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Loss probability in a finite queue with service interruptions and queue length distribution in the corresponding infinite queue☆

Fumio Ishizaki*

Department of Mathematics and Telecommunication Mathematics Research Center, Korea University, 1, 5-ka, Anam-dong, Sungbuk-gu, Seoul 136-701, Republic of Korea

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Abstract

In this paper, we consider a discrete-time finite-buffer queue with correlated arrivals and service interruptions and the corresponding infinite-buffer queue. Under some assumptions, we derive an exact relation holding between the loss probability in the finite-buffer queue and the queue length distribution in the corresponding infinite-buffer queue. The exact relation is considered as an integration/generalization of the exact relations which have been derived in previous papers. By applying the exact relation, we also develop formulas to estimate the loss probability.

Keywords: Loss probability; Finite-buffer queue; Infinite-buffer queue; Proportional relation; Exact relation

1. Introduction

In this paper, we consider a discrete-time finite-buffer queue with correlated arrivals and service interruptions and the corresponding infinite-buffer queue. Under some assumptions, we derive an exact relation holding between the loss probability in the finite-buffer queue and the queue length distribution in the corresponding infinite-buffer queue. By applying the exact relation, we also develop two formulas to estimate the loss probability. The first formula, which uses Ramaswami's recursion [14] in the exact relation,

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* Tel.: +82 2 3290 4297.
E-mail address: fumio@ieee.org.

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computes the exact loss probability. The second formula, which uses the geometric asymptotic expression [3] of the queue length distribution in the exact relation, calculates the asymptotic loss probability.

Similar exact relations between the loss probability in the finite-buffer queue and the queue length distribution in the corresponding infinite-buffer queue have been studied by several researchers. Kang et al. [10] have considered discrete-time queueing systems where the arrival process is a superposition of Bernoulli sources (i.e., the arrival process is i.i.d.) and the service is available every R slots where R is a positive integer. For such queueing systems, they have established an exact relation holding between the loss probability in a finite-buffer queue and the queue length distribution in the corresponding infinite-buffer queue. Ishizaki and Takine [8] have considered discrete-time queueing systems where the arrival process is similar to (but a little restrictive compared to) the arrival process considered in this paper and the service is always available. For such queueing systems, they have established a proportional relation between the stationary queue length distribution in the finite-buffer queue and that in the corresponding infinite-buffer queue. Using this proportional relation, they have obtained an exact relation holding between the loss probability in a finite-buffer queue and the queue length distribution in the corresponding infinite-buffer queue. Ishizaki [6] has considered discrete-time queueing systems where the arrival process is similar to the arrival process considered in this paper and the service is available every R slots. A similar exact relation has been obtained for such queueing systems. The exact relation derived in this paper is considered as an integration/generalization of those exact relations in the following sense: the service process considered in this paper is governed by a special Markov chain and it includes the ones considered in [6,8,10] as special cases. Also, the arrival process considered in this paper can have some correlations and it generalizes the one considered in [10].

Queueing systems with correlated arrivals and service interruptions have a wide range of applications to manufacturing, computer and telecommunication systems where the server is subject to breakdown or some scheduling mechanism such as round-robin (see, e.g. [9,2,18] and references therein). In packet/cell networks, the arrival process at statistical multiplexer is usually a superposition of sources which typically generate time-correlated and bursty traffic due to their origin (e.g., periodic sampling of voice traffic or MPEG encoded real-time video traffic) or traffic shaping. The service process at statistical multiplexer may be subject to a scheduling mechanism (e.g., round-robin scheduling). For instance, the service for packet/cell transmission may be available every R slots, where R is a positive integer. The dynamics of such a multiplexer is modeled as a discrete-time single-server finite-buffer queue with correlated arrivals and service interruptions, and the dynamics of such a queue may be described as a finite-state Markov chain. The loss probability is then obtained from the stationary distribution of the Markov chain. Although we can directly apply the standard algorithm (e.g., as shown in [4,11,16]) to compute the stationary distribution of the Markov chain, the following difficulties arise in the computation. Usually, the number of states to describe the dynamics of multiplexer becomes prohibitively large. This makes the computation with enough accuracy very difficult. In addition, the standard algorithms such as block Gaussian elimination include subtractions and this often makes the algorithms unstable, especially, when the size of the matrices is large. Thus numerical algorithms to estimate the loss probability efficiently and stably should be developed. The first formula derived in this paper stably computes the exact loss probability. The second formula derived in this paper can estimate the loss probability more easily than the standard algorithm when the size of the matrices is large.

The remainder of this paper is organized as follows. In Section 2, we consider an M/G/1-type Markov chain with some regenerative structure and a corresponding truncated Markov chain which is obtained from the M/G/1-type Markov chain by limiting its maximum level to K where K is a nonnegative integer.

Under some assumptions, we derive a preliminary result (Theorem 1) for a proportional relation holding between the stationary distribution of the M/G/1-type Markov chain and that of the corresponding truncated Markov chain. The result is interpreted as a generalization of the proportional relation [8] between the stationary queue length distribution in the finite-buffer queue and that in the corresponding infinite-buffer queue. Section 3 shows a discrete-time single-server infinite-buffer queue with correlated arrivals and service interruptions whose dynamics is described as the M/G/1-type Markov chain considered in Section 2 and its corresponding finite-buffer queue whose dynamics is described as the corresponding truncated Markov chain. In Section 4, using the preliminary result derived in Section 2, we establish an exact relation (Theorem 2) holding between the loss probability in a finite-buffer queue and the stationary queue length distribution in the corresponding infinite-buffer queue. Using Theorem 2 in the geometric expression of the asymptotic tail distribution in the infinite-buffer queue, Section 5 derives a formula (Theorem 3) to compute the asymptotic loss probability in the finite-buffer queue.

2. Preliminary result

In this section, we consider an M/G/1-type Markov chain with some regenerative structure and a corresponding truncated Markov chain which is obtained from the M/G/1-type Markov chain by limiting its maximum level to K where K is a nonnegative integer. We then provide a preliminary result for a proportional relation holding between the stationary distribution of the M/G/1-type Markov chain and that of the corresponding truncated Markov chain.

Throughout this paper, we use the following notation. For any matrix $\boldsymbol{C}$, $[\boldsymbol{C}]_{i,j}$ denotes the (i, j)th element of the matrix $\boldsymbol{C}$, and the row and column index numbers of any matrix are labeled from 0. Similarly, for any vector $\boldsymbol{c}$, $[\boldsymbol{c}]_i$ denotes the ith element of the vector $\boldsymbol{c}$, and the row or column index numbers of any vector are labeled from 0.

We consider a discrete-time Markov chain $\{(X_n, V_n)\}_{n=0}^{\infty}$ whose state space is $\mathcal{S} = \{(k, l) \mid k = 0, 1, \ldots; l = 0, \ldots, L\}$, where L is a nonnegative integer. We assume that the Markov chain $\{(X_n, V_n)\}_{n=0}^{\infty}$ is an M/G/1-type Markov chain and its down-shift matrices have some special structures shown in the assumption below. Given that sequences of $(L + 1) \times (L + 1)$ matrices $\{\boldsymbol{A}_i\}$ $(i = 0, 1, \ldots)$ and $\{\boldsymbol{B}_i\}$ $(i = 0, 1, \ldots)$, we consider the Markov chain whose transition probability matrix $\boldsymbol{Q}^{(\infty)}$ has the following block structure:

$$\boldsymbol{Q}^{(\infty)} = \begin{bmatrix} \boldsymbol{B}_0 & \boldsymbol{B}_1 & \boldsymbol{B}_2 & \boldsymbol{B}_3 & \cdots \\ \boldsymbol{A}_0 & \boldsymbol{A}_1 & \boldsymbol{A}_2 & \boldsymbol{A}_3 & \cdots \\ \boldsymbol{O} & \boldsymbol{A}_0 & \boldsymbol{A}_1 & \boldsymbol{A}_2 & \cdots \\ \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{A}_0 & \boldsymbol{A}_1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \tag{1}$$

where $\boldsymbol{O}$ denotes the $(L + 1) \times (L + 1)$ zero matrix. We also consider a discrete-time Markov chain $\{(Y_n, V_n)\}_{n=0}^{\infty}$ which is obtained from the Markov chain $\{(X_n, V_n)\}_{n=0}^{\infty}$ by limiting its maximum level to K where K is a nonnegative integer. In other words, its state space is $\mathcal{S} = \{(k, m) \mid k = 0, \ldots, K; m =$

$0, \ldots, L\}$ and its transition probability matrix $\boldsymbol{Q}^{(K)}$ has the following block structure:

$$\boldsymbol{Q}^{(K)} = \begin{bmatrix} \boldsymbol{B}_0 & \boldsymbol{B}_1 & \boldsymbol{B}_2 & \boldsymbol{B}_3 & \cdots & \boldsymbol{B}_{K-1} & \boldsymbol{B}_K^* \\ \boldsymbol{A}_0 & \boldsymbol{A}_1 & \boldsymbol{A}_2 & \boldsymbol{A}_3 & \cdots & \boldsymbol{A}_{K-1} & \boldsymbol{A}_K^* \\ \boldsymbol{O} & \boldsymbol{A}_0 & \boldsymbol{A}_1 & \boldsymbol{A}_2 & \cdots & \boldsymbol{A}_{K-2} & \boldsymbol{A}_{K-1}^* \\ \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{A}_0 & \boldsymbol{A}_1 & \cdots & \boldsymbol{A}_{K-3} & \boldsymbol{A}_{K-2}^* \\ \vdots & & \ddots & \ddots & & & \vdots \\ \vdots & & & \ddots & \ddots & & \vdots \\ \boldsymbol{O} & \cdots & \cdots & \cdots & \boldsymbol{O} & \boldsymbol{A}_0 & \boldsymbol{A}_1^* \end{bmatrix}, \tag{2}$$

where $\boldsymbol{A}_k^* = \sum_{m=k}^{\infty} \boldsymbol{A}_m$, $\boldsymbol{B}_k^* = \sum_{m=k}^{\infty} \boldsymbol{B}_m$, and $\boldsymbol{O}$ denotes the $(L+1) \times (L+1)$ zero matrix.

