The true Cramer-Rao bound for phase-independent carrier frequency estimation from a PSK signal

Nele Noels, Heidi Steendam and Marc Moeneclaey
Telecommunications and Information Processing (TELIN) department, Ghent University,
St-Pietersnieuwstraat 41, B-9000 Gent, Belgium
{nnoels, hs, Marc.Moeneclaey}@telin.rug.ac.be
Tel: ++32-9-264 34 12; Fax: ++32-9-264 42 95

Abstract **- This contribution considers the Cramer-Rao bound (CRB) related to phase-independent carrier frequency estimation from a noisy PSK signal. Instead of estimating the frequency *jointly* with the carrier phase, we treat the phase as a *nuisance parameter*. Ideal symbol timing is assumed. Both cases of known data (training sequence) and random data are considered. We show that frequency estimation irrespective of the carrier phase yields a larger CRB than does joint frequency and phase estimation; the penalty resulting from the former strategy vanishes with increasing observation interval.**

I. INTRODUCTION

The Cramer-Rao bound (CRB) is a lower bound on the error variance of any unbiased estimate, and as such serves as a useful benchmark for practical estimators [1]. The CRB is formulated in terms of the likelihood function of the scalar parameter to be estimated. In many cases, the statistics of the observation depend not only on the parameter to be estimated, but also on a number of *nuisance* parameters we do not want to estimate. The presence of the nuisance parameters makes the analytical computation of the CRB very hard, if not impossible.

In order to avoid the computational complexity caused by the nuisance parameters, a modified CRB (MCRB) has been derived in [2]. The MCRB is much simpler to evaluate than the CRB, but is in general looser than the CRB. In [3], the high-SNR limit of the CRB has been evaluated analytically, and has been shown to coincide with the MCRB when estimating the delay, the frequency offset or the carrier phase of a linearly modulated waveform.

The true Cramer-Rao bound related to *joint* carrier phase and frequency estimation (with the random data symbols considered as nuisance parameters) has been derived for BPSK and QPSK in [4] and for QAM in [5]. In [6], the low-SNR limit of the CRB for carrier phase and frequency estimation, has been obtained analytically for M-PSK, M-QAM and M-PAM.

In this contribution we investigate the true CRB related to frequency estimation *irrespective* of the carrier phase: the carrier phase is included in the nuisance parameter vector. The data symbols are either known (training sequence) or random, and taken from an M-PSK constellation. The low-SNR limit of the corresponding CRB is presented as well. As in [4-6], timing is assumed to be known. Results are presented for several transmitted sequence lengths and various PSK constellations, and compared with the true CRB related to joint phase and frequency estimation, as derived in [4]. One of the main conclusions is that frequency estimation irrespective of the carrier phase exhibits a performance penalty as compared to joint frequency and phase estimation; this penalty decreases with increasing observation interval.

II. PROBLEM FORMULATION

Consider a linearly modulated signal, obtained by applying a data symbol sequence to a square-root Nyquist transmit filter, that is transmitted over an additive Gaussian noise (AWGN) channel. The resulting noisy signal is applied to a receiver filter, matched to the transmit filter. The receiver filter output signal is sampled at the decision instants, which yields the observation vector $\mathbf{r} = (r_{-K}, \ldots, r_K)$, with

$$r_k = \varepsilon\, a_k \exp\big(j(2\pi\, kFT + \theta)\big) + w_k \qquad (1)$$

In (1), $\mathbf{a} = (a_{-K}, \ldots a_K)$ is a vector of L = 2K+1 M-PSK symbols ($|a_k|^2 = 1$); F is the carrier frequency offset; θ is the carrier phase at k = 0; T is the symbol interval; $\{w_k\}$ is a sequence of independent zero-mean complex-valued Gaussian random variables, with independent real and imaginary parts that each have variance equal to ½; finally, $\varepsilon = (E_s/N_0)^{1/2}$, with E_s and N_0 denoting the symbol energy and the noise power spectral density, respectively. Depending on the scenario to be considered, the M-PSK symbols are either a priori known to the receiver (training sequence), or they are statistically independent and uniformly distributed over the M-PSK constellation (random data).

