The Impact of the Observation Model on the Cramer-Rao Bound for Carrier Phase and Frequency Synchronization

Nele Noels, Heidi Steendam and Marc Moeneclaey
Telecommunications and Information Processing (TELIN) department
Ghent University,
Gent, Belgium
{nnoels, hs, mm}@telin.rug.ac.be

***Abstract*— This contribution considers the Cramer-Rao bound (CRB) related to the joint estimation of the carrier phase and frequency of a noisy linearly modulated signal with random data symbols, using the correct continuous-time model of the received signal. We compare our results with the existing results obtained from a (commonly used) simplified discrete-time model of the matched filter output samples [1,2,3], that ignores useful signal reduction and ISI caused by nonzero frequency offset. We show that both models yield the same CRB for phase estimation. As far as frequency estimation is concerned, we point out that (i) the correct observation model yields the smaller CRB, and (ii) the difference between the CRBs resulting from the two models is apparent only at very small SNR. This indicates that at practical SNR values, the frequency offset affects the likelihood function mainly through the signal rotation (rather than signal magnitude) at the matched filter output.**

Carrier Synchronization, Cramer-Rao Lower Bound, Phase Estimation, Frequency Estimation

I. INTRODUCTION

The Cramer-Rao bound (CRB) is a lower bound on the error variance of any unbiased estimate, and as such serves as a useful benchmark for practical estimators [4]. In many cases, the statistics of the observation depend not only on the vector parameter to be estimated, but also on a *nuisance* vector parameter we do not want to estimate. The presence of this nuisance parameter makes the analytical computation of the CRB very hard, if not impossible.

In order to avoid the computational complexity caused by the nuisance parameters, a modified CRB (MCRB) has been derived in [5,6]. The MCRB is much simpler to evaluate than the CRB, but is in general looser than the CRB. In [7], the high-SNR limit of the CRB related to estimating a scalar synchronization parameter has been evaluated analytically; and has been shown to coincide with the MCRB when estimating the frequency offset or the carrier phase of a linearly modulated waveform.

The 'true' (as opposed to 'the modified') Cramer-Rao bound related to carrier phase or frequency estimation has been derived for M-PSK, M-PAM and M-QAM in [1,2,3], assuming a *simplified* observation model. In this model, the received signal is fed directly to the matched filter (without first applying frequency error correction), and the observation consists of the matched filter output samples (taken at the decision instants). The simplification consists of neglecting the signal reduction and the ISI that occur at the matched filter output when the frequency offset is nonzero. In [8], the low-SNR limit of the CRB for carrier phase or frequency estimation, again assuming this simplified observation model, has been obtained analytically for M-PSK, M-QAM and M-PAM.

In this contribution we consider a noisy linearly modulated signal with random data symbols (PSK, QAM or PAM), an arbitrary square-root Nyquist transmit pulse, and timing assumed to be known. We derive the associated CRBs and their low-SNR limit for the joint estimation of the carrier frequency and phase, based on the *correct* model of the received signal. Comparing our results with those from [1,2,3] that correspond to the simplified model, our main conclusion is that both models, although considerably different, yield essentially the same CRBs at SNR values of practical interest.

II. PROBLEM FORMULATION

Let us consider the complex baseband representation r(t) of a noisy linearly modulated signal:

$$r(t) = \varepsilon \sum_{k=-K}^{K} a_k h(t-kT) e^{j(2\pi F t + \theta)} + w(t) \qquad (1)$$

where $\mathbf{a} = (a_{-K}, \ldots a_K)$ is a vector of L = 2K+1 data symbols ($E[|a_k|^2] = 1$); h(t) is a real-valued unit-energy square-root Nyquist pulse; θ is the carrier phase at t = 0; F is the frequency offset; T is the symbol interval; w(t) is complex-valued zero-mean Gaussian noise with independent real and imaginary parts, each having a normalized power spectral density of 1/2; $\varepsilon = (E_s/N_0)^{1/2}$, with E_s and N_0 denoting the symbol energy and the noise power spectral density, respectively. The data symbols are statistically independent and uniformly distributed over the M-PSK, M-PAM or M-QAM constellation. Based on the

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observation model (1), the CRBs for joint phase and frequency estimation irrespective of the data symbols will be derived and investigated in this paper.