For the structure of the down-shift matrix $\boldsymbol{A}_0$, the following assumption is made.

Assumption 1. There exists a $1 \times (L+1)$ probability vector $\boldsymbol{a}$ such that $\boldsymbol{A}_0 = \boldsymbol{A}_0\boldsymbol{e}\boldsymbol{a}$, where $\boldsymbol{e}$ is an $(L+1) \times 1$ column vector with unit elements.

Note that Assumption 1 is equivalent to the following statement: For $i = 0, \ldots, L$, $l = 0, \ldots, L$ and $k = 0, 1, \ldots$, we have $\mathrm{P}(V_n = l \mid X_{n-1} = k+1, X_n = k) = \mathrm{P}(V_n = l \mid X_{n-1} = k+1, X_n = k, V_{n-1} = i)$. For $i = 0, \ldots, L$, $l = 0, \ldots, L$ and $k = 0, 1, \ldots$, we then have $[\boldsymbol{a}]_l = \mathrm{P}(V_n = l \mid X_{n-1} = k+1, X_n = k, V_{n-1} = i)$. In other words, Assumption 1 means that when the down-ward shift of the level $\{X_n\}$ (or $\{Y_n\}$) occurs, $\{V_n\}$ regenerates.

We also made the following assumption.

Assumption 2. The Markov chains $\{(X_n, V_n)\}$ and $\{(Y_n, V_n)\}$ are irreducible and positive recurrent.

Under Assumption 2, the stationary distribution of the Markov chain $\{(X_n, V_n)\}$ and the stationary distribution of the Markov chain $\{(Y_n, V_n)\}$ exist, and they are uniquely determined [1].

Let $\boldsymbol{x}$ and $\boldsymbol{y}$ denote the stationary distribution of the Markov chain $\{(X_n, V_n)\}$ and that of the Markov chain $\{(Y_n, V_n)\}$, respectively. We then have

$$\boldsymbol{x} = \boldsymbol{x}\boldsymbol{Q}^{(\infty)}, \qquad \boldsymbol{y} = \boldsymbol{y}\boldsymbol{Q}^{(K)}, \tag{3}$$

where

$$\boldsymbol{x} = (\boldsymbol{x}_0, \boldsymbol{x}_1, \ldots), \qquad \boldsymbol{y} = (\boldsymbol{y}_0, \boldsymbol{y}_1, \ldots, \boldsymbol{y}_K),$$

$\boldsymbol{x}_j$ is a $1 \times (L+1)$ vector whose lth element $[\boldsymbol{x}_j]_l$ is given by $[\boldsymbol{x}_j]_l = \mathrm{P}(X_n = j, V_n = l)$, and $\boldsymbol{y}_j$ is a $1 \times (L+1)$ vector whose lth element $[\boldsymbol{y}_j]_l$ is given by $[\boldsymbol{y}_j]_l = \mathrm{P}(Y_n = j, V_n = l)$.

The following lemma for the stationary distribution $\boldsymbol{x}$ is readily obtained by using the results shown in [13,14].

Lemma 1. *Under Assumptions 1 and 2, the stationary distribution $\boldsymbol{x}$ satisfies the equations*

$$\boldsymbol{x}_0 = \boldsymbol{x}_0\bar{\boldsymbol{B}}_0, \tag{4}$$

$$\boldsymbol{x}_i = \left(\boldsymbol{x}_0\bar{\boldsymbol{B}}_i + \sum_{k=1}^{i-1}\boldsymbol{x}_k\bar{\boldsymbol{A}}_{i-k+1}\right)(\boldsymbol{I}-\bar{\boldsymbol{A}}_1)^{-1}, \quad i = 1, 2, \ldots, \tag{5}$$

where for $\nu = 0, \ldots, K-1$, *we define* $\bar{\boldsymbol{A}}_\nu$ *and* $\bar{\boldsymbol{B}}_\nu$ *as*

$$\bar{\boldsymbol{A}}_\nu = \boldsymbol{A}_\nu + \boldsymbol{A}^*_{\nu+1}\boldsymbol{ea}, \qquad \bar{\boldsymbol{B}}_\nu = \boldsymbol{B}_\nu + \boldsymbol{B}^*_{\nu+1}\boldsymbol{ea},$$

and $\boldsymbol{e}$ *is a column vector with unit elements.*

Proof. Since $\{(X_n, V_n)\}$ is a Markov chain of M/G/1-type, we have [14]

$$\boldsymbol{x}_k = \left(\boldsymbol{x}_0\tilde{\boldsymbol{B}}_k + \sum_{j=1}^{k-1}\boldsymbol{x}_j\tilde{\boldsymbol{A}}_{k+1-j}\right)\left(\boldsymbol{I}-\tilde{\boldsymbol{A}}_1\right)^{-1}, \quad k = 1, 2, \ldots, \tag{6}$$

where $\tilde{\boldsymbol{A}}_k$ $(k = 1, 2, \ldots)$ and $\tilde{\boldsymbol{B}}_k$ $(k = 1, 2, \ldots)$ are substochastic matrices, which are given by

$$\tilde{\boldsymbol{A}}_k = \sum_{j=k}^{\infty}\boldsymbol{A}_j\boldsymbol{G}^{j-k}, \qquad \tilde{\boldsymbol{B}}_k = \sum_{j=k}^{\infty}\boldsymbol{B}_j\boldsymbol{G}^{j-k},$$

and $\boldsymbol{G}$ is an $(L+1)\times(L+1)$ stochastic matrix whose (i, j)th element denotes the conditional probability that the Markov chain $\{(X_n, V_n)\}$ starting in state $(l+1, i)$ (for any level l) will reach level l eventually and end up in phase j when it reaches level l. On the other hand, we see that for any $i = 0, \ldots, L$ and $j = 0, \ldots, L$, we have $[\boldsymbol{a}]_j = [\boldsymbol{G}]_{i,j}$ from Assumption 1. We thus obtain

$$\boldsymbol{G} = \boldsymbol{ea}. \tag{7}$$

From (6) and (7), we see that (5) holds.

Let $\boldsymbol{H}$ denote an $(L+1)\times(L+1)$ stochastic matrix whose (i, j)th element denotes the conditional probability that the Markov chain $\{(X_n, V_n)\}$ starting in state $(0, i)$ will reach level 0 eventually and end up in phase j when it reaches level 0. From (7), we then have [13]

$$\boldsymbol{H} = \sum_{k=0}^{\infty}\boldsymbol{B}_k\boldsymbol{G}^k = \boldsymbol{B}_0 + \boldsymbol{B}^*_1\boldsymbol{ea} = \bar{\boldsymbol{B}}_0. \tag{8}$$

Since $\boldsymbol{x}_0$ is an invariant vector for $\boldsymbol{H}$ [13], we obtain

$$\boldsymbol{x}_0 = \boldsymbol{x}_0\boldsymbol{H}. \tag{9}$$

From (8) and (9), we see that (4) holds. □

The following lemma for the stationary distribution $\boldsymbol{y}$ is readily obtained as a special case of the result shown in [5].

Lemma 2. *Under Assumptions* 1 *and* 2, *the stationary distribution* $\boldsymbol{y}$ *is determined by the equations*

$$\boldsymbol{y}_0 = \boldsymbol{y}_0\bar{\boldsymbol{B}}_0, \tag{10}$$

$$\boldsymbol{y}_i = \left(\boldsymbol{y}_0 \bar{\boldsymbol{B}}_i + \sum_{k=1}^{i-1} \boldsymbol{y}_k \bar{\boldsymbol{A}}_{i-k+1}\right)(\boldsymbol{I} - \bar{\boldsymbol{A}}_1)^{-1}, \quad i = 1, \ldots, K-1, \tag{11}$$

$$\boldsymbol{y}_K = \left(\boldsymbol{y}_0 \boldsymbol{B}_K + \sum_{k=1}^{K-1} \boldsymbol{y}_k \boldsymbol{A}^*_{K-k+1}\right)(\boldsymbol{I} - \boldsymbol{A}^*_1)^{-1}, \tag{12}$$

$$\sum_{n=0}^{K} \boldsymbol{y}_n \boldsymbol{e} = 1,$$

where for $\nu = 0, \ldots, K-1$, *we define* $\bar{\boldsymbol{A}}_\nu$ *and* $\bar{\boldsymbol{B}}_\nu$ as

$$\bar{\boldsymbol{A}}_\nu = \boldsymbol{A}_\nu + \boldsymbol{A}^*_{\nu+1}\boldsymbol{e}\boldsymbol{a}, \qquad \bar{\boldsymbol{B}}_\nu = \boldsymbol{B}_\nu + \boldsymbol{B}^*_{\nu+1}\boldsymbol{e}\boldsymbol{a},$$

and $\boldsymbol{e}$ *is a column vector with unit elements.*

The following theorem, which establishes a proportional relation holding between the stationary distribution of the M/G/1-type Markov chain and that of the corresponding truncated Markov chain, is a conclusion of Lemmas 1 and 2.