Suppose that one is able to produce from the observation vector **r** an *unbiased* estimate $\hat{F}$ of the carrier frequency offset *F*. Then the estimation error variance is lower bounded by the Cramer-Rao bound (CRB) [1]: $E_{\mathbf{r}}[(\hat{F} - F)^2] \geq CRB_F$, where

$$CRB_F^{-1} = E_{\mathbf{r}}\left[\left(\frac{\partial}{\partial F}\ln(p(\mathbf{r};F))\right)^2\right] \qquad (2)$$

The probability density p(**r**;F) of **r**, corresponding to a given value of F, is called the *likelihood function* of F, while ln(p(**r**;F)) is the *log-likelihood function* of F. The expectation $E_{\mathbf{r}}[.]$ in (2) is with respect to p(**r**;F).

When the observation **r** depends not only on the frequency offset F to be estimated but also on a nuisance vector parameter **v**, the likelihood function of F is obtained by averaging the *joint* likelihood function p(**r**|**v**;F) of the vector (**v**,F) over the a priori distribution of the nuisance parameter : $p(\mathbf{r};F) = E_{\mathbf{v}}[p(\mathbf{r} \mid \mathbf{v};F)]$. In the context of this paper, **v** = θ or **v** = (**a**, θ), depending on whether the data symbols are known or random.

From (1) it follows that the joint likelihood function p(**r**|**v**;F) is, within a factor not depending on (**v**,F), given by

$$p(\mathbf{r} \mid \mathbf{v};F) = \prod_{k=-K}^{K} F\left(a_k, \breve{r}_k e^{-j\theta}\right) \tag{3}$$

where

$$F\left(a_k, \breve{r}_k e^{-jq}\right) = \exp\left(e\, a_k \breve{r}_k^* e^{jq} + e\, a_k^* \breve{r}_k e^{-jq}\right) \tag{4}$$

and

$$\breve{r}_k = r_k \exp(-j2pFkT)\,. \tag{5}$$

The log-likelihood function ln(p(**r**;F)) resulting from (3) is given by

$$\ln p(\mathbf{r};F) = \ln\left(E_{\mathbf{v}}\left[\prod_{k=-K}^{K} F\left(a_k, \breve{r}_k e^{-jq}\right) \right] \right). \tag{6}$$

Computation of the CRB requires the substitution of (6) into (2), and the evaluation of the various expectations included in (6) and (2).

As the evaluation of the expectations involved in CRB_F and p(**r**;F) is quite tedious, a simpler lower bound, called the modified CRB (MCRB), has been derived in [2], i.e., $E_{\mathbf{r}}[(\hat{F}-F)^2] \geq CRB_F \geq MCRB_F$. The MCRB for frequency estimation, is given by [2]

$$MCRB_F = \frac{3}{2\pi^2 \frac{E_s}{N_0} L(L^2-1)} \cdot \frac{1}{T^2} \tag{7}$$

where L = 2K+1 denotes the number of symbols transmitted within the observation interval. In [3] it has been shown that for *high* SNR (i.e., $E_s/N_0 \to \infty$) the CRB for frequency estimation converges to the MCRB given by (7).

Also, a closed-form expression can be derived for the *low*-SNR limit (i.e. $E_s/N_0 \to 0$) of the CRB, which we call the asymptotic CRB (ACRB). In [6] this has been accomplished for the joint estimation of F and θ (but not for the estimation of F irrespective of θ) based on random data:

$$ACRB_F^{(J),ra} = \frac{3(M!)}{2p^2 \left(\frac{E_s}{N_0}\right)^M M^2 L(L^2-1)} \frac{1}{T^2} \tag{8}$$

The two superscripts in (8) and in the subsequent equations related to the CRBs refer to the estimation of the frequency offset *jointly with* (J) or *irrespective of* (I) the carrier phase and to the receiver's knowledge about the data symbols (ra: random, kn: known). The $ACRB_F^{(J),ra}$ (8) is proportional to $(E_S/N_0)^{-M}$ and to L^{-3}.

It is easily shown that when the data are not random but a priori known at the receiver side the parameters F and θ are decoupled ($J_{\theta F}$ = 0). The resulting $CRB^{(J),kn}=ACRB^{(J),kn}$ is nothing but the $MCRB_F$ given by (7).