In [1,2,3,8], CRBs related to frequency and phase estimation have been derived, assuming the following observation model:

$$r_k = \varepsilon a_k e^{j(2\pi FkT + \theta)} + w_k \qquad (2)$$

with k=-K, …, K. In (2), $\{w_k\}$ is a sequence of independent zero-mean complex-valued Gaussian random variables, with independent real and imaginary parts that each have variance equal to 1/2. The quantity r_k stands for the matched filter output sample taken at the correct decision instant kT, when r(t) from (1) is applied to the matched filter (see Fig. 1) and the frequency offset is assumed to be so small (i.e., $|FT| \ll 1$), that signal reduction and ISI at the matched filter output can be neglected. It is important to realize that the observations r(t) from (1) and $\{r_k\}$ from (2) are not equivalent for nonzero F, as will be pointed out in the sequel.

Suppose that one is able to produce from an observation vector **r** an *unbiased* estimate $(\hat{F}, \hat{\theta})$ of a deterministic vector parameter (F, θ). Then the estimation error variance is lower bounded by the Cramer-Rao bound (CRB) [4]:

$$E_{\mathbf{r}}[(\hat{F} - F)^2] \geq CRB_F$$

$$E_{\mathbf{r}}[(\hat{\theta} - \theta)^2] \geq CRB_{\theta}$$

where $CRB_{\theta} = J_{FF}/(J_{FF}J_{\theta\theta} - J_{F\theta}^2)$ and $CRB_F = J_{\theta\theta}/(J_{FF}J_{\theta\theta} - J_{F\theta}^2)$. $J_{\theta\theta}$, J_{FF} and $J_{F\theta}$ are the elements of the symmetrical *Fisher information matrix* **J**, and are given by

$$J_{uv} = E_{\mathbf{r}}\left[\left(\frac{\partial}{\partial u}\ln(p(\mathbf{r}; F, \theta))\right)\left(\frac{\partial}{\partial v}\ln(p(\mathbf{r}; F, \theta))\right)\right] \qquad (3)$$

where u and v must be replaced by F or θ. The probability density p(**r**;F,θ) of **r**, corresponding to a given value of (F,θ), is called the *likelihood function* of (F,θ), while ln(p(**r**;F,θ)) is the *log-likelihood function* of (F,θ). The expectation $E_{\mathbf{r}}[.]$ in (3) is with respect to p(**r**;F,θ). When using the simplified observation model (2), the vector **r** is given by $(r_{-K}, \ldots, r_K)$. When using the correct observation model (1), **r** is a vector representation of the signal r(t) from (1).

As the observation **r** depends not only on the parameters F and θ to be estimated but also on a nuisance vector parameter **a**, the likelihood function of (F,θ) is obtained by averaging the *joint* likelihood function p(**r**|**a**;F,θ) of the vector (F,θ,**a**) over the a priori distribution of the nuisance parameter: $p(\mathbf{r}; F, \theta) = E_{\mathbf{a}}[p(\mathbf{r} \mid \mathbf{a}; F, \theta)]$.

The joint likelihood function p(**r**|**a**;F,θ) is, within a factor not depending on (F,θ,**a**), given by

$$p(\mathbf{r} \mid a; F, \theta) = \prod_{k=-K}^{K} F(a_k, x_k) \qquad (4)$$

where

$$F(a_k, x_k) = \exp\left(2\varepsilon \operatorname{Re}(a_k^* x_k) - \varepsilon^2 |a_k|^2\right) \qquad (5)$$

and x_k depends on the observation model used.