Theorem 1. *Under Assumptions* 1 *and* 2, *there exists a constant c such that*

$$\boldsymbol{y}_i = c\boldsymbol{x}_i, \quad i = 0, 1, \ldots, K-1. \tag{13}$$

In addition, the proportional constant c in (13) *can be expressed as*

$$c = \frac{\boldsymbol{\pi} \boldsymbol{A}_0 \boldsymbol{e}}{\sum_{i=0}^{K} \boldsymbol{x}_i \boldsymbol{A}_0 \boldsymbol{e}}, \tag{14}$$

where $\boldsymbol{\pi}$ *is a* $1 \times (L+1)$ *vector whose jth element is given by* $[\boldsymbol{\pi}]_j = \mathrm{P}(V_n = j)$.

Proof. First we recursively show that (13) holds. Recall that from (4), (8) and (10), both $\boldsymbol{x}_0$ and $\boldsymbol{y}_0$ are invariant vectors for the stochastic matrix $\boldsymbol{H} = \bar{\boldsymbol{B}}_0$. Since the Markov chains $\{(X_n, V_n)\}$ and $\{(Y_n, V_n)\}$ are irreducible and positive recurrent from Assumption 2, there exists some constant satisfying $\boldsymbol{y}_0 = c\boldsymbol{x}_0$. We thus see that (13) holds for $i = 0$. Suppose that (13) holds for $i = 0, \ldots, k-1$, where k is some integer satisfying $k \in \{1, \ldots, K-1\}$. Then, since (5) and (11) are identical recursions, we see that $\boldsymbol{y}_k = c\boldsymbol{x}_k$ is satisfied. We therefore show that there exists a constant c such that $\boldsymbol{y}_i = c\boldsymbol{x}_i$ for $i = 0, 1, \ldots, K-1$.

Next we show (14) along with similar lines of the proof shown in [8]. From (3), we have

$$\boldsymbol{y}_K = \boldsymbol{y}_0 \boldsymbol{B}^*_K + \sum_{i=1}^{K} \boldsymbol{y}_i \boldsymbol{A}^*_{K-i+1}. \tag{15}$$

Using (13) and noting $\sum_{i=0}^{K} \boldsymbol{y}_i = \boldsymbol{\pi}$, we rewrite (15) to

$$\left(\boldsymbol{\pi} - c\sum_{i=0}^{K-1} \boldsymbol{x}_i\right)\boldsymbol{e} = c\boldsymbol{x}_0 \boldsymbol{B}^*_K \boldsymbol{e} + c\sum_{i=1}^{K-1} \boldsymbol{x}_i \boldsymbol{A}^*_{K-i+1}\boldsymbol{e} + \left(\boldsymbol{\pi} - c\sum_{i=0}^{K-1} \boldsymbol{x}_i\right)\boldsymbol{A}^*_1 \boldsymbol{e}, \tag{16}$$

where c is a constant appearing in (13). From (16), it follows that

$$\boldsymbol{\pi}(\boldsymbol{I} - \boldsymbol{A}_1^*)\boldsymbol{e} = c\left[\boldsymbol{x}_0\boldsymbol{B}_K^* + \sum_{i=1}^{K-1}\boldsymbol{x}_i\boldsymbol{A}_{K-i+1}^* + \sum_{i=0}^{K-1}\boldsymbol{x}_i(\boldsymbol{I} - \boldsymbol{A}_1^*)\right]\boldsymbol{e}. \tag{17}$$

Note that we have

$$(\boldsymbol{I} - \boldsymbol{A}_1^*)\boldsymbol{e} = \boldsymbol{A}_0\boldsymbol{e}. \tag{18}$$

Also note that the following equilibrium equation holds:

$$\boldsymbol{x}_0\boldsymbol{B}_K^* + \sum_{i=1}^{K-1}\boldsymbol{x}_i\boldsymbol{A}_{K-i+1}^* = \boldsymbol{x}_K\boldsymbol{A}_0, \tag{19}$$

where the left-hand side of (19) denotes the total flow from a macro-state which composes of the states that the level is less than K into a macro-state which composes of the states that the level is greater than or equal to K, and the right-hand side of (19) denotes the total flow from the latter macro-state into the former macro-state. Using (18) and (19) in (17), we obtain

$$\boldsymbol{\pi}\boldsymbol{A}_0\boldsymbol{e} = c\sum_{i=0}^{K}\boldsymbol{x}_i\boldsymbol{A}_0\boldsymbol{e}. \tag{20}$$

Note here that $\sum_{i=0}^{K}\boldsymbol{x}_i\boldsymbol{A}_0\boldsymbol{e} > 0$ under Assumption 2. We therefore derive (14) from (20). $\square$

3. Queueing model

In this section, we consider two queueing models, i.e., a discrete-time finite-buffer queue with correlated arrivals and service interruptions and an infinite-buffer queue that is obtained from the finite-buffer queue by eliminating the queue length constraint. Both queueing systems are identical except for the buffer capacity. The queueing models are considered as an extension of ones considered in [8,6]. The dynamics of the infinite-buffer queue and that of the finite-buffer queue will be described by the Markov chains $\{(X_n, V_n)\}$ and $\{(Y_n, V_n)\}$ which are defined in the previous section, respectively.

We first sketch the queueing systems below. Time is slotted and the slot length is equal to a unit time. The arrivals of batches occur at the beginnings of slots immediately after departures (i.e., early arrival model [17]). The service times of customers are i.i.d. (independent and identically distributed) and they follow a geometric distribution with mean $1/\gamma\,(\gamma > 0)$. As described later, the service is interrupted in the nth slot if the Markov chain for the service process is in a particular state. The service of a customer starts at the beginning of a slot and ends at the end of the slot (i.e., on slot boundaries). The finite-buffer queue accommodates at most K customers including the one in service. Thus, if m ($m \geq K - k + 1$) customers arrive to find k customers (including the one in service) in the system, only $K - k$ customers are accommodated in the system, and the remaining $m - (K - k)$ customers are discarded. On the other hand, the infinite-buffer queue accommodates all arriving customers and no customers are discarded.

We now describe the queueing systems in more detail. We begin with the description of the service process. To describe the service process, we introduce a Markov chain. Let $\{S_n\}_{n\in\mathbb{Z}_+}$ denote a Markov chain on $\mathcal{R} = \{0, \ldots, R\}$ where R is a nonnegative integer. The service is available in the nth slot if and only if

$S_n = 0$. We call the Markov chain $\{S_n\}$ the underlying Markov chain for the service process. We assume that the underlying Markov chain for the service process is stationary and ergodic. We next describe the arrival process. Let $\{A_n\}_{n\in\mathbb{Z}}$ denote a stochastic sequence where A_n represents the number of arrivals in the nth slot. We assume that $\{A_n\}_{n\in\mathbb{Z}}$ is governed by a Markov chain $\{P_n\}_{n\in\mathbb{Z}_+}$ on $\mathcal{M} = \{0, \dots, M\}$ where M denotes a nonnegative integer. More precisely, we assume that given P_n, A_n is conditionally independent of all other random variables. We call the Markov chain $\{P_n\}$ the underlying Markov chain for the arrival process. We assume that the underlying Markov chain for the arrival process is stationary and ergodic. We now consider the queueing processes. Let $\{X_n\}_{n\in\mathbb{Z}_+}$ and $\{Y_n\}_{n\in\mathbb{Z}_+}$ denote a stochastic sequence representing the queue length (including a customer in service) in the infinite-buffer queue and that in the finite-buffer queue, respectively. Let $\{D_n\}_{n\in\mathbb{Z}_+}$ denote a Bernoulli sequence on $\{0, 1\}$ where $\mathrm{P}(D_n = 1) = \gamma$ and $\mathrm{P}(D_n = 0) = 1 - \gamma$ for $n \in \mathbb{Z}_+$. We assume that $\{S_n\}$, $\{P_n\}$ and $\{D_n\}$ are independent with each other. The queueing processes $\{X_n\}$ and $\{Y_n\}$ evolve according to the following recursions with initial queue length X_0 and Y_0:

$$X_{n+1} = (X_n - 1_{\{S_n=0\}} D_n)^+ + A_{n+1}, \qquad Y_{n+1} = \min[(Y_n - 1_{\{S_n=0\}} D_n)^+ + A_{n+1}, K],$$

where $(\cdot)^+ = \min(\cdot, 0)$ and 1 denotes the indicator function. Let $Z_n (n \in \mathbb{Z})$ denote a random variable representing the number of lost customers in the nth slot in the finite-buffer queue. $Z_n (n \in \mathbb{Z})$ is given by

$$Z_n = ((Y_{n-1} - 1_{\{S_{n-1}=0\}} D_{n-1})^+ + A_n - K)^+.$$

Now we describe a stochastic setting for the arrival and service processes. First, we describe the stochastic setting for the arrival process. Let $\hat{\boldsymbol{U}}$ denote the transition matrix of the underlying Markov chain for the arrival process, i.e. $[\hat{\boldsymbol{U}}]_{i,j} = \mathrm{P}(P_{n+1} = j \mid P_n = i)$ for $i, j \in \mathcal{M}$. Let $\hat{\boldsymbol{\pi}}$ denote the stationary vector of the underlying Markov chain for the arrival process, i.e. $[\hat{\boldsymbol{\pi}}]_i = \mathrm{P}(P_n = i)$. $\hat{\boldsymbol{\pi}}$ then satisfies $\hat{\boldsymbol{\pi}} = \hat{\boldsymbol{\pi}}\hat{\boldsymbol{U}}$ and $\hat{\boldsymbol{\pi}}\boldsymbol{e} = 1$. We denote by $\hat{a}_j(k)$ the conditional probability that k customers arrive given that the underlying Markov chain is in state j:

$$\hat{a}_j(k) = \mathrm{P}(A_n = k \mid P_n = j), \quad j \in \mathcal{M},\ k = 0, 1, 2, \dots.$$

Let $\hat{A}_{i,j}(k)$ denote the conditional joint probability of the following events: k customers arrive in the $(n + 1)$st slot and the underlying Markov chain is in state j in the $(n + 1)$st slot, given that the underlying Markov chain was in state i in the nth slot. Namely,

$$\hat{A}_{i,j}(k) = \mathrm{P}(A_{n+1} = k, P_{n+1} = j \mid P_n = i) = \hat{a}_j(k)[\hat{\boldsymbol{U}}]_{i,j}, \quad i, j \in \mathcal{M}.$$

Let $\hat{\boldsymbol{A}}_k$ and $\hat{\boldsymbol{A}}_k^*$ $(k = 0, 1, \dots)$ denote $(M + 1) \times (M + 1)$ matrices whose (i, j)th elements are given by $\hat{A}_{i,j}(k)$ and $\sum_{m=k}^{\infty} \hat{A}_{i,j}(m)$, respectively. Note that $\hat{\boldsymbol{A}}_k$ (resp. $\hat{\boldsymbol{A}}_k^*$) represents the transition matrix of the underlying Markov chain when k customers (resp. more than or equal to k customers) arrive at the system.

