, du\right]+\delta(x),$$

whence

$$\bar{p}_i(x, s) = \left[\frac{(\lambda x)^{N-i}}{(N-i)!}\bar{p}_N(0, s)+\frac{(\lambda x)^{N-i-1}}{(N-i-1)!}\bar{p}_{N-1}(0, s)+\dots+\bar{p}_i(0, s)+1\right]$$

$$\times \exp[-(\lambda+s)x] \exp\left[-\int_0^x \eta(u)\,du\right]. \qquad (27)$$

Proceeding similarly, the differential equation for $\bar{p}_{-M}(x, s)$ will be

$$\frac{\partial \bar{p}_{-M}(x, s)}{\partial x}+[s+\eta(x)]\bar{p}_{-M}(x, s)$$

$$= \lambda\left[\sum_{r=1}^{M+N}\frac{(\lambda x)^{r-1}}{(r-1)!}\bar{p}_{r-M}(0, s)+\frac{(\lambda x)^{M+i-1}}{(M+i-1)!}\right]$$

$$\times \exp[-(\lambda+s)x] \exp\left[-\int_0^x \eta(u)\,du\right],$$

which gives

$$\bar{p}_{-M}(x, s) = \lambda \exp(-sx) \exp\left(-\int_0^x \eta(u)\,du\right)\left[\sum_{r=1}^{M+N}\int_0^x \bar{p}_{r-M}(0, s)\frac{(\lambda y)^{r-1}}{\Gamma(r)}\exp(-\lambda y)\,dy\right.$$

$$\left.+\int_0^x \frac{(\lambda y)^{M+i-1}}{\Gamma(M+i)}\exp(-\lambda y)\,dy\right]. \qquad (28)$$

Therefore, eqn. (16) becomes

$$\frac{\partial \bar{f}}{\partial x}+[\lambda+s+\eta(x)-\lambda/\alpha]\bar{f}$$

$$= \alpha^i\delta(x)+\lambda^2\alpha^{-M-1}(\alpha-1)\exp(-sx)\exp\left(-\int_0^x \eta(u)\,du\right)\left[\sum_{r=1}^{M+N}\int_0^x \bar{p}_{r-M}(0, s)\right.$$

$$\left.\times\frac{(\lambda y)^{r-1}}{\Gamma(r)}\exp(-\lambda y)\,dy+\int_0^x \frac{(\lambda y)^{M+i-1}}{\Gamma(M+i)}\exp(-\lambda y)\,dy\right]. \qquad (29)$$

Solution of (29) is

$$\bar{f}(x, \alpha, s) = \exp[-(\lambda+s-\lambda/\alpha)x]\exp\left(-\int_0^x \eta(u)\,du\right)$$

$$\times\left[\alpha^i+\bar{f}(0, \alpha, s)+\lambda^2\alpha^{-M}(1-1/\alpha)\int_0^x \exp[\lambda(1-1/\alpha)z]\sum_{r=1}^{M+N}\bar{p}_{r-M}(0, s)\int_0^z \frac{(\lambda y)^{r-1}}{\Gamma(r)}\right.$$

$$\left.\times\exp(-\lambda y)\,dy\,dz+\int_0^x \exp[\lambda(1-1/\alpha)z]\int_0^z \frac{(\lambda y)^{M+i-1}}{\Gamma(M+i)}\exp(-\lambda y)\,dy\,dz\right]. \qquad (30)$$

Now, changing the order of integration, we have

$$\int_0^x \int_0^z \exp\,[\lambda(1-1/\alpha)z]\,\frac{(\lambda y)^{r-1}}{\Gamma(r)}\exp\,(-\lambda y)\,dy\,dz$$

$$=\int_0^x \int_y^x \exp\,[\lambda(1-1/\alpha)z]\,\frac{(\lambda y)^{r-1}}{\Gamma(r)}\exp\,(-\lambda y)\,dz\,dy$$

$$=\int_0^x \frac{\exp\,[\lambda(1-1/\alpha)x]-\exp\,[\lambda(1-1/\alpha)y]}{\lambda(1-1/\alpha)}\,\frac{(\lambda y)^{r-1}}{\Gamma(r)}\exp\,(-\lambda y)\,dy$$