In this paper we compute the CRB corresponding to the estimation of F *irrespective of* the carrier phase, and present the expression for the corresponding ACRB.

III. Evaluation of the True CRB

A. Random data symbols

Taking in (6) **v** = (**a**, θ), the log-likelihood function ln(p(**r**;F)) is (within an arbitrary constant) given by

$$\ln\left(p(\mathbf{r};F)\right) = \ln\left(\int_{-p}^{p} K(\breve{\mathbf{r}}\, e^{-jq})\, dq \right) \tag{9}$$

where $\breve{\mathbf{r}} = (\breve{r}_{-K}, ..., \breve{r}_K)$,

$$K\left(\breve{\mathbf{r}}\, e^{-jq}\right) = \prod_{k=-K}^{K} I\left(\breve{r}_k e^{-jq}\right) \tag{10}$$

$$I\left(\breve{r}_k e^{-jq}\right) = \sum_{i=0}^{M-1} F\left(a_i, \breve{r}_k e^{-jq}\right) \tag{11}$$

and $\{\alpha_0, \alpha_1, ..., \alpha_{M-1}\}$ is the set of PSK constellation points. Differentiation of (9) yields

$$\frac{d}{dF} \ln p(r;F) = e \int_{-p}^{p} \sum_{k=-K}^{K} \sum_{i=0}^{M-1} M\left(\breve{\mathbf{r}}\, e^{-jq}\right) H\left(a_i, \breve{r}_k e^{-jq}\right) dq \tag{12}$$

where

$$M\left(\breve{\mathbf{r}}\, e^{-jq}\right) = \frac{K\left(\breve{\mathbf{r}}\, e^{-jq}\right)}{\int_{-p}^{p} K\left(\breve{\mathbf{r}}\, e^{-jq}\right) dq} \tag{13}$$

$$H\left(a_i, \breve{r}_k e^{-jq}\right) = G\left(a_i, \breve{r}_k e^{-jq}\right) 4p\ \operatorname{Im}(a_i^* \breve{r}_k e^{-jq}) \tag{14}$$

$$G\left(\alpha_i, \breve{r}_k e^{-j\theta}\right) = \frac{F\left(\alpha_i, \breve{r}_k e^{-j\theta}\right)}{I\left(\breve{r}_k e^{-j\theta}\right)} \qquad (15)$$

As the variable F in (5) corresponds to the actual frequency offset, the quantity $\breve{r}_k$ can be decomposed as

$$\breve{r}_k = \varepsilon\, a_k + N(k) \qquad (16)$$

where {N(k)} is a sequence of zero-mean complex Gaussian random variables, with $E\left[N(k)N^*(m)\right] = \delta_{k-m}$. As $M\left(\breve{\mathbf{r}}\, e^{-j\theta}\right)$, $H\left(\alpha_i, \breve{r}_k e^{-j\theta}\right)$ and $K\left(\breve{\mathbf{r}}\, e^{-j\theta}\right)$ are periodic in θ with period equal to 2π, there is no need to include a carrier phase in the first term of (16).

Taking (12) into account, we derive from (2):

$$\frac{1}{CRB_F^{(I),ra}} = \frac{E_s}{N_0} \frac{4\pi^2 L\left(L^2-1\right)T^2}{3} \cdot A\left(\frac{E_s}{N_0}, M, L\right) \qquad (17)$$

where

$$A\left(\frac{E_s}{N_0}, M, L\right) = E\left[N^2(k) - N(k)N(m)\right] \qquad (18)$$

$$N(k) = \mathrm{Im}\left[\breve{r}_k \int_{-\pi}^{\pi} \sum_{i=0}^{M-1} M(\breve{\mathbf{r}} e^{-j\theta}) G(\alpha_i, \breve{r}_k e^{-j\theta}) \alpha_i^* e^{-j\theta} d\theta\right] \qquad (19)$$

and E[.] denotes averaging over the data symbols and the noise. The variables $\breve{r}_k$ and $\breve{r}_m$ involved in (18) represent two statistically independent quantities (k≠m).