Next, we describe the stochastic setting for the service process. Let $\tilde{\boldsymbol{U}}$ denote the one-step state transition matrix of the underlying Markov chain for the service process, i.e. $[\tilde{\boldsymbol{U}}]_{i,j} = \mathrm{P}(S_{n+1} = j \mid S_n = i)$ for

$i, j \in \mathcal{R}$. Further, we define $\tilde{\boldsymbol{U}}_0$ and $\tilde{\boldsymbol{U}}_1$ as

$$[\tilde{\boldsymbol{U}}_0]_{i,j} = \begin{cases} [\tilde{\boldsymbol{U}}]_{i,j} & (j = 0), \\ 0 & (j \neq 0), \end{cases} \qquad [\tilde{\boldsymbol{U}}_1]_{i,j} = \begin{cases} [\tilde{\boldsymbol{U}}]_{i,j} & (j \neq 0), \\ 0 & (j = 0), \end{cases}$$

Note here that we have $\tilde{\boldsymbol{U}} = \tilde{\boldsymbol{U}}_0 + \tilde{\boldsymbol{U}}_1$. Let $\tilde{\boldsymbol{\pi}}$ denote the stationary vector of the underlying Markov chain for the service process, i.e. $[\tilde{\boldsymbol{\pi}}]_i = \mathrm{P}(S_n = i)$. $\tilde{\boldsymbol{\pi}}$ then satisfies $\tilde{\boldsymbol{\pi}} = \tilde{\boldsymbol{\pi}}\tilde{\boldsymbol{U}}$ and $\tilde{\boldsymbol{\pi}}\boldsymbol{e} = 1$.

Finally we will give a series of definitions and assumptions in order that the dynamics of the queues are described by the Markov chains presented in Section 2. For this purpose, we begin with the definition of random variables $V_n (n = 0, 1, \ldots)$. We first consider a mapping $f : \mathcal{S} \times \mathcal{M} \to \mathcal{V}$ defined as

$$f(x, y) = (M + 1)x + y,$$

where $\mathcal{V} = \{0, \ldots, (M + 1)(R + 1) - 1\}$. We then define random variables $V_n (n = 0, 1, \ldots)$ on $\mathcal{V}$ by

$$V_n = f(S_n, P_n) = (M + 1)S_n + P_n.$$

We next define an $(M + 1)(R + 1) \times (M + 1)(R + 1)$ matrix $\boldsymbol{A}_i (i = 0, 1, \ldots)$ by

$$\boldsymbol{A}_i = \gamma\tilde{\boldsymbol{U}}_0 \otimes \hat{\boldsymbol{A}}_i + ((1 - \gamma)\tilde{\boldsymbol{U}}_0 + \tilde{\boldsymbol{U}}_1) \otimes \hat{\boldsymbol{A}}_{i-1}, \tag{21}$$

where $\otimes$ denotes the Kronecker product and for notational convenience, we define $\hat{\boldsymbol{A}}_{-1}$ as $\hat{\boldsymbol{A}}_{-1} = \boldsymbol{O}$. Similarly, we define an $(M + 1)(R + 1) \times (M + 1)(R + 1)$ matrix $\boldsymbol{B}_i$ $(i = 0, 1, \ldots)$ by

$$\boldsymbol{B}_i = \tilde{\boldsymbol{U}} \otimes \hat{\boldsymbol{A}}_i. \tag{22}$$

Note that obviously $\{(X_n, V_n)\}$ and $\{(Y_n, V_n)\}$ defined in this section become Markov chains whose transition matrices are given by (1) and (2), respectively, with $L = (M + 1)(R + 1) - 1$. Let $\boldsymbol{\pi}$ denote the stationary vector of the Markov chain $\{V_n\}$, i.e. $[\boldsymbol{\pi}]_i = \mathrm{P}(V_n = i)$. Since the underlying Markov chain $\{P_n\}$ for the arrival process and the underlying Markov chain $\{S_n\}$ for the service process are independent, $\boldsymbol{\pi}$ is given by

$$\boldsymbol{\pi} = \tilde{\boldsymbol{\pi}} \otimes \hat{\boldsymbol{\pi}}. \tag{23}$$

We assume that the Markov chains $\{(X_n, V_n)\}$ and $\{(Y_n, V_n)\}$ defined in this section satisfy Assumptions 1 and 2.

As shown below, we can also replace Assumption 1 with the following two assumptions, which are more directly associated with the queueing models.

Assumption 3. There exists a $1 \times (M + 1)$ probability vector $\hat{\boldsymbol{a}}$ such that $\hat{\boldsymbol{A}}_0 = \hat{\boldsymbol{A}}_0\boldsymbol{e}\hat{\boldsymbol{a}}$.

Assumption 4. There exists a $1 \times (R + 1)$ probability vector $\tilde{\boldsymbol{a}}$ such that $\tilde{\boldsymbol{U}}_0 = \tilde{\boldsymbol{U}}_0\boldsymbol{e}\tilde{\boldsymbol{a}}$.

Assumption 3 means that an active period of the arrival process is subject to a geometric distribution and an inactive period to a phase-type distribution. Similarly, Assumption 4 means that an active period of the server is subject to a geometric distribution and an inactive period to a phase-type distribution. We here define a $1 \times (M + 1)(R + 1)$ probability vector $\boldsymbol{a}$ by $\boldsymbol{a} = \tilde{\boldsymbol{a}} \otimes \hat{\boldsymbol{a}}$. We can readily confirm that if Assumptions 3 and 4 are satisfied, Assumptions 1 is satisfied.

Proposition 1. *Under Assumptions 3 and 4, we have* $\boldsymbol{A}_0 = \boldsymbol{A}_0\boldsymbol{e}\boldsymbol{a}$.

Proof. From the definition (21) of $\boldsymbol{A}_0$ and Assumptions 3 and 4, we have

$$\boldsymbol{A}_0\boldsymbol{e}\boldsymbol{a} = \gamma(\tilde{\boldsymbol{U}}_0 \otimes \hat{\boldsymbol{A}}_0)\boldsymbol{e}(\tilde{\boldsymbol{a}} \otimes \hat{\boldsymbol{a}}) = \gamma(\tilde{\boldsymbol{U}}_0\boldsymbol{e}\tilde{\boldsymbol{a}}) \otimes (\hat{\boldsymbol{A}}_0\boldsymbol{e}\hat{\boldsymbol{a}}) = \gamma\tilde{\boldsymbol{U}}_0 \otimes \hat{\boldsymbol{A}}_0 = \boldsymbol{A}_0. \qquad \square$$

In the setting made in this section and under Assumptions 2–4 (or Assumptions 1 and 2), Theorem 1 holds for the Markov chains $\{(X_n, V_n)\}$ and $\{(Y_n, V_n)\}$ defined in this section, and it establishes a proportional relation between the stationary queue length distribution in the finite-buffer queue and that in the corresponding infinite-buffer queue.

4. Exact relation between loss probability and queue length

In this section, we will establish an exact relation holding between the loss probability in the finite-buffer queue and the queue length distribution in the infinite-buffer queue. The exact relation is directly derived from the proportional relation (Theorem 1).

We define the loss probability P_{loss} in the finite-buffer queue as

$$P_{\mathrm{loss}} \stackrel{\triangle}{=} \frac{\mathrm{E}[Z_n]}{\mathrm{E}[A_n]}. \tag{24}$$

Let ρ denote the traffic intensity which is given by

$$\rho \stackrel{\triangle}{=} \frac{1}{\gamma}\mathrm{E}[A_n] = \frac{1}{\gamma}\hat{\boldsymbol{\pi}}\sum_{k=1}^{\infty} k\hat{\boldsymbol{A}}_k\boldsymbol{e}. \tag{25}$$

We assume that $\rho < [\tilde{\boldsymbol{\pi}}]_0$. This assumption guarantees that the infinite-buffer queue is stable and the Markov chain $\{(X_n, V_n)\}$ is positive recurrent.

The following formula for the loss probability immediately follows.