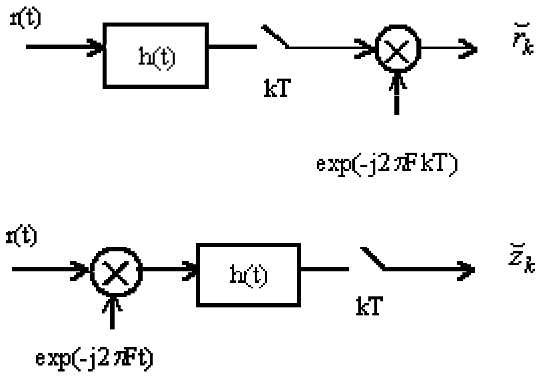

Figure 1. Simplified observation model (top) versus correct observation model (bottom)

When using the simplified observation model (2), x_k equals $\breve{r}_k$ given by

$$\breve{r}_k = r_k \exp(-j2\pi FkT) \exp(-j\theta) \qquad (6)$$

As indicated in the top part of Fig. 1, the quantity r_k is obtained by feeding r(t) to a filter matched to the transmit pulse h(t), and sampling the matched filter output at instant kT. The quantity $\breve{r}_k$ is obtained by applying to r_k a rotation of –(2πFkT + θ) rad. When using the correct observation model (1) x_k equals $\breve{z}_k$ given by

$$\breve{z}_k = \int r(t) \exp(-j2\pi Ft) h(t - kT) dt \, \exp(-j\theta) \qquad (7)$$

As indicated in the bottom part of Fig. 1, the quantity $\breve{z}_k$ is obtained by first applying to r(t) a constant-speed rotation of -2πF rad/s, feeding the result to a filter matched to the transmit pulse h(t), sampling the matched filter output at instant kT and applying to the result a rotation of -θ rad. Note the similarity between expressions (6) and (7). However, unless F=0, it follows from Fig. 1 that $\breve{r}_k \neq \breve{z}_k$. Actually, $\{\breve{z}_k\}$ cannot be computed from $\{\breve{r}_k\}$ when F is nonzero, and therefore estimating F from $\{r_k\}$ instead of r(t) is suboptimum.

The log-likelihood function ln(p(**r**;**u**)) resulting from (4) is given by

$$\ln p(\mathbf{r}; F, \theta) = \ln\left(E_{\mathbf{a}}\left[\prod_{k=-K}^{K} F(a_k, x_k)\right]\right) \qquad (8)$$

Computation of the CRB requires the substitution of (8) into (3), and the evaluation of the various expectations included in (8) and (3).

As the evaluation of the expectations involved in **J** and p(**r**;F,θ) are quite tedious, a simpler lower bound, called the modified CRB (MCRB), has been derived in [5,6], i.e.,

$$E_{\mathbf{r}}[(\hat{F}-F)^2] \geq CRB_F \geq MCRB_F$$

$$E_{\mathbf{r}}[(\hat{\theta}-\theta)^2] \geq CRB_\theta \geq MCRB_\theta$$

The MCRBs for phase and frequency estimation happen to be independent of the observation model, and are given by [5,6]

$$MCRB_F = \frac{3}{2\pi^2 \frac{E_s}{N_0} L(L^2-1)} \cdot \frac{1}{T^2} \tag{9}$$

$$MCRB_\theta = \frac{1}{2L\frac{E_s}{N_0}} \tag{10}$$

In [7] it has been shown that the high-SNR limits (i.e., $E_s/N_0 \rightarrow \infty$) of the CRBs related to the estimation of the carrier phase or frequency converge to the corresponding MCRB.

