Carrier Phase Recovery in Turbo Receivers: Cramer-Rao Bound and Synchronizer Performance

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ABSTRACT

This contribution considers carrier phase recovery in turbo receivers. We point out that the resulting Cramer-Rao bound (CRB) is close to its *high* signal-to-noise ratio (SNR) asymptote, even at the typical *low* operating SNR of turbo systems. We show that '*code-unaware*' phase estimators are unable to perform closely to the CRB at the normal operating SNR of powerful codes. The advantage of '*code-aware*' carrier recovery is illustrated with examples of existing phase estimation techniques applied to a turbo-coded 4-PSK system. Comparison with the CRB reveals that the 'code-aware' soft decision-directed phase estimator presented in [1] performs very close to the CRB.

Keywords: Carrier recovery, Phase estimation, Coded systems, Cramer-Rao bound, Synchronizer performance

INTRODUCTION

The impressive performance of turbo receivers implicitly assumes coherent detection, i.e. the carrier reference must be recovered accurately before data detection. Synchronization of turbo receivers yet is a very challenging task since such systems usually operate at extremely low SNR values. The development of accurate carrier phase estimation techniques has therefore recently received a lot of attention in the technical literature. A common approach to judge the performance of such parameter estimators consists in comparing their resulting mean-square error with the Cramer-Rao bound (CRB), which is a fundamental lower bound on the error variance of unbiased estimators [2]. In order to avoid the computational complexity related to the *true* CRB, a *modified* CRB (MCRB) has been derived in [3]. The MCRB is much simpler to evaluate than the CRB, but is in general looser (i.e. lower) than the CRB, especially at low SNR. In [4], the $CRB_{uncoded}$ for carrier phase estimation from *uncoded* data symbols has been obtained. In [5], the CRB for carrier phase estimation from *coded* data has been expressed in terms of the marginal a posteriori probabilities (APPs) of the coded symbols, allowing the numerical evaluation of the bound for a wide range of coded systems, including schemes with iterative detection (turbo schemes).

This contribution further examines the CRB for carrier phase recovery in turbo receivers, and compares the bound to the performance of actual synchronizers. We point out that at the normal operating SNR of turbo receivers, carrier phase estimation should exploit the encoding properties in order to achieve an MSE that is close to the CRB.

CODED SYSTEM IN THE PRESENCE OF A CARRIER PHASE OFFSET

We consider the observation vector **r** =(r_{-K}, …, r_K) with

$$r_k = \varepsilon \, a_k \exp(j\theta) + w_k \qquad (1)$$

In (1), **a**=(a_{-K},…,a_K) is a vector of L = 2K+1 data symbols taken from an M-PSK, M-QAM or M-PAM constellation ($E[|a_k|^2]=1$) according to a combination of an encoding rule and a mapping rule. The vector **w**=(w_{-K},…,w_K) consists of independent identically distributed zero-mean complex Gaussian noise variables, with independent real and imaginary parts each having a variance of ½. The parameter θ represents the deterministic but unknown carrier phase. Finally, $\varepsilon = (E_s/N_0)^{1/2}$, with E_s and N_0 the energy per coded symbol and the noise power spectral density, respectively.

TRUE AND MODIFIED CRB

Suppose that one is able to produce from the observation vector **r** an unbiased estimate $\hat{\theta}$ of the carrier phase θ. Then the estimation error variance is lower bounded by the CRB: $E_{\mathbf{r}}[(\hat{\theta}-\theta)^2] \geq CRB_{\theta}$, where CRB_{θ} is given by [2]

$$CRB_{\theta} = \left(E_{\mathbf{r}}\left[\left(\frac{d}{d\theta} \ln\left(p(\mathbf{r};\theta) \right) \right)^2 \right] \right)^{-1} \qquad (2)$$

The probability density p(**r**;θ) of **r**, corresponding to a given value of θ, is called the *likelihood function* of θ. The expectation $E_{\mathbf{r}}$[.] in (2) is with respect to p(**r**;θ). Since the observation **r** depends not only on the parameter θ to be estimated but also on the nuisance vector **a** of data symbols,

This work has been supported by the Interuniversity Attraction Poles Program – Belgian State – Federal Office for Scientific, Technical and Cultural Affairs.

p($\mathbf{r}$;θ) is obtained by averaging the likelihood function p($\mathbf{r}|\mathbf{a}$;θ) of the vector (θ,$\mathbf{a}$) over the a priori distribution of the nuisance parameter vector $\mathbf{a}$, i.e. $p(\mathbf{r};\theta) = E_{\mathbf{a}}[p(\mathbf{r}|\mathbf{a};\theta)]$.