Proposition 2. *Under Assumption 2, P_{loss} is given by*

$$P_{\mathrm{loss}} = 1 - \frac{1}{\rho}[[\tilde{\boldsymbol{\pi}}]_0 - \boldsymbol{y}_0[(\tilde{\boldsymbol{U}}_0\boldsymbol{e}) \otimes \boldsymbol{e}]]. \tag{26}$$

Proof. From Rate Conservation Law (see, e.g. [12]) or Little's formula, it immediately follows that

$$\mathrm{E}[A_n] - \mathrm{E}[Z_n] = \gamma\sum_{k=1}^{K} \boldsymbol{y}_k(\tilde{\boldsymbol{U}}_0 \otimes \hat{\boldsymbol{U}})\boldsymbol{e}, \tag{27}$$

where the left-hand side is the expected upward drift of the queue length and the right-hand side is the expected downward drift of the queue length. From (24), (25) and (27), we have

$$P_{\mathrm{loss}} = 1 - \frac{1}{\rho}\sum_{k=1}^{K} \boldsymbol{y}_k(\tilde{\boldsymbol{U}}_0 \otimes \hat{\boldsymbol{U}})\boldsymbol{e}. \tag{28}$$

Noting $\sum_{k=0}^{K} \boldsymbol{y}_k = \boldsymbol{\pi}$ and (23), we rewrite $\sum_{k=1}^{K} \boldsymbol{y}_k(\tilde{\boldsymbol{U}}_0 \otimes \hat{\boldsymbol{U}})\boldsymbol{e}$ as

$$\begin{aligned}\sum_{k=1}^{K} \boldsymbol{y}_k(\tilde{\boldsymbol{U}}_0 \otimes \hat{\boldsymbol{U}})\boldsymbol{e} &= \sum_{k=1}^{K} \boldsymbol{y}_k[(\tilde{\boldsymbol{U}}_0\boldsymbol{e}) \otimes (\hat{\boldsymbol{U}}\boldsymbol{e})] = \sum_{k=1}^{K} \boldsymbol{y}_k[(\tilde{\boldsymbol{U}}_0\boldsymbol{e}) \otimes \boldsymbol{e}] \\ &= \sum_{k=0}^{K} \boldsymbol{y}_k[(\tilde{\boldsymbol{U}}_0\boldsymbol{e}) \otimes \boldsymbol{e}] - \boldsymbol{y}_0[(\tilde{\boldsymbol{U}}_0\boldsymbol{e}) \otimes \boldsymbol{e}] = \boldsymbol{\pi}[(\tilde{\boldsymbol{U}}_0\boldsymbol{e}) \otimes \boldsymbol{e}] - \boldsymbol{y}_0[(\tilde{\boldsymbol{U}}_0\boldsymbol{e}) \otimes \boldsymbol{e}] \\ &= (\tilde{\boldsymbol{\pi}} \otimes \hat{\boldsymbol{\pi}})[(\tilde{\boldsymbol{U}}_0\boldsymbol{e}) \otimes \boldsymbol{e}] - \boldsymbol{y}_0[(\tilde{\boldsymbol{U}}_0\boldsymbol{e}) \otimes \boldsymbol{e}] = \tilde{\boldsymbol{\pi}}\tilde{\boldsymbol{U}}_0\boldsymbol{e} - \boldsymbol{y}_0[(\tilde{\boldsymbol{U}}_0\boldsymbol{e}) \otimes \boldsymbol{e}] \\ &= [\tilde{\boldsymbol{\pi}}]_0 - \boldsymbol{y}_0[(\tilde{\boldsymbol{U}}_0\boldsymbol{e}) \otimes \boldsymbol{e}]. \end{aligned} \tag{29}$$

Substituting (29) into (28), we obtain (26). □

The following theorem establishes an exact relation holding between the loss probability in the finite-buffer queue and the stationary queue length distribution in the infinite-buffer queue, and the exact relation expresses the loss probability in the finite-buffer queue as a function of the stationary queue length distribution in the infinite-buffer queue.

Theorem 2. *Under Assumptions 2–4, the loss probability P_{loss} is given in terms of the stationary distribution $\boldsymbol{x}$ as follows*:

$$P_{\text{loss}} = \frac{([\tilde{\boldsymbol{\pi}}]_0 - \rho)\sum_{i=K+1}^{\infty} \boldsymbol{x}_i \boldsymbol{A}_0\boldsymbol{e}}{\rho \sum_{i=0}^{K} \boldsymbol{x}_i \boldsymbol{A}_0\boldsymbol{e}}.$$

Proof. By similar argument when we derived (27), we obtain

$$\frac{1}{\gamma}\mathrm{E}[A_n] = \sum_{k=1}^{\infty} \boldsymbol{x}_k(\tilde{\boldsymbol{U}}_0 \otimes \hat{\boldsymbol{U}})\boldsymbol{e}. \tag{30}$$

By similar argument when we derived (29), the right-hand side of (30) can be rewritten as

$$\sum_{k=1}^{\infty} \boldsymbol{x}_k(\tilde{\boldsymbol{U}}_0 \otimes \hat{\boldsymbol{U}})\boldsymbol{e} = [\tilde{\boldsymbol{\pi}}]_0 - \boldsymbol{x}_0\left[(\tilde{\boldsymbol{U}}_0\boldsymbol{e}) \otimes \boldsymbol{e}\right]. \tag{31}$$

Using (25) and (31) in (30), we derive

$$\boldsymbol{x}_0[(\tilde{\boldsymbol{U}}_0\boldsymbol{e}) \otimes \boldsymbol{e}] = [\tilde{\boldsymbol{\pi}}]_0 - \rho. \tag{32}$$

From Theorem 1 and Proposition 2, using (32) and noting $\sum_{i=0}^{\infty} \boldsymbol{x}_i = \boldsymbol{\pi}$, we have

$$\begin{aligned} P_{\text{loss}} &= 1 - \frac{1}{\rho}\left[[\tilde{\boldsymbol{\pi}}]_0 - \frac{\boldsymbol{\pi}\boldsymbol{A}_0\boldsymbol{e}\boldsymbol{x}_0[(\tilde{\boldsymbol{U}}_0\boldsymbol{e}) \otimes \boldsymbol{e}]}{\sum_{i=0}^{K} \boldsymbol{x}_i \boldsymbol{A}_0\boldsymbol{e}}\right] = 1 - \frac{1}{\rho}\left[[\tilde{\boldsymbol{\pi}}]_0 - \frac{\boldsymbol{\pi}\boldsymbol{A}_0\boldsymbol{e}([\tilde{\boldsymbol{\pi}}]_0 - \rho)}{\sum_{i=0}^{K} \boldsymbol{x}_i \boldsymbol{A}_0\boldsymbol{e}}\right] \\ &= 1 + \frac{[\tilde{\boldsymbol{\pi}}]_0 \sum_{i=K+1}^{\infty} \boldsymbol{x}_i \boldsymbol{A}_0\boldsymbol{e} - \rho\boldsymbol{\pi}\boldsymbol{A}_0\boldsymbol{e}}{\rho \sum_{i=0}^{K} \boldsymbol{x}_i \boldsymbol{A}_0\boldsymbol{e}} = \frac{[\tilde{\boldsymbol{\pi}}]_0 \sum_{i=K+1}^{\infty} \boldsymbol{x}_i \boldsymbol{A}_0\boldsymbol{e} - \rho \sum_{i=K+1}^{\infty} \boldsymbol{x}_i \boldsymbol{A}_0\boldsymbol{e}}{\rho \sum_{i=0}^{K} \boldsymbol{x}_i \boldsymbol{A}_0\boldsymbol{e}} \\ &= \frac{([\tilde{\boldsymbol{\pi}}]_0 - \rho)\sum_{i=K+1}^{\infty} \boldsymbol{x}_i \boldsymbol{A}_0\boldsymbol{e}}{\rho \sum_{i=0}^{K} \boldsymbol{x}_i \boldsymbol{A}_0\boldsymbol{e}}. \quad \square \end{aligned}$$

Remark 1. The exact relation (Theorem 2) is considered as a generalization/integration of the exact relations shown in [6,8,10], and Theorem 2 includes those exact relations as special cases.

Remark 2. From Theorem 2, we immediately obtain

$$P_{\rm loss} = \frac{([\tilde{\boldsymbol{\pi}}]_0 - \rho)\left(\boldsymbol{\pi} \boldsymbol{A}_0 \boldsymbol{e} - \sum_{i=0}^{K} \boldsymbol{x}_i \boldsymbol{A}_0 \boldsymbol{e}\right)}{\rho \sum_{i=0}^{K} \boldsymbol{x}_i \boldsymbol{A}_0 \boldsymbol{e}}. \tag{33}$$

Using this equation with the recursion given in Lemma 1, we can efficiently compute the exact loss probability $P_{\rm loss}$ for any positive integer K.

5. Asymptotic loss probability

When we use Theorem 2 or Remark 2 to calculate the loss probability in the finite-buffer queue, we need to compute $\boldsymbol{x}$. The computation of the $\boldsymbol{x}$ is not an easy task when M or R are large. In this section, we develop a formula which can estimate the loss probability more easily even when M or R are large. For this purpose, we exploit the property that the tail distribution of the queue length in infinite-buffer queues has a rather simple asymptotic form in many cases. In particular, since $\{(X_n, V_n)\}$ is an M/G/1 type Markov chain, the tail distribution $\sum_{k=N+1}^{\infty} \boldsymbol{x}_k$ has a simple geometric asymptotic form under some conditions. Exploiting this property, the formula derived in this section computes the *asymptotic loss probability* [6,8].