Also, a closed-form expression can be derived for the *low*-SNR limit (i.e. $E_s/N_0 \rightarrow 0$) of the CRB, which we call the asymptotic CRB (ACRB). In [8] this has been accomplished for the CRB related to carrier synchronization using the simplified observation model (2), yielding

$$ACRB_F^{si} = \frac{3(N!)}{2\pi^2\left(\frac{E_s}{N_0}\right)^N N^2 L(L^2-1)|A_N|^2} \frac{1}{T^2} \tag{11}$$

$$ACRB_\theta^{si} = \frac{N!}{2\left(\frac{E_s}{N_0}\right)^N N^2\, L|A_N|^2} \tag{12}$$

where $A_N = E[a_k^N]$, and N is related to the symmetry angle ($2\pi/N$) of the constellation : N=2 for M-PAM, N=4 for M-QAM, and N=M for M-PSK. The superscript si denotes that the result was obtained with the simplified observation model (2). Note that $ACRB_F^{si}$ is proportional to $(E_s/N_0)^{-N}$ and to L^{-3}, whereas $ACRB_\theta^{si}$ is proportional to $(E_s/N_0)^{-N}$ and to L. Both ACRBs (11)-(12) depend on the constellation (through N and A_N), but not on the shape of the transmit pulse h(t); obviously, this is because the simplified observation model (2) does not depend on h(t).

Similarly, one can derive the ACRBs related to the correct observation model (1). For carrier phase estimation we obtain $ACRB_\theta^{co} = ACRB_\theta^{si}$, where the superscript co denotes that the results where obtained with the correct observation model (1). For frequency estimation we must distinguish between N>2 and N=2.

- For N=2 (real-valued constellation), we find that for large L, $ACRB_F^{co}$ converges to $ACRB_F^{si}$ from (13), evaluated for N=2; hence, for N=2 both observation models yield essentially the same ACRB.
- For N>2 (complex-valued constellation), we obtain

$$ACRB_F^{co} = \frac{1}{8\pi^2\left(\frac{E_s}{N_0}\right)^2 L\int_{-\infty}^{+\infty} t^2 h^2(t)\, dt}, \; N>2 \tag{13}$$

which is only proportional to $(E_s/N_0)^{-2}$ and to L^{-1}. Note also that (13) is independent of the type and size of the complex-valued constellation, but is affected by the shape of the transmit pulse h(t).

Comparing equations (11) and (13) reveals that, for N>2, the correct and the simplified observation models result in considerably different CRB_F-values as $E_s/N_0 \rightarrow 0$; this indicates that the simplified observation model is highly inaccurate, as far as frequency estimation at very low E_s/N_0 is concerned. In this contribution we study the effect of simplifying the observation model on the CRB at moderate and high SNR. Therefore, we derive in Section III the true CRB related to the correct observation model (2), and compare it to the results corresponding to the simplified observation model [1,2,3].

It should be stressed that the ACRBs (11)-(13) do not necessarily provide a lower bound on the actual frequency error variance for moderate and large SNR, but only indicate the behavior of the CRBs at small SNR.

III. Evaluation of the True CRB

We obtain for the log-likelihood function $\ln(p(\mathbf{r}|F,\theta))$ of (F,θ)

$$\ln(p(\mathbf{r}\,|\,F,\theta)) = \sum_{k=-K}^{K} \ln(I(x_k)) \tag{14}$$

where

$$I(x_k) = \sum_{i=0}^{M-1} F(\alpha_i, x_k) \tag{15}$$

and $\{\alpha_0, \alpha_1, ..., \alpha_{M-1}\}$ is the set of constellation points. Differentiation of (14) yields

$$\frac{\partial}{\partial F}\ln p(\mathbf{r}\,|\,F,\theta) = \varepsilon \sum_{k=-K}^{K}\sum_{i=0}^{M-1} H(\alpha_i, x_k, x_{Fk}) \tag{16}$$

$$\frac{\partial}{\partial \theta}\ln p(\mathbf{r}\,|\,F,\theta) = \varepsilon \sum_{k=-K}^{K}\sum_{i=0}^{M-1} H(\alpha_i, x_k, -jx_k) \tag{17}$$