$$=\frac{1}{\lambda(1-1/\alpha)\Gamma(r)}\left[\exp\,[\lambda(1-1/\alpha)x]\int_0^x (\lambda y)^{r-1}\exp\,(-\lambda y)\,dy\right.$$
$$\left.-\alpha^{r-1}\int_0^x \exp\left(-\frac{\lambda}{\alpha}y\right)\left(\frac{\lambda}{\alpha}y\right)^{r-1}dy\right]$$

$$=\frac{1}{\lambda^2(1-1/\alpha)\Gamma(r)}\left[\exp\,[\lambda(1-1/\alpha)x]\gamma(r,\,\lambda x)-\alpha^r\gamma\left(r,\frac{\lambda}{\alpha}x\right)\right],$$

where γ is the Incomplete Gamma function defined by

$$\gamma(\alpha,\,x)=\int_0^x \exp\,[-z]z^{\alpha-1}dz=\alpha^{-1}x^{\alpha}{}_1F_1(\alpha\,;\,\alpha+1\,;\,-x).$$

Therefore, (30) gives

$$\bar f(x,\,\alpha,\,s)=\exp\,[-(\lambda+s-\lambda/\alpha)x]\,\exp\left[-\int_0^x \eta(u)\,du\right]\left[\alpha^i+\bar f(0,\,\alpha,\,s)\right.$$
$$+\alpha^{-M}\left\{\sum_{r=1}^{M+N}\frac{1}{\Gamma(r)}\bar p_{r-M}(0,\,s)\left(\exp\,[\lambda(1-1/\alpha)x]\gamma(r,\,\lambda x)-\alpha^r\gamma\left(r,\,\frac{\lambda x}{\alpha}\right)\right)\right.$$
$$\left.\left.+\frac{1}{\Gamma(M+i)}\left(\exp\,[\lambda(1-1/\alpha)x]\gamma(M+i,\,\lambda x)-\alpha^{M+i}\gamma\left(M+i,\,\frac{\lambda x}{\alpha}\right)\right)\right\}\right]. \quad (31)$$

Substituting this value of $\bar f(x,\,\alpha,\,s)$ in (23), we get

$$\bar f(0,\,\alpha,\,s)=[\alpha^{i+1}+\alpha\bar f(0,\,\alpha,\,s)]\bar S(\lambda+s-\lambda/\alpha)-\alpha^N(\alpha-1)\bar p_N(0,\,s)\bar S(\lambda+s)$$
$$+\alpha^{-M+1}\left[\sum_{r=1}^{M+N}\left\{\frac{1}{\Gamma(r)}\bar p_{r-M}(0,\,s)\int_0^\infty S(x)\exp\,[-(\lambda+s-\lambda/\alpha)x]\right.\right.$$
$$\left.\times\left(\exp\,[\lambda(1-1/\alpha)x]\gamma(r,\,\lambda x)-\alpha^r\gamma\left(r,\,\frac{\lambda x}{\alpha}\right)\right)dx\right\}$$
$$+\frac{1}{\Gamma(M+i)}\int_0^\infty S(x)\exp\,[-(\lambda+s-\lambda/\alpha)x]\left(\exp\,[\lambda(1-1/\alpha)x]\gamma(M+i,\,\lambda x)\right.$$
$$\left.\left.-\alpha^{M+i}\gamma\left(M+i,\,\frac{\lambda x}{\alpha}\right)\right)dx\right]. \quad .. \quad (32)$$

Now, substituting for $\bar{p}_n(x, s)$, $(-M \leqslant n \leqslant N)$ from equations (24) through (28) in (21), (22), we get $M+N$ equations in the $M+N$ unknowns $\bar{p}_n(0, s)$, $(-M+1 \leqslant n \leqslant N)$, which are sufficient to determine them.

Substituting for $\bar{f}(0, \alpha, s)$ from (32) in (31), we get $\bar{f}(x, \alpha, s)$.