We begin with the definition of notations which will appear in the formula. We define an $(M+1)(R+1) \times (M+1)(R+1)$ matrix generating function $\boldsymbol{A}(z)$, an $(M+1) \times (M+1)$ matrix generating function $\hat{\boldsymbol{A}}(z)$ and an $(R+1) \times (R+1)$ matrix generating function $\tilde{\boldsymbol{U}}_\gamma(z)$ as

$$\boldsymbol{A}(z) = \sum_{k=0}^{\infty} \boldsymbol{A}_k z^k, \qquad \hat{\boldsymbol{A}}(z) = \sum_{k=0}^{\infty} \hat{\boldsymbol{A}}_k z^k, \qquad \tilde{\boldsymbol{U}}_\gamma(z) = (\gamma + (1-\gamma)z)\tilde{\boldsymbol{U}}_0 + z\tilde{\boldsymbol{U}}_1. \tag{34}$$

Note here that from (21), we have

$$\boldsymbol{A}(z) = \tilde{\boldsymbol{U}}_\gamma(z) \otimes \hat{\boldsymbol{A}}(z). \tag{35}$$

Let $\delta(z)$ denote the Perron–Frobenius eigenvalue of $\boldsymbol{A}(z)$, and $\boldsymbol{u}(z)$ and $\boldsymbol{v}(z)$ denote its left and right eigenvectors which satisfy the normalizing conditions: $\boldsymbol{u}(z)\boldsymbol{v}(z) = 1$ and $\boldsymbol{u}(z)\boldsymbol{e} = 1$. Also, let $\hat{\delta}(z)$ denote the Perron–Frobenius eigenvalue of $\hat{\boldsymbol{A}}(z)$, and $\hat{\boldsymbol{u}}(z)$ and $\hat{\boldsymbol{v}}(z)$ denote its left and right eigenvectors which satisfy the normalizing conditions: $\hat{\boldsymbol{u}}(z)\hat{\boldsymbol{v}}(z) = 1$ and $\hat{\boldsymbol{u}}(z)\boldsymbol{e} = 1$. Similarly, let $\tilde{\delta}(z)$ denote the Perron–Frobenius eigenvalue of $\tilde{\boldsymbol{U}}_\gamma(z)$, and $\tilde{\boldsymbol{u}}(z)$ and $\tilde{\boldsymbol{v}}(z)$ denote its left and right eigenvectors which satisfy the normalizing conditions: $\tilde{\boldsymbol{u}}(z)\tilde{\boldsymbol{v}}(z) = 1$ and $\tilde{\boldsymbol{u}}(z)\boldsymbol{e} = 1$. Note here that from (35), we have [13]

$$\delta(z) = \tilde{\delta}(z)\hat{\delta}(z), \qquad \boldsymbol{u}(z) = \tilde{\boldsymbol{u}}(z) \otimes \hat{\boldsymbol{u}}(z), \qquad \boldsymbol{v}(z) = \tilde{\boldsymbol{v}}(z) \otimes \hat{\boldsymbol{v}}(z). \tag{36}$$

Now we make several assumptions [3,7] to ensure that the queue length has a simple asymptotic expression.

Assumption 5.

- There exists at least one zero of $\det[z\boldsymbol{I} - \boldsymbol{A}(z)]$ outside the unit disk, where $\boldsymbol{I}$ denotes the $(M+1)(R+1) \times (M+1)(R+1)$ identity matrix.
- Among those, there exists a real and positive zero z^*, and the absolute value of z^* is strictly smaller than those of other zeros.
- $\boldsymbol{O} < \boldsymbol{A}(z) \ll +\infty$, $1 \le z \le z^*$, $z \in \mathbb{R}$, where $\mathbb{R}$ denotes the set of all real numbers.

The following formula enables us to compute the asymptotic loss probability.

Theorem 3. *Under Assumptions 2–5, the loss probability P_{loss} is asymptotically expressed as*

$$P_{\text{loss}} \approx \left(\frac{1}{\rho} - \frac{1}{[\tilde{\boldsymbol{\pi}}]_0}\right) \frac{\boldsymbol{x}_0[(\tilde{\boldsymbol{U}}_0\tilde{\boldsymbol{v}}(z^*)) \otimes \hat{\boldsymbol{v}}(z^*)]}{\tilde{\delta}(z^*)(\delta'(z^*) - 1)} \frac{\boldsymbol{u}(z^*)\boldsymbol{A}_0\boldsymbol{e}}{\hat{\boldsymbol{\pi}}\hat{\boldsymbol{A}}_0\boldsymbol{e}}(z^*)^{-K}. \tag{37}$$

Remark 3. The formula (37) to compute the asymptotic loss probability is a generalization of the formula (Corollary 5) derived in [8]. When $R = 0$, $\tilde{\boldsymbol{U}}_0 = 1$ and $\tilde{\boldsymbol{U}}_1 = 0$, the formula (37) is reduced to the formula in [8].

To prove Theorem 3, we need a proposition and a corollary. The following proposition shows that the tail distribution of the queue length in the infinite-buffer queue has a simple geometric expression. The proof is provided in Appendix A.

Proposition 3. *Under Assumptions 2 and 5, $\sum_{n=N+1}^{\infty} \boldsymbol{x}_n$ is expressed as*

$$\sum_{n=N+1}^{\infty} \boldsymbol{x}_n = \frac{\gamma \boldsymbol{x}_0[(\tilde{\boldsymbol{U}}_0\tilde{\boldsymbol{v}}(z^*)) \otimes \hat{\boldsymbol{v}}(z^*)]}{\tilde{\delta}(z^*)(\delta'(z^*) - 1)}(z^*)^{-N}\boldsymbol{u}(z^*) + o((z^*)^{-N})\boldsymbol{e}^{\mathrm{T}}, \quad N \ge 0, \tag{38}$$

where $\boldsymbol{e}^{\mathrm{T}}$ denotes a $1 \times (M+1)(R+1)$ row vector with unit elements, and z^* is the minimum real solution of $z = \delta(z)$ for $z \in (1, \infty)$.

The following corollary is directly obtained from Theorem 2.

Corollary 1. *Under Assumptions 2–4, the loss probability P_{loss} is expressed as*

$$P_{\text{loss}} = \frac{([\tilde{\boldsymbol{\pi}}]_0 - \rho)\sum_{i=K+1}^{\infty} \boldsymbol{x}_i\boldsymbol{A}_0\boldsymbol{e}}{\rho\left(\gamma[\tilde{\boldsymbol{\pi}}]_0\hat{\boldsymbol{\pi}}\hat{\boldsymbol{A}}_0\boldsymbol{e} - \sum_{i=K+1}^{\infty} \boldsymbol{x}_i\boldsymbol{A}_0\boldsymbol{e}\right)}. \tag{39}$$

Proof. First, note that from the definition (21) of $\boldsymbol{A}_i$, $\boldsymbol{A}_0$ is expressed as

$$\boldsymbol{A}_0 = \gamma\tilde{\boldsymbol{U}}_0 \otimes \hat{\boldsymbol{A}}_0.$$

$\boldsymbol{A}_0\boldsymbol{e}$ is thus expressed as

$$\boldsymbol{A}_0\boldsymbol{e} = \gamma(\tilde{\boldsymbol{U}}_0 \otimes \hat{\boldsymbol{A}}_0)\boldsymbol{e} = \gamma(\tilde{\boldsymbol{U}}_0\boldsymbol{e}) \otimes (\hat{\boldsymbol{A}}_0\boldsymbol{e}). \tag{40}$$

From (23) and (40), we have

$$\boldsymbol{\pi}\boldsymbol{A}_0\boldsymbol{e} = \gamma(\tilde{\boldsymbol{\pi}} \otimes \hat{\boldsymbol{\pi}})\left[(\tilde{\boldsymbol{U}}_0\boldsymbol{e}) \otimes (\hat{\boldsymbol{A}}_0\boldsymbol{e})\right] = \gamma(\tilde{\boldsymbol{\pi}}\tilde{\boldsymbol{U}}_0\boldsymbol{e})(\hat{\boldsymbol{\pi}}\hat{\boldsymbol{A}}_0\boldsymbol{e}) = \gamma[\tilde{\boldsymbol{\pi}}]_0\hat{\boldsymbol{\pi}}\hat{\boldsymbol{A}}_0\boldsymbol{e}. \tag{41}$$

Using $\sum_{i=0}^{\infty} \boldsymbol{x}_i = \boldsymbol{\pi}$ and (41) in Theorem 2, we obtain (39). $\square$

Now we are ready to prove Theorem 3.

Proof. Using (38) in (39), we obtain

$$P_{\text{loss}} = \frac{[\tilde{\boldsymbol{\pi}}]_0 - \rho}{\rho} \sum_{l=1}^{\infty} \left(\frac{\sum_{i=K+1}^{\infty} \boldsymbol{x}_i \boldsymbol{A}_0 \boldsymbol{e}}{\gamma [\tilde{\boldsymbol{\pi}}]_0 \hat{\boldsymbol{\pi}} \hat{\boldsymbol{A}}_0 \boldsymbol{e}} \right)^l \approx \frac{[\tilde{\boldsymbol{\pi}}]_0 - \rho}{\rho} \frac{\sum_{i=K+1}^{\infty} \boldsymbol{x}_i \boldsymbol{A}_0 \boldsymbol{e}}{\gamma [\tilde{\boldsymbol{\pi}}]_0 \hat{\boldsymbol{\pi}} \hat{\boldsymbol{A}}_0 \boldsymbol{e}}$$

$$\approx \left(\frac{1}{\rho} - \frac{1}{[\tilde{\boldsymbol{\pi}}]_0} \right) \frac{\boldsymbol{x}_0 \left[(\tilde{\boldsymbol{U}}_0 \tilde{\boldsymbol{v}}(z^*)) \otimes \hat{\boldsymbol{v}}(z^*) \right]}{\tilde{\delta}(z^*)(\delta'(z^*) - 1)} \frac{\boldsymbol{u}(z^*) \boldsymbol{A}_0 \boldsymbol{e}}{\hat{\boldsymbol{\pi}} \hat{\boldsymbol{A}}_0 \boldsymbol{e}} (z^*)^{-K}. \quad \square$$

6. Numerical example

In this section, to demonstrate the numerical feasibility of the formulas (33) and (37), we study a special case of the queueing model considered in Section 3 and we provide a numerical example. The special case studied in this section is reduced to the queueing model considered in [6] when $\gamma = 1$ (i.e., the service time of a customer is deterministic and equal to one slot). Further, if $R = 0$, the special case is reduced to the queueing model considered in [8].