where

$$H(\alpha_i, x_k, y) = G(\alpha_i, x_k)\, 2\,\mathrm{Re}(\alpha_i^* y) \tag{18}$$

$$G(\alpha_i, x_k) = \frac{F(\alpha_i, x_k)}{I(x_k)} \tag{19}$$

and the subscript F denotes differentiation with respect to F. In these expressions $x_k = \breve{r}_k$ or $x_k = \breve{z}_k$ and $x_{Fk} = -\,j\,2\,\pi\,k\,T\,\breve{r}_k$ or $x_{Fk} = \breve{z}_{Fk}$, according to the observation model being used.

As the variables F and θ in (3) correspond to the actual frequency and phase offsets, the quantities $\breve{r}_k$, $\breve{z}_k$ and $\breve{z}_{Fk}$ to be used in (16)-(17) can be decomposed as

$$\breve{r}_k = \breve{z}_k = \varepsilon\, a_k + N(k) \tag{20}$$

$$\breve{z}_{Fk} = -\, j\, 2\pi k T \varepsilon\, a_k + N_F(k) \tag{21}$$

where N(k) and $N_F(k)$ are zero-mean Gaussian random variables, with

$$\begin{aligned} &E\left[N(m)N^*(n)\right] = \delta_{m-n} \\ &E\left[N_F(m)N^*(n)\right] = -j\,2\pi mT\,\delta_{m-n} \\ &E\left[N_F(m)N_F^*(n)\right] = 4\pi^2 m^2 T^2 \delta_{m-n} \\ &\qquad + 4\pi^2 \int_{-\infty}^{+\infty} t^2 h(t)\, h(t-(m-n)T)\, dt \end{aligned} \tag{22}$$

Taking (16) and (17) into account, J_{FF}, $J_{\theta\theta}$ and $J_{F\theta}$ from (3) can be represented as

$$J_{uv} = \frac{E_s}{N_0} \sum_{k_1,k_2=-K}^{K} \sum_{i_1,i_2=0}^{M-1} E\begin{bmatrix} H(\alpha_{i_1}, x_{k_1}, y_u(k_1)) \\ H(\alpha_{i_2}, x_{k_2}, y_v(k_2)) \end{bmatrix} \tag{23}$$

where u and v must be replaced by F or θ, $y_\theta(k) = -jx_k$, $y_F(k) = x_{Fk}$, and E[.] denotes averaging over the data symbols and the noise. Note that (x_{k_1}, x_{k_2}) depends only on $(a_{k_1}, a_{k_2}, N_{k_1}, N_{k_2})$. Expression (23) can be further simplified by making use of the statistical properties of x_{Fk1} and x_{Fk2}, conditioned on $(a_{k_1}, a_{k_2}, N_{k_1}, N_{k_2})$. The average in (23) is computed by first taking the expectation conditioned on (x_{k_1}, x_{k_2}), and then averaging over (x_{k_1}, x_{k_2}).

We obtain $J_{F\theta} = 0$, indicating that the parameters F and θ are *uncoupled*; this implies that the joint estimation of the parameters F and θ yields the same CRB_θ (= $J_{\theta\theta}^{-1}$) and CRB_F (= J_{FF}^{-1}) as in the case of estimating one of these parameters, assuming the other parameter to be known.

Substituting (20) to (17) gives rise to an expression that is the same for both observation models. Hence, the correct observation model and the simplified observation model yield the same CRB_θ, the expression of which has been presented in [1,2,3]. Hence, in the sequel we mainly concentrate on CRB_F.