Hence $\bar{\pi}(\alpha, s)$, the L.T. of the probability generating function $\pi(\alpha, t)$, is given by

$$\bar{\pi}(\alpha, s) = \int_0^\infty \bar{f}(x, \alpha, s)\, dx. \qquad \ldots \quad \ldots \quad \ldots \quad (33)$$

5. Particular Cases

I. *k-Erlang arrivals of taxis*

Let us assume that the arrivals of taxis are according to a k-Erlang distribution, so that

$$S(x) = \frac{\mu^k x^{k-1} \exp[-\mu x]}{\Gamma(k)} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (34)$$

$$\bar{S}(s) = \left(\frac{\mu}{\mu+s}\right)^k \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (35)$$

$$\eta(x) = \frac{\mu^k x^{k-1} \exp[-\mu x]}{\Gamma(k, \mu x)} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (36)$$

$$\exp\left[-\int_0^x \eta(u)\, du\right] = \frac{\Gamma(k, \mu x)}{\Gamma(k)}, \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (37)$$

where $\Gamma(k, \mu x)$ is the Incomplete Gamma function defined by

$$\Gamma(\alpha, x) = \int_x^\infty \exp[-z]\, z^{\alpha-1}\, dz = \Gamma(\alpha) - \gamma(\alpha, x).$$

Relations (34), (35) are well known. Proofs for (36), (37) are given in the Appendix.

On using (34), (35) and Erdèlyi *et al.* (1954*b*, p. 308, eqn. 15), viz.

$$\int_0^\infty x^{\mu-1} \exp[-\beta x]\, \gamma(\nu, \alpha x)\, dx = \frac{\alpha^\nu \Gamma(\mu+\nu)}{\nu(\alpha+\beta)^{\mu+\nu}}\, {}_2F_1\left(1, \mu+\nu;\, \nu+1;\, \frac{\alpha}{\alpha+\beta}\right),$$

$$Re(\alpha+\beta) > 0,\ Re\, \beta > 0,\ Re(\mu+\nu) > 0, \quad \ldots \quad (38)$$

equation (32) yields

$$\bar{f}(0, \alpha, s) = \frac{1}{1-\alpha\left(\dfrac{\mu}{\mu+\lambda+s-\lambda/\alpha}\right)^k}\left[\alpha^{i+1}\left(\frac{\mu}{\mu+\lambda+s-\lambda/\alpha}\right)^k - \alpha^N(\alpha-1)\bar{p}_N(0, s)\right.$$

$$\times\left(\frac{\mu}{\mu+\lambda+s}\right)^k + \alpha^{-M+1}\sum_{r=1}^{M+N}\frac{1}{\Gamma(r+1)}\bar{p}_{r-M}(0, s)\frac{\mu^k}{\Gamma(k)}\frac{\lambda^r\Gamma(r+k)}{(\lambda+\mu+s)^{r+k}}$$

$$\times\left\{{}_2F_1\left(1, r+k; r+1; \frac{\lambda}{\mu+\lambda+s}\right) - {}_2F_1\left(1, r+k; r+1; \frac{\lambda/\alpha}{\mu+\lambda+s}\right)\right\}$$

$$+\frac{\alpha^{-M+1}}{\Gamma(M+i+1)}\frac{\mu^k}{\Gamma(k)}\frac{\lambda^{M+i}\Gamma(M+i+k)}{(\lambda+\mu+s)^{M+i+k}}\left\{{}_2F_1\left(1, M+i+k; M+i+1;\right.\right.$$

$$\left.\left.\times\frac{\lambda}{\mu+\lambda+s}\right) - {}_2F_1\left(1, M+i+k; M+i+1; \frac{\lambda/\alpha}{\mu+\lambda+s}\right)\right\}\Bigg]. \quad \ldots \quad \ldots \quad (39)$$

Substituting for $\bar{f}(0, \alpha, s)$ from (39) in (31), we get $\bar{f}(x, \alpha, s)$. Then (33) yields

$$\bar{\pi}(\alpha, s) = \{\alpha^i + \bar{f}(0, \alpha, s)\}\left[\frac{1-\left(\dfrac{\mu}{\mu+\lambda+s-\lambda/\alpha}\right)^k}{\lambda+s-\dfrac{\lambda}{\alpha}}\right] + \phi, \quad \ldots \quad \ldots \quad (40)$$

where

$$\phi(\alpha, s) = \alpha^{-M}\sum_{r=1}^{M+N}\frac{1}{\Gamma(r)}\bar{p}_{r-M}(0, s)\left[\int_0^\infty \exp[-sx]\,\gamma(r, \lambda x)\exp\left[-\int_0^x \eta(u)\,du\right]dx\right.$$