For low SNR, (17) converges to the corresponding ACRB. The computation of this ACRB is very similar to the derivation of the $ACRB^{(J),ra}$ in [6]. Expanding the exponential functions in (2) into a Taylor series, averaging each resulting term with respect to the data symbols and the carrier phase, and keeping only the relevant terms that correspond to the smallest powers of ε, we obtain

$$ACRB_F^{(I),ra} = \frac{3\left((M-1)!\right)^2}{2\pi^2 \left(\frac{E_s}{N_0}\right)^{2M} L^2(L^2-1)} \frac{1}{T^2} \qquad (20)$$

where L= 2K+1 and M represents the number of constellation points. The ACRB (20) is proportional to $(E_s/N_0)^{-2M}$ and to L^{-4}.

B. Known data symbols

When the transmitted data symbols are known at the receiver, the nuisance parameter is given by **v** = θ; and no averaging over the data is required. Equation (17) remains valid (with the superscript ra substituted by kn), provided we remove in (19) the summations over the constellation points and replace α_i by the actual symbol a_k.

The corresponding low-SNR limit can again be obtained as in [6]. We obtain

$$ACRB_F^{(I),kn} = \frac{3}{2\pi^2 \left(\frac{E_s}{N_0}\right)^2 L^2\left(L^2-1\right)} \cdot \frac{1}{T^2} \qquad (21)$$

which is proportional to $(E_s/N_0)^{-2}$ and to L^{-4}.

IV. NUMERICAL RESULTS AND DISCUSSION

As no further analytical simplification of (17) seems possible, we have to resort to numerical computation. This involves numerical integration with respect to θ in (19), and replacing the statistical expectations in (18) by arithmetical averages over a number of computer-generated vectors $\breve{\mathbf{r}}$.

A. Known data symbols

In the case of known data symbols, the CRB_F resulting from the joint estimation of F and θ equals $MCRB_F$. Hence, for any other scenario the ratio CRB/MCRB is a measure of the penalty occurred by not knowing the data symbols and/or not estimating θ jointly with F.

Fig. 1 shows the ratio CRB/MCRB related to the estimation of F independently of θ, along with the corresponding ACRB/MCRB. These results do not depend on the constellation size. For small (large) SNR, the CRBs converge to the corresponding ACRB (to the MCRB). Increasing the observation interval from L_1 to L_2 shifts the curve of CRB/MCRB to the left by an amount of $10\log(L_2/L_1)$ dB; hence the value of E_s/N_0 at which the CRB comes close to the MCRB is shifted by the same amount. Note that for SNR values of practical interest, CRB/MCRB is close to 1, even for a moderate length of the observation interval. Hence, the penalty, caused by treating the carrier phase as a nuisance parameter, can be safely ignored in practice.

B. Random data symbols

It follows from (8) and (20) that the ACRB corresponding to the joint estimation of F and θ and the ACRB corresponding to the estimation of F irrespective of θ are much different.

Fig. 2 shows the ratios CRB/MCRB and the corresponding ACRB/MCRB, for BPSK and QPSK, and for both joint estimation of F and θ and estimation of F irrespective of θ.

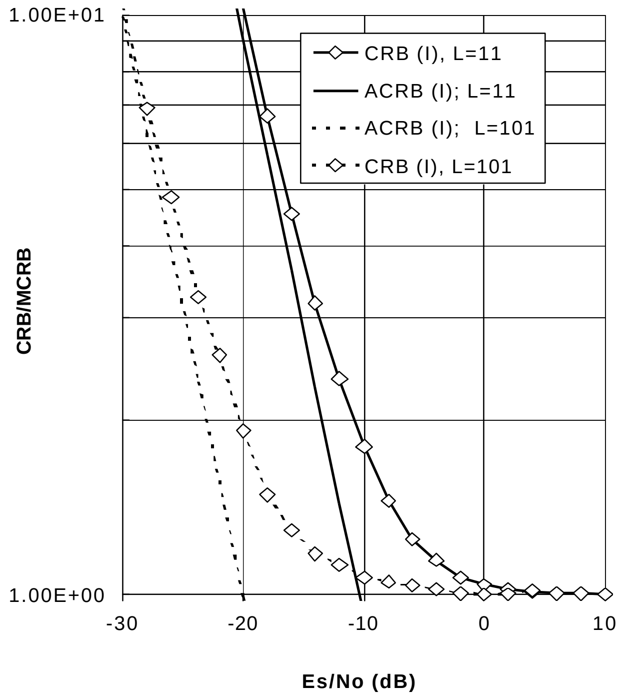

Fig. 1. CRB and ACRB related to frequency estimation irrespective of the carrier phase (I), based on known data.