For the CRB_F we obtain

$$\frac{1}{CRB_F^{si}} = \frac{E_s}{N_0} \frac{4\pi^2 L(L^2-1)T^2}{3} \cdot A\left(\frac{E_s}{N_0}, N\right) \tag{24}$$

$$\begin{aligned} \frac{1}{CRB_F^{co}} &= \frac{1}{CRB_F^{si}} \\ &+ \frac{E_s}{N_0} 8\pi^2 L \int_{-\infty}^{+\infty} t^2 h^2(t)\, dt \cdot B\left(\frac{E_s}{N_0}, N\right) \end{aligned} \tag{25}$$

where

$$A\left(\frac{E_s}{N_0}, N\right) = E\left[\left(\mathrm{Im}\left(\sum_{i=1}^{M-1} G(\alpha_i, x_k)\, \alpha_i^* x_k\right)\right)^2\right] \tag{26}$$

$$B\left(\frac{E_s}{N_0}, N\right) = E\left[\left|\sum_{i=0}^{M-1} G(\alpha_i, x_k)\, \alpha_i^*\right|^2\right] \tag{27}$$

We observe that $1/CRB_F^{co}$ equals the sum of $1/CRB_F^{si}$ and a positive term. Hence, $CRB_F^{co} < CRB_F^{si}$, which indicates that frequency estimation from $\{r_k\}$ is suboptimum. We have verified that the term $1/CRB_F^{si}$ dominates at high E_s/N_0 (assuming large L), whereas at low SNR the additional term becomes the largest (this is consistent with the expressions for the ACRBs). Consequently, both observation models yield essentially the same CRB_F at sufficiently high SNR. The quantities $A(E_s/N_0, N)$ and $B(E_s/N_0, N)$ depend on the SNR and on the constellation, but not on the shape of the Nyquist transmit pulse. The shape of the transmit pulse affects only the coefficient of $B(E_s/N_0, N)$ in (25). Note that evaluating CRB_F^{si} from (24) for BPSK, QPSK and QAM yields the CRB for frequency estimation presented in [1,2,3].

IV. NUMERICAL RESULTS AND DISCUSSION

As no further analytical simplification of (24)-(27) seems possible, we have to resort to numerical computation. This involves replacing the statistical expectations in (26)-(27) by arithmetical averages over a number of computer-generated vectors **x**.

In the case of known data symbols, the CRB_F resulting from the joint estimation of F and θ equals $MCRB_F$. Hence, for any other scenario the ratio $CRB_F/MCRB_F$ is a measure of the penalty occurred by not knowing the data symbols. Fig. 2 shows, for BPSK and QPSK, the ratios $CRB_F/MCRB_F$ and $ACRB_F/MCRB_F$, for both observation models. The behavior of the various curves is as follows.

- For small (large) SNR, the CRBs converge to the corresponding ACRBs (to the MCRB). The value of E_s/N_0 at which the CRB comes close to the MCRB is larger for QPSK than for BPSK. We have verified that for all constellations considered (PAM, PSK, QAM) this value of E_s/N_0 increases with the constellation size M.
- For N>2, the simplified observation model yields the larger CRBF. Indeed, as the transformation from r(t) to $\{r_k\}$ cannot be inverted, estimating F from $\{r_k\}$ instead of r(t) is suboptimum. We have pointed out in section III that both observation models yield the same CRBF for large values of E_s/N_0. Our numerical results indicate that the two observation models yield essentially the same CRBF for all SNR values of practical interest. The difference between these observation models becomes apparent only at (very) small SNR. Consequently, the shape of the transmit pulse has no effect on the CRB at moderate and high SNR.