$$\left.-\alpha^r\int_0^\infty \exp\left[-\left(\lambda+s-\frac{\lambda}{\alpha}\right)x\right]\gamma\left(r, \frac{\lambda x}{\alpha}\right)\exp\left[-\int_0^x \eta(u)\,du\right]dx\right]$$

$$+\frac{\alpha^{-M}}{\Gamma(M+i)}\left[\int_0^\infty \exp[-sx]\gamma(M+i, \lambda x)\exp\left[-\int_0^x \eta(u)\,du\right]dx\right.$$

$$\left.-\alpha^{M+i}\int_0^\infty \exp\left[-\left(\lambda+s-\frac{\lambda}{\alpha}\right)x\right]\gamma\left(M+i, \frac{\lambda x}{\alpha}\right)\exp\left[-\int_0^x \eta(u)\,du\right]dx\right].$$

$$\ldots \quad (41)$$

Using (37), we have

$$\frac{1}{\Gamma(r)}\int_0^\infty \exp[-sx]\gamma(r,\lambda x)\exp\left[-\int_0^x \eta(u)\,du\right]dx$$

$$=\frac{1}{\Gamma(r)\Gamma(k)}\int_0^\infty \exp[-sx]\,\gamma(r,\lambda x)\Gamma(k,\mu x)\,dx$$

$$=\frac{1}{\Gamma(r)}\int_0^\infty \exp[-sx]\gamma(r,\lambda x)\,dx-\frac{1}{\Gamma(r)\Gamma(k)}\int_0^\infty \exp[-sx]\gamma(r,\lambda x)\gamma(k,\mu x)\,dx$$

$$=\frac{1}{s}\left(\frac{\lambda}{\lambda+s}\right)^r-\frac{\lambda^r\mu^k}{\Gamma(r+1)\Gamma(k+1)}\int_0^\infty \exp[-sx]x^{r+k}{}_1F_1(r;r+1;-\lambda x)$$
$$\times{}_1F_1(k;k+1;-\mu x)\,dx,$$

using Erdèlyi *et al.* (1954*a*, p. 179, eqn. 34) for evaluation of the first integral.

Also, from Erdèlyi *et al.* (1954*a*, p. 216, eqn. 14), we have

$$\int_0^\infty \exp[-sx]x^{r+k}{}_1F_1(r;r+1;-\lambda x){}_1F_1(k;k+1;-\mu x)\,dx$$

$$=\frac{\Gamma(r+k+1)}{(s+\lambda+\mu)^{r+k+1}}F_2\left(r+k+1;1,1;r+1,k+1;\frac{\lambda}{s+\lambda+\mu},\frac{\mu}{s+\lambda+\mu}\right),$$

where $Re(r+k+1)>0$, $Re(s)>0$, $Re(s+\lambda+\mu)>0$ and F_2 is Appell's function defined (Erdèlyi *et al.* 1954*a*, p. 384) by:

$$F_2(\alpha;\beta,\beta';\gamma,\gamma';x,y)=\sum_{m=0}^\infty\sum_{n=0}^\infty\frac{(\alpha)_{m+n}(\beta)_m(\beta')_n}{(\gamma)_m(\gamma')_n m!\;n!}x^my^n.$$