First we will specify the service process. In the special case, the underlying Markov chain $\{S_n\}$ for the service process is described as follows:

$$\mathrm{P}(S_0 = l) = \begin{cases} \dfrac{1}{R+1} & (l = 0, \ldots, R), \\ 0 & (\text{otherwise}), \end{cases}$$

and for $n = 1, 2, \ldots, S_n$ is given by

$$S_n = (S_0 + n) \bmod (R+1).$$

Then, in the queues, the service is periodically available every $(R+1)$ slots. In this case, $\tilde{\boldsymbol{U}}_0$ and $\tilde{\boldsymbol{U}}_1$ are given by

$$\tilde{\boldsymbol{U}}_0 = \begin{pmatrix} 0 & \cdots & \cdots & \cdots & 0 \\ \vdots & & & & \vdots \\ \vdots & & & & \vdots \\ 0 & \cdots & \cdots & \cdots & 0 \\ 1 & 0 & \cdots & \cdots & 0 \end{pmatrix}, \qquad \tilde{\boldsymbol{U}}_1 = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ \vdots & & & \ddots & 1 \\ 0 & \cdots & \cdots & \cdots & 0 \end{pmatrix}.$$

Note that this setting of the service process satisfies Assumption 4.

Next we will specify the arrival process. We consider a superposition of W independent homogeneous on-off sources as the arrival process. We assume here that the lengths of on-periods in each source are i.i.d. and geometrically distributed, and each on-off source generates exactly one customer in each slot during on-periods. Further, we assume that the lengths of off-periods in each source are also i.i.d. and geometrically distributed and each source does not generate customers during off-periods. We denote the mean length of on-periods (resp. off-periods) by $(1-\alpha)^{-1}$ (resp. $(1-\beta)^{-1}$). For the queues with homogeneous sources, with a sophisticated labeling of states [15], we can set $M = W$ and

$$[\hat{\boldsymbol{U}}]_{i,j} = \sum_{k=\max(0,i+j-W)}^{\min(i,j)} \binom{i}{k} \alpha^k (1-\alpha)^{i-k} \cdot \binom{W-i}{j-k} (1-\beta)^{j-k} \beta^{W-i-j+k},$$

$$\hat{a}_j(k) = \begin{cases} 1, & k = j, \\ 0, & k \neq j. \end{cases}$$

We then have

$$[\hat{\boldsymbol{A}}_0]_{i,j} = \begin{cases} \hat{a}_j(0)[\hat{\boldsymbol{U}}]_{i,j} & (i \in \mathcal{M};\ j = 0), \\ 0 & (i \in \mathcal{M};\ j \neq 0). \end{cases}$$

We see that the setting of the arrival process satisfies Assumption 3.

We now apply the numerical formula (33) to compute the exact loss probability. We immediately obtain $\rho = W(1-\beta)/(2-\alpha-\beta)$ and $[\tilde{\boldsymbol{\pi}}]_0 = 1/(R+1)$. Substituting these into (33), we can readily calculate the exact loss probability.

Next we apply the numerical formula (37) to compute the asymptotic loss probability. In the special case, the formula (37) is reduced to

$$P_{\text{loss}} \approx \frac{\gamma}{\gamma + (1-\gamma)z^*} \frac{(1-\rho(R+1))^2}{\rho(R+1)^2} \frac{[\hat{\boldsymbol{v}}(z^*)]_0}{\delta'(z^*) - 1} \frac{\hat{\boldsymbol{u}}(z^*)\hat{\boldsymbol{A}}_0\boldsymbol{e}}{\tilde{\boldsymbol{\pi}}\hat{\boldsymbol{A}}_0\boldsymbol{e}} (z^*)^{-K}, \tag{42}$$

where

$$[\hat{\boldsymbol{v}}(z^*)]_0 = \left(\frac{\bar{\delta}(z^*) - (\alpha+\beta-1)z^*}{2\bar{\delta}(z^*) - \alpha z^* - \beta} \right)^W, \qquad \hat{\boldsymbol{u}}(z^*)\hat{\boldsymbol{A}}_0\boldsymbol{e} = \left(\frac{(1-\alpha)\bar{\delta}(z^*)}{\bar{\delta}(z^*) - (\alpha+\beta-1)} \right)^W,$$

$$\delta'(z^*) = \frac{\mathrm{d}}{\mathrm{d}z} \tilde{\delta}(z)(\bar{\delta}(z))^W \,|_{z=z^*} = z^* \left(W \frac{\bar{\delta}'(z^*)}{\bar{\delta}(z^*)} + \frac{\tilde{\delta}'(z^*)}{\tilde{\delta}(z^*)} \right),$$

$$\bar{\delta}(z) = \frac{\alpha z + \beta}{2} + \sqrt{\left(\frac{\alpha z + \beta}{2} \right)^2 - (\alpha+\beta-1)z}, \qquad \tilde{\delta}(z) = (\gamma + (1-\gamma)z)^{1/(R+1)} z^{R/(R+1)}.$$

Now we show the exact loss probability computed by (33) and the asymptotic loss probability computed by (42) as a function of the buffer capacity K. In the numerical example, we set $R = 2$, $W = 30$, $\gamma = 0.8000$, $\alpha = 0.6900$ and $\beta = 0.9975$ ($\rho = 0.3$). For comparison, we also plot the exact tail distribution $T_{K+1} = \sum_{k=K+1}^{\infty} \boldsymbol{x}_k \boldsymbol{e}$.

In Fig. 1, we observe that the asymptotic loss probability quickly approaches the exact loss probability with the increase of the buffer capacity K. For example, for $K \geq 79$, they are almost identical and the relative error is within 5% if we regard the asymptotic loss probability as the approximation of the exact loss probability. On the other hand, the exact tail distribution is considerably greater than the exact loss probability irrespective of K. For example, at $K = 79$, the relative error is 760% if we regard the tail

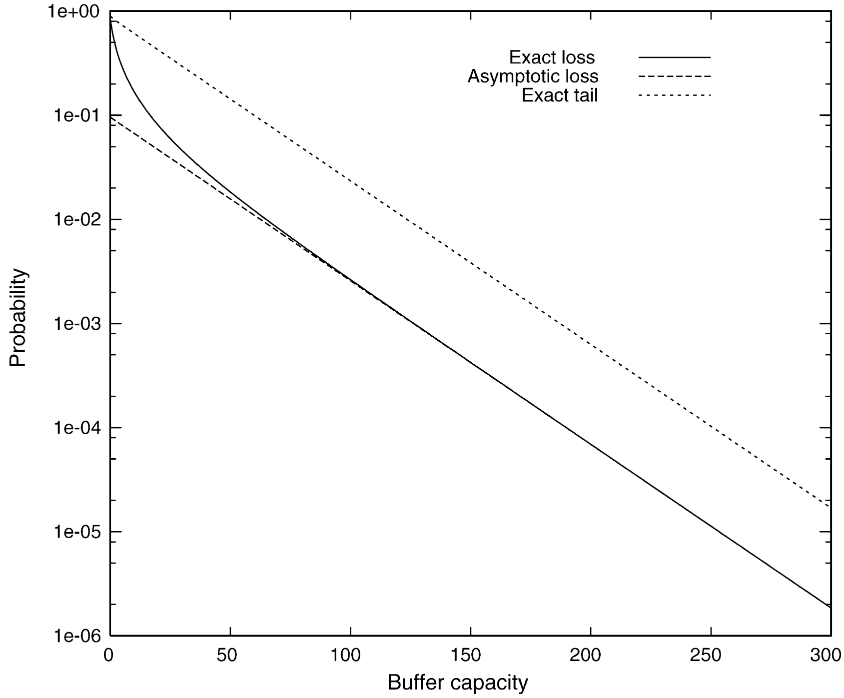

Fig. 1. Loss probability.

distribution as the approximation of the exact loss probability. We thus conclude that the asymptotic loss probability becomes a more accurate and efficient estimation of the exact loss probability than the tail distribution of the queue length in the corresponding infinite-buffer, though many studies regard the tail distribution as the approximation of the loss probability.

Appendix A. Proof of Proposition 3

In this section, we provide the proof of Proposition 3.

Proof. We define the $(M+1)(R+1) \times (M+1)(R+1)$ probability matrix generating function $\boldsymbol{B}(z)$ as

$$\boldsymbol{B}(z) = \sum_{k=0}^{\infty} \boldsymbol{B}_k z^k.$$

Applying Theorem 3.5 in [3] and noting $\boldsymbol{x}_0 = \boldsymbol{x}_0 \boldsymbol{B}_0 + \boldsymbol{x}_1 \boldsymbol{A}_0$ from (3), it immediately follows that

$$\boldsymbol{x}_n = \frac{\boldsymbol{x}_0(\boldsymbol{B}(z^*) - \boldsymbol{I})\boldsymbol{v}(z^*)}{\boldsymbol{u}(z^*)\boldsymbol{A}'(z^*)\boldsymbol{v}(z^*) - 1}(z^*)^{-n}\boldsymbol{u}(z^*) + o\left((z^*)^{-n}\right)\boldsymbol{e}^{\mathrm{T}}, \quad n \geq 1, \tag{43}$$

where $\boldsymbol{I}$ denotes the $(M+1)(R+1) \times (M+1)(R+1)$ identity matrix and $\boldsymbol{e}^{\mathrm{T}}$ denotes a $1 \times (M+1)(R+1)$ row vector with unit elements.