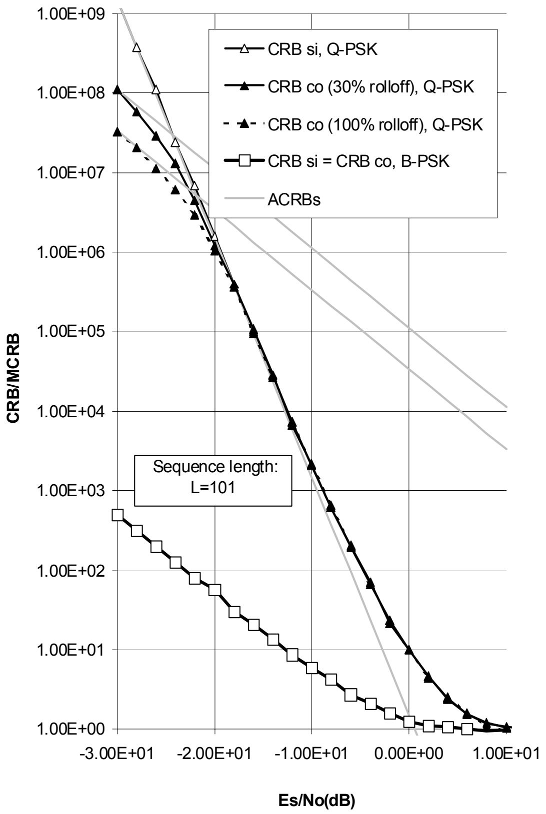

Figure 2. Effect of the observation model on the ratio $CRB_F/MCRB_F$

- We have pointed out in section II that in case N=2 (PAM), both observation models yield essentially the same ACRB. Our results show that both observation models also yield essentially the same CRB.

Let us give an interpretation of the above results in terms of maximum-likelihood (ML) estimation of (F, θ). Denoting by (F', θ') a trial value of the estimate, the main point is that $|x_k|$ does not depend on F' when using the simplified model. The ML estimate F' maximizes L(F'), which denotes the maximum of the log-likelihood function over θ' at given F'. For high SNR, it can be verified from (8) that L(F') is determined by

$$L^2(F') = \sum_{k,\ell=-K}^{K} |a_k a_\ell x_k x_\ell| \exp\left(j \arg(a_k^* a_\ell x_k x_\ell^*)\right) \tag{28}$$

For both observation models, we have $\arg(a_k^* a_\ell x_k x_\ell^*) \cong (\ell - k)(F' - F)$, whereas $|a_k a_\ell x_k x_\ell|$ depends on F' only when using the correct model. However, as both models yield essentially the same CRB for sufficiently high SNR, it follows that the main dependence of L(F') on F' is through $\arg(a_k^* a_\ell x_k x_\ell^*)$, with $k \neq \ell$. For low SNR and N>2, the function to be maximized becomes

$$L(F') = \sum_{k=-K}^{K} |x_k|^2 + \begin{matrix} Nth-order \\ terms \end{matrix} \tag{29}$$

with the Nth-order terms decreasing with decreasing SNR. When using the correct model, the quadratic term in (29) depends on F', and dominates at low SNR. In case of the simplified model, L(F') from (29) depends on F' only through the higher order terms, yielding noise enhancement and resulting in a CRB_F that is substantially larger than when using the correct model. For low SNR and N=2, the function to be maximized is given by

$$L(F') = \sum_{k=-K}^{K} |x_k|^2 + \left| \sum_{k=-K}^{K} x_k^2 \right| \tag{30}$$

In this case, both observation models yield essentially the same CRB_F; hence, a similar reasoning as for large SNR indicates that L(F') from (30) depends on F' mainly through $\arg(x_k^2 x_\ell^{*2})$, with $k \neq \ell$.

V. CONCLUSIONS AND REMARKS

In this contribution, we have considered the CRB related to joint carrier phase and frequency estimation of a noisy linearly modulated signal with arbitrary square-root Nyquist transmit pulse. We have contrasted the results obtained from the correct observation model with those resulting from a simplified model, used in [1,2,3,8].

For both observation models the frequency and phase are uncoupled (i.e., $J_{F\theta} = 0$). As far as phase estimation is concerned, both observation models yield the same CRB, which can be found in [1,2,3]. For complex data symbols, the CRBs related to frequency estimation at practical SNR are essentially the same for both observation models; substantial differences occur only at very low E_s/N_0. This is because in the simplified model the frequency offset does not affect the magnitude of the matched filter output signal. For real data symbols both observation models yield essentially the same CRB, irrespective of SNR.

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