The behavior of the various curves is as follows:

- For small (large) SNR, the CRBs converge to the corresponding ACRBs (to the MCRB).
- For a given constellation size, the joint estimation of F and θ yields the smaller CRB_F. This indicates that estimating F jointly with θ is potentially more accurate than estimating F irrespective of θ. Indeed, it follows from [4-5] that F and θ are uncoupled, so that the joint estimation of F and θ yields the same CRB_F as estimating F when θ is a priori known. Therefore, $CRB_F^{(J),ra}$ is smaller than $CRB_F^{(I),ra}$. For given E_s/N_0, both the $CRB^{(J),ra}$ and the $CRB^{(I),ra}$ increase with increasing M. This indicates that frequency estimation becomes more difficult for larger constellations. This effect is more pronounced for the estimation of F irrespective of θ (with ACRB being proportional to $(E_s/N_0)^{-2M}$), than for joint estimation of F and θ (with ACRB being proportional to $(E_s/N_0)^{-M}$).
- When E_s/N_0 is sufficiently large, the CRBs for both scenarios (I) and (J) converge to the MCRB. The value of E_s/N_0 at which the CRB comes close to the MCRB increases with the number M of constellation points, but is independent of the considered scenario.

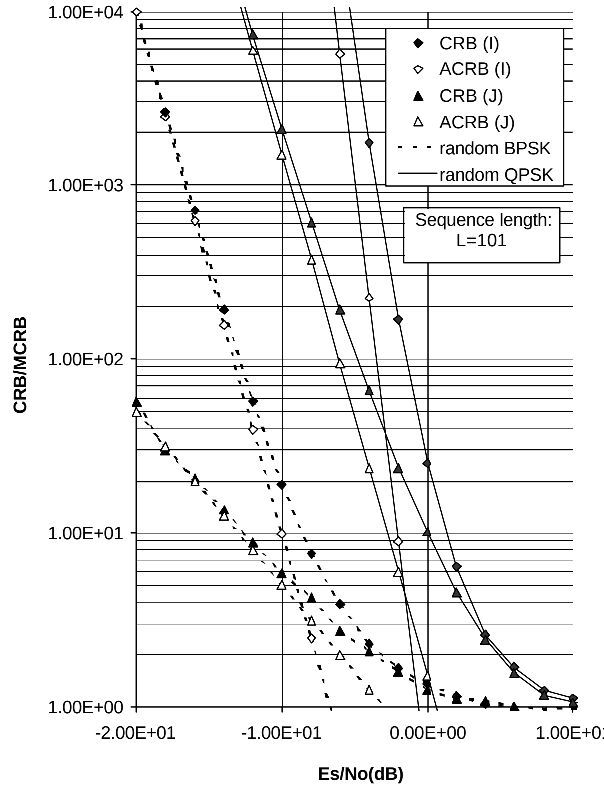

Fig. 2. Comparison of CRB and ACRB related to joint phase and frequency estimation (J) and frequency estimation irrespective of the carrier phase (I).

C. Effect of observation interval (random data)

Fig. 3 shows the ratios CRB/MCRB and the corresponding ACRB/MCRB for different lengths L of the observation interval, assuming random QPSK modulation (we have verified that the same behaviour applies to other M-PSK constellations).