Thus (41) simplifies to

$$\phi(\alpha,s)=\alpha^{-M}\sum_{r=1}^{M+N}\bar{p}_{r-M}(0,s)\left[\left(\frac{1}{s}-\frac{1}{\lambda+s-\lambda/\alpha}\right)\left(\frac{\lambda}{s+\lambda}\right)^r-\frac{(r+k)!\;\lambda^r\mu^k}{r!\;k!\;(s+\lambda+\mu)^{r+k+1}}\right.$$
$$\times\left\{F_2\left(r+k+1;1,1;r+1,k+1;\frac{\lambda}{s+\lambda+\mu},\frac{\mu}{s+\lambda+\mu}\right)\right.$$
$$\left.\left.-F_2\left(r+k+1;1,1;r+1,k+1;\frac{\lambda/\alpha}{s+\lambda+\mu},\frac{\mu}{s+\lambda+\mu}\right)\right\}\right]$$
$$+\alpha^{-M}\left[\left(\frac{1}{s}-\frac{1}{\lambda+s-\lambda/\alpha}\right)\left(\frac{\lambda}{s+\lambda}\right)^{M+i}-\frac{(M+i+k)!\;\lambda^{M+i}\mu^k}{(M+i)!\;k!\;(s+\lambda+\mu)^{M+i+k+1}}\right.$$
$$\times\left\{F_2\left(M+i+k+1;1,1;M+i+1,k+1;\frac{\lambda}{s+\lambda+\mu},\frac{\mu}{s+\lambda+\mu}\right)\right.$$
$$\left.\left.-F_2\left(M+i+k+1;1,1;M+i+1,k+1;\frac{\lambda/\alpha}{s+\lambda+\mu},\frac{\mu}{s+\lambda+\mu}\right)\right\}\right].\qquad\ldots\quad(42)$$

II. *Poisson arrivals of taxis*

Setting $k = 1$ in (39), we get

$$\bar{f}(0, \alpha, s) = \frac{\mu+\lambda+s-\lambda/\alpha}{\mu+\lambda+s-\lambda/\alpha-\mu\alpha}\left[\frac{\mu\alpha^{i+1}}{\mu+\lambda+s-\lambda/\alpha} - \alpha^N(\alpha-1)\bar{p}_N(0, s)\frac{\mu}{\mu+\lambda+s}\right.$$

$$\left. + \mu\alpha^{-M+1}\left(\frac{1}{\mu+s} - \frac{1}{s+\lambda+\mu-\lambda/\alpha}\right)\left\{\sum_{r=1}^{M+N}\left(\frac{\lambda}{\lambda+\mu+s}\right)^r \bar{p}_{r-M}(0, s) + \left(\frac{\lambda}{\lambda+\mu+s}\right)^{M+i}\right\}\right]$$

$$= \frac{1}{\mu+\lambda+s-\lambda/\alpha-\mu\alpha}\left[\mu\alpha^{i+1} - \alpha^N(\alpha-1)\bar{p}_N(0, s)\frac{\mu}{\lambda+\mu+s}(\lambda+\mu+s-\lambda/\alpha)\right.$$

$$\left. + \frac{\lambda\mu\alpha^{-M}(\alpha-1)}{\mu+s}\left\{\sum_{r=1}^{M+N}\left(\frac{\lambda}{\lambda+\mu+s}\right)^r \bar{p}_{r-M}(0, s) + \left(\frac{\lambda}{\lambda+\mu+s}\right)^{M+i}\right\}\right]. \qquad (43)$$

And from (40), (42), setting $k = 1$, in which case F_2 simplifies in terms of Binomial expansions, we get

$$\bar{\pi}(\alpha, s) = \{\alpha^i + \bar{f}(0, \alpha, s)\}\frac{1}{\mu+\lambda+s-\lambda/\alpha}$$

$$+ \alpha^{-M}\left\{\frac{1}{s+\mu} - \frac{1}{s+\lambda+\mu-\lambda/\alpha}\right\}\left\{\sum_{r=1}^{M+N}\left(\frac{\lambda}{\lambda+\mu+s}\right)^r \bar{p}_{r-M}(0, s)\right.$$

$$\left. + \left(\frac{\lambda}{\lambda+\mu+s}\right)^{M+i}\right\}. \qquad \ldots \quad (44)$$

Now, the numerator of (43) must vanish at the two zeros of the denominator, i.e. the roots, α_1, α_2, of the equation

$$\mu\alpha^2 - (s+\lambda+\mu)\alpha + \lambda = 0.$$

Thus we get two equations, which are sufficient to determine

$$\bar{p}_N(0, s) \text{ and } \sum_{r=1}^{M+N}\left(\frac{\lambda}{\lambda+\mu+s}\right)^r \bar{p}_{r-M}(0, s).$$