First, we focus on the term $(\boldsymbol{B}(z^*) - \boldsymbol{I})\boldsymbol{v}(z^*)$. Since $\hat{\boldsymbol{v}}(z^*)$ is the right eigenvector of $\hat{\boldsymbol{A}}(z^*)$ and the associated eigenvalue is $\hat{\delta}(z^*)$, we have

$$\hat{\boldsymbol{A}}(z^*)\hat{\boldsymbol{v}}(z^*) = \hat{\delta}(z)\hat{\boldsymbol{v}}(z^*). \tag{44}$$

Using (22), (36) and (44), we can rewrite $\boldsymbol{B}(z^*)\boldsymbol{v}(z^*)$ as

$$\boldsymbol{B}(z^*)\boldsymbol{v}(z^*) = (\tilde{\boldsymbol{U}} \otimes \hat{\boldsymbol{A}}(z^*))(\tilde{\boldsymbol{v}}(z^*) \otimes \hat{\boldsymbol{v}}(z^*)) = (\tilde{\boldsymbol{U}}\tilde{\boldsymbol{v}}(z^*)) \otimes (\hat{\boldsymbol{A}}(z^*)\hat{\boldsymbol{v}}(z^*)) = \hat{\delta}(z^*)(\tilde{\boldsymbol{U}}\tilde{\boldsymbol{v}}(z^*)) \otimes \hat{\boldsymbol{v}}(z^*). \tag{45}$$

We will further rewrite (45) below. From (34), for $z > 0$, we have

$$\tilde{\boldsymbol{U}} = \frac{\tilde{\boldsymbol{U}}_\gamma(z) - \gamma(1-z)\tilde{\boldsymbol{U}}_0}{z}. \tag{46}$$

Since z^* is a solution of $z = \delta(z)$, from (36), it follow that

$$z^* = \tilde{\delta}(z^*)\hat{\delta}(z^*), \tag{47}$$

Note that since $\tilde{\boldsymbol{v}}(z^*)$ is the right eigenvector of $\tilde{\boldsymbol{U}}_\gamma(z^*)$ and its associated eigenvalue is $\tilde{\delta}(z^*)$, we have

$$\tilde{\boldsymbol{U}}_\gamma(z^*)\tilde{\boldsymbol{v}}(z^*) = \tilde{\delta}(z)\tilde{\boldsymbol{v}}(z^*). \tag{48}$$

Using (46)–(48) in (45), we further rewrite (45) as

$$\begin{aligned}
\boldsymbol{B}(z^*)\boldsymbol{v}(z^*) &= \hat{\delta}(z^*)\left[\frac{\tilde{\boldsymbol{U}}_\gamma(z^*) - \gamma(1-z^*)\tilde{\boldsymbol{U}}_0}{z^*}\tilde{\boldsymbol{v}}(z^*)\right] \otimes \hat{\boldsymbol{v}}(z^*) \\
&= \hat{\delta}(z^*)\left[\frac{\tilde{\delta}(z^*)\tilde{\boldsymbol{v}}(z^*) - \gamma(1-z^*)\tilde{\boldsymbol{U}}_0\tilde{\boldsymbol{v}}(z^*)}{z^*}\right] \otimes \hat{\boldsymbol{v}}(z^*) \\
&= \boldsymbol{v}(z^*) - \frac{\gamma(1-z^*)\hat{\delta}(z^*)}{z^*}(\tilde{\boldsymbol{U}}_0\tilde{\boldsymbol{v}}(z^*)) \otimes \hat{\boldsymbol{v}}(z^*) = \boldsymbol{v}(z^*) - \frac{\gamma(1-z^*)}{\tilde{\delta}(z^*)}(\tilde{\boldsymbol{U}}_0\tilde{\boldsymbol{v}}(z^*)) \otimes \hat{\boldsymbol{v}}(z^*).
\end{aligned} \tag{49}$$

Thus, from (49), we finally obtain

$$(\boldsymbol{B}(z^*) - \boldsymbol{I})\boldsymbol{v}(z^*) = \frac{\gamma(z^*-1)}{\tilde{\delta}(z^*)}(\tilde{\boldsymbol{U}}_0\tilde{\boldsymbol{v}}(z^*)) \otimes \hat{\boldsymbol{v}}(z^*). \tag{50}$$

Next we focus on the term $\boldsymbol{u}(z)\boldsymbol{A}'(z)\boldsymbol{v}(z)$. Since $\boldsymbol{u}(z)$ satisfies $\boldsymbol{u}(z)[\delta(z)\boldsymbol{I} - \boldsymbol{A}(z)] = \boldsymbol{0}$, we have

$$\boldsymbol{u}'(z)[\delta(z)\boldsymbol{I} - \boldsymbol{A}(z)] + \boldsymbol{u}(z)[\delta'(z)\boldsymbol{I} - \boldsymbol{A}'(z)] = \boldsymbol{0}. \tag{51}$$

On the other hand, since $\boldsymbol{v}(z)$ satisfies $[\delta(z)\boldsymbol{I} - \boldsymbol{A}(z)]\boldsymbol{v}(z) = \boldsymbol{0}$, post-multiplying the both sides of (51) by $\boldsymbol{v}(z)$ and noting $\boldsymbol{u}(z)\boldsymbol{v}(z) = 1$, we obtain

$$\boldsymbol{u}(z)\boldsymbol{A}'(z)\boldsymbol{v}(z) = \delta'(z). \tag{52}$$

Using (50) and (52) in (43), we derive (38). □

References

[1] S. Asmussen, Applied Probability and Queues, Wiley, Chichester, 1987.
[2] H. Bruneel, Analysis of discrete-time buffer with one single output channel subject to a general interruption process, in: Proceedings of the Performance'84, 1984, pp. 103–115.
[3] E. Falkenberg, On the asymptotic behavior of the stationary distribution of Markov chains of M/G/1-type, Stochastic Mod. 10 (1994) 75–98.
[4] W.K. Grassman, M.L. Taksar, D.P. Heyman, Regenerative analysis and steady state distribution for Markov chains, Oper. Res. 33 (1985) 1107–1116.
[5] F. Ishizaki, Numerical method for discrete-time finite-buffer queues with some regenerative structure, Stochastic Mod. 18 (2002) 25–39.
[6] F. Ishizaki, Exact relation between loss probability and queue length in an ATM multiplexer with correlated arrivals and periodic vacations, in: Proceedings of the 14th ICOIN, 2000, pp. 1A-3.1–1A-38.
[7] F. Ishizaki, T. Takine, Bounds for the tail distribution in a queue with the superposition of general periodic Markov sources, in: Proceedings of the INFOCOM'97, 1997, pp. 1088–1095.
[8] F. Ishizaki, T. Takine, Loss probability in a finite discrete-time queue in terms of the steady state distribution of an infinite queue, Queueing Syst. 31 (1999) 317–326.
[9] F. Ishizaki, T. Takine, Y. Takahashi, T. Hasegawa, A generalized SBBP/G/1 queue and its applications, Perform. Eval. 21 (1994) 163–181.
[10] K. Kang, B. Steyaert, C. Kim, A simple relation between loss performance and buffer contents in a statistical multiplexer with periodic vacations, IEICE Trans. Commun. E80-B (1997) 1749–1752.
[11] G. Latouche, P.A. Jacobs, D.P. Gaver, Finite Markov chain models skip-free in one direction, Naval Res. Logistics Quart. 31 (1984) 571–588.
[12] M. Miyazawa, Y. Takahashi, Rate conservation principle for discrete-time queues, Queueing Syst. 12 (1992) 215–230.
[13] M.F. Neuts, Structured Stochastic Matrices of M/G/1 Type and their Applications, Marcel Dekker, New York, 1989.
[14] V. Ramaswami, A stable recursion for steady state vector in Markov chains of M/G/1 type, Stochastic Mod. 4 (1988) 183–188.
[15] V. Ramaswami, J.L. Wang, A discrete time queueing analysis of ATM systems with heterogeneous traffic sources, in: Proceedings of the GLOBECOM'96, 1996, pp. 623–628.
[16] W.J. Stewart, Introduction to the Numerical Solution of Markov Chains, Princeton University Press, Princeton, 1994.
[17] H. Takagi, Queueing Analysis, vol. 3, North-Holland, Amsterdam, 1993.
[18] D. Towsley, The analysis of a statistical multiplexer with nonindependent arrivals and errors, IEEE Trans. Commun. 28 (1980) 65–72.

Fumio Ishizaki received the B. Eng., M. Eng., and Dr. Eng. from Kyoto University, Kyoto, Japan, in 1990, 1992 and 1996, respectively. From 1993 to 1995, he was an Assistant Professor in the Department of Electronic Engineering at Kinki University, Higashiosaka, Japan. From 1995 to 2000, he was an Assistant Professor in the Department of Information Science and Intelligent Systems at the University of Tokushima, Tokushima, Japan. From May 1998 to October 1998, he was a Visiting Researcher in the Department of Information and Computer Science, University of California, Irvine, CA, USA. From 2000 to 2005, he was an Associate Professor in the Department of Information and Telecommunication Engineering at Nanzan University, Seto, Japan. From August 2001 to July 2002, he was an Invited Member of Engineering Staff at Electronics and Telecommunication Research Institute (ETRI), Daejeon, Korea. Since 2005, he has been an Invited Associate Professor in the Department of Mathematics and Telecommunication Mathematics Research Center at Korea University, Seoul, Korea. His research interests include queueing theory, stochastic discrete event systems, performance evaluation of communication systems, and control of communication networks.