Note that MCRB from (7) is proportional to L^{-3}. For moderate and small SNR, the ratio $CRB^{(I),ra}$/MCRB decreases with L, whereas the ratio $CRB^{(J),ra}$/MCRB does not depend on L. This is consistent with the observation that the $ACRB^{(I),ra}$ ($ACRB^{(J),ra}$) is proportional to L^{-4} (to L^{-3}), so that $ACRB^{(I),ra}$/MCRB ($ACRB^{(J),ra}$/MCRB) is proportional to L^{-1} (independent of L). For given E_s/N_0, increasing L makes $CRB^{(I),ra}$ approach $CRB^{(J),ra}$, and hence reduces the penalty, caused by treating θ as a nuisance parameter; for given L this penalty is larger than in the case of known data symbols. The ratio $CRB^{(J),ra}$/MCRB being independent of L indicates that the penalty, caused by treating the *data symbols* as nuisance parameters, cannot be reduced by increasing the observation interval.

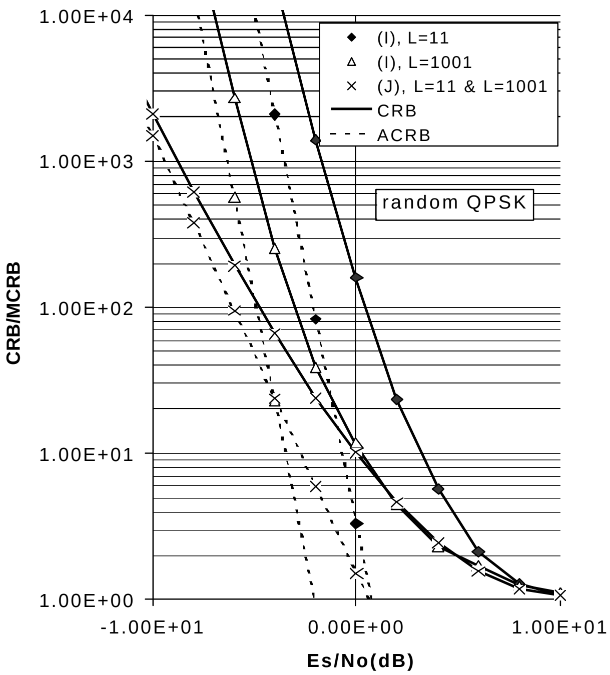

Fig. 3. Effect of the observation interval on the CRB and ACRB related to joint phase and frequency estimation (J) and frequency estimation irrespective of the carrier phase (I).

V. Conclusions and remarks

In this contribution, we have considered the $CRB^{(I)}$ related to the frequency estimation irrespective of the carrier phase of a noisy linearly modulated signal with either known or random data symbols. We have compared our results to the $CRB^{(J)}$ related to joint phase and frequency estimation, derived in [4-5]. The numerical evaluation of the $CRB^{(I)}$ requires numerical integration, and the approximation of each statistical expectation by an arithmetical average. These averages and numerical integration depend on E_s/N_0, on the constellation size and on the transmitted sequence length.

Frequency estimation irrespective of the carrier phase yields a larger CRB than does joint frequency and phase estimation. This implies that the latter strategy is potentially the better one. For given SNR, the penalty of the former strategy with respect to the latter decreases with increasing observation interval. When the data symbols are known, this penalty can be neglected for practical values of SNR, even for moderate observation intervals. In the case of random data symbols, considerably longer observation intervals are required to make the penalty very small.

For small and moderate E_s/N_0, the CRBs that assume random data symbols increase with increasing constellation size. This effect is strongest when the frequency is estimated irrespective of the carrier phase.

The observations $\{r_k\}$ from (2) denote the matched filter output samples taken at the decision instants. Strictly speaking, the expression (2) is not correct, because it ignores the reduction of the useful signal component and the introduction of ISI, both caused by the received signal being ill-centered (because of the frequency offset F) with respect to the receiver filter. Hence, the observation model (2) is valid only for small frequency offsets (say, $|FT| < 0.05$). When the frequency offset is so large that (2) is no longer accurate, one should consider as the observation the received complex baseband signal r(t), given by:

$$r(t) = e \sum_{k=-K}^{K} a_k h(t-kT) \exp\big(j(2p\ F\,t+q)\big) + w(t) \qquad (22)$$

where h(t) is a real-valued unit-energy square-root Nyquist pulse and w(t) is complex-valued zero-mean Gaussian noise with independent real and imaginary parts, each having a normalized power spectral density of 1/2. The behavior of the true CRB for frequency estimation based on the observation model (22) is a topic for further research.

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