On solving these equations, we get

$$\bar{p}_N(0, s) = \frac{\lambda+\mu+s}{s}\cdot\frac{\alpha_2^{M+i+1}(1-\alpha_1)+\alpha_1^{M+i+1}(\alpha_2-1)}{\alpha_2^{M+N+1}-\alpha_1^{M+N+1}}, \quad \ldots \quad (45)$$

$$\sum_{r=1}^{M+N}\left(\frac{\lambda}{\lambda+\mu+s}\right)^r \bar{p}_{r-M}(0, s) + \left(\frac{\lambda}{\lambda+\mu+s}\right)^{M+i}$$

$$= \frac{\mu(\mu+s)}{\lambda s}\frac{(\alpha_1\alpha_2)^{M+i+1}[\alpha_2^{N-i}(\alpha_2-1)+\alpha_1^{N-i}(1-\alpha_1)]}{\alpha_2^{M+N+1}-\alpha_1^{M+N+1}}. \quad \ldots \quad (46)$$

Substituting these values in (43) and using (44), we obtain

$$\bar{\pi}(\alpha, s) = \frac{1}{s+\lambda+\mu-\lambda/\alpha-\mu\alpha}\left[\alpha^{i}+\alpha^{N}(1-\alpha)\frac{\mu}{s}\cdot\frac{\alpha_2^{M+i+1}(1-\alpha_1)+\alpha_1^{M+i+1}(\alpha_2-1)}{\alpha_2^{M+N+1}-\alpha_1^{M+N+1}}\right.$$

$$\left.+\alpha^{-M}(1-1/\alpha)\cdot\frac{\mu}{s}\cdot(\alpha_1\alpha_2)^{M+i+1}\cdot\frac{\alpha_2^{N-i}(\alpha_2-1)+\alpha_1^{N-i}(1-\alpha_1)}{\alpha_2^{M+N+1}-\alpha_1^{M+N+1}}\right], \quad \ldots \quad (47)$$

which agrees with equation (31) of Kashyap (1965).

Remark: A comparison of (45), (46) with corresponding results of Kashyap (1965) shows that

$$\bar{p}_N(s) = \frac{1}{\lambda+\mu+s}\bar{p}_N(0, s) \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (48)$$

and

$$\bar{p}_{-M}(s) = \frac{1}{\mu+s}\left[\sum_{r=1}^{M+N}\left(\frac{\lambda}{\lambda+\mu+s}\right)^{r}\bar{p}_{r-M}(0, s)+\left(\frac{\lambda}{\lambda+\mu+s}\right)^{M+i}\right], \quad \ldots \quad (49)$$

which are easy to verify from (24) and (28).

Appendix

That equations (36), (37) follow from (34) is shown below.

$\eta(x)\Delta$ = Prob {inter-arrival time lies in $(x, x+\Delta)$/ inter-arrival time $> x$}

$$= \frac{\int_x^{x+\Delta} S(u)\,du}{\int_x^{\infty} S(u)\,du},$$

where the right-hand side is simplified to the first power of Δ.

Now,
$$\int_x^{x+\Delta} S(u)\,du = D(x+\Delta)-D(x),$$

where $D(x) = \int_0^x S(u)\,du$ is the distribution function of inter-arrival times

$$= \left(\frac{d}{dx}D(x)\right)\Delta, \text{ up to the first order terms in } \Delta$$

$$= S(x)\,\Delta.$$

Therefore,
$$\eta(x) = \frac{\dfrac{\mu^k x^{k-1}\exp[-\mu x]}{\Gamma(k)}}{\displaystyle\int_x^{\infty}\frac{\mu^k u^{k-1}\exp[-\mu u]}{\Gamma(k)}\,du}$$

$$= \frac{\mu^k x^{k-1}\exp[-\mu x]}{\Gamma(k, \mu x)}.$$

Now from the definition of $\Gamma(\alpha, x)$, we have

$$\frac{d}{dx}\Gamma(\alpha, x) = -\exp[-x]x^{\alpha-1}.$$

Therefore, $$\int_0^x \eta(u)\,du = \int_0^x \frac{\mu^k u^{k-1}\exp[-\mu u]}{\Gamma(k, \mu u)}\,du$$

$$= \int_0^{\mu x} \frac{v^{k-1}\exp[-v]\,dv}{\Gamma(k, v)} = -\int_0^{\mu x} \frac{d\,\Gamma(k, v)}{\Gamma(k, v)}$$

$$= -\log\frac{\Gamma(k, \mu x)}{\Gamma(k)},$$

from which (37) follows.

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