The presence of the nuisance data symbols makes the evaluation of the likelihood function of θ and of the corresponding CRB (2) quite tedious (in particular for coded data sequences). An alternative lower bound, the modified CRB (MCRB), has been proposed in [3]

$$MCRB_\theta = \left(E_{\mathbf{a},\mathbf{r}}\left[\left(\frac{\mathrm{d}}{\mathrm{d}\theta} \ln\left(p(\mathbf{r}\,|\,\mathbf{a};\theta)\right) \right)^2 \right] \right)^{-1} \qquad (3)$$

where

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\ln\left(p(\mathbf{r}\,|\,\mathbf{a};\theta)\right) = 2e \sum_{k=-K}^{K} \mathrm{Im}(a_k^* r_k e^{-j\theta}) \qquad (4)$$

The MCRB for carrier phase estimation is easily evaluated analytically, in spite of the presence of the nuisance data symbols $\mathbf{a}$. One obtains: $MCRB_\theta = N_0/2LE_s$. Since the computation of the MCRB makes no specific assumptions regarding the correlation between the data symbols, this result is valid for uncoded as well as for coded transmission, *regardless of the encoding rule*. However, the reduced computational effort of the MCRB as compared to the true CRB goes at the cost of degraded tightness, i.e., the MCRB is in general looser then the CRB: $E_{\mathbf{r}}[(\hat{\theta}-\theta)^2] \geq CRB_\theta \geq MCRB_\theta$. In [6], it has been shown that the high-SNR limit of the CRB coincides with the MCRB. Though the MCRB is a good approximation of the true bound for higher SNR, it departs significantly from the CRB at low SNR [4]. To analyze and design carrier phase estimators operating at low SNR it is therefore important to know and understand the behavior of the ratio CRB/MCRB. In [5], we have expressed the CRB for carrier phase estimation from a noisy linearly modulated signal with encoded data symbols as

$$\frac{d}{d\theta}\ln\left(p(\mathbf{r};\theta)\right) = 2e \sum_{k=-K}^{K} \sum_{m=0}^{M-1} \Pr[A_k = a_m \,|\, \mathbf{r};\theta]\, \mathrm{Im}\left(a_m^* r_k e^{-j\theta}\right) \qquad (5)$$

where $\Pr[A_k = a_m \,|\, \mathbf{r};\theta]$ is the a posteriori probability (APP) of the k-th coded symbol, $(\alpha_0, \alpha_1, ..., \alpha_{M-1})$ denotes the set of constellation points, and $\mathbf{r}$ is given by (1). Substitution of (5) into (2) allows the numerical evaluation of the CRB for turbo receivers by means of computer simulation. The fact that for turbo receivers the APPs needed in (5) are obtained through iterations makes the computation of the CRB quite computationally intensive.

Note that (5) reduces to (4) when $\Pr[A_k = a_k \,|\, \mathbf{r};\theta] = 1$. This indicates that CRB converges to MCRB provided that the ratios $R(\mathbf{r},k,\alpha_m) = \Pr[A_k = a_m \,|\, \mathbf{r};\theta] / \Pr[A_k = a_k \,|\, \mathbf{r};\theta]$ are likely to be much smaller than 1 for all $a_m \neq a_k$. The probability that the random variable $R(\mathbf{r},k,\alpha_m)$ exceeds y for at least one constellation point $a_m \neq a_k$ is given by

$$P_y = \frac{1}{M^{rL}} \sum_{\mathbf{a}\in x} \Pr\left[\bigcup_{a_m \neq a_k} R(\mathbf{r},k,a_m) \geq y \right] \qquad (6)$$

where ξ denotes the set of legitimate coded sequences of length L. We assume that $\Pr[\mathbf{a}] = M^{-rL}$ for $\mathbf{a} \in \xi$ and $\Pr[\mathbf{a}] = 0$ otherwise, where the quantities r and M denote the rate of the code and the number of constellation points, respectively. With y=1, Eq(6) is nothing but the symbol error rate (SER) resulting from an optimal maximum a posteriori probability (MAP) symbol decoder [7]. This establishes a relation between the convergence of the CRB to the MCRB and the SER of the code. This relation is illustrated in Fig.1. The ratio CRB/MCRB for carrier phase estimation based on the observation of L=2K+1 symbols (the right ordinate) and the SER corresponding to perfect synchronization and MAP decoding (the left ordinate, solid lines) are presented as a function of E_s/N_0 per coded symbol. The following scenarios are considered: (i) *uncoded* (U) transmission, (ii) a memory order 4 *turbo* (T) code with interleaver size I and overall rate r, and (iii) a rate ½ *LDPC* code with block length 1000. We observe that, for large E_s/N_0, the CRB converges to the MCRB; this is consistent with [6]. When E_s/N_0 decreases, a critical value $(E_s/N_0)_{crit}$ is reached where the CRB starts to diverge from the MCRB. Fig.1 shows that this critical value corresponds to a SER of about 10^{-3}. Hence, as far as the CRB for carrier phase estimation is concerned, transmission at a SER of less than 10^{-3} is nearly equivalent to transmitting a sequence of known training symbols. At the normal operating SNR of digital communication systems, the CRB is therefore very well approximated by the MCRB. Finally, it may be useful to note that, for small constellations like 2-PSK and 4-PSK, SER≈BER (see also Fig.1, with BER in dashed line). The bit error rate (BER) is a more important code performance parameter than the SER, and is therefore more often available in technical literature.

PHASE ESTIMATOR PERFORMANCE

In this section we consider the mean square error (MSE) resulting from some existing phase estimators operating on the rate ½ turbo-encoded 4-PSK scheme from the previous section. This MSE is compared to the CRB for both coded and uncoded transmission. The real part of the k^{th} transmitted symbol corresponds to the k^{th} information bit (i_k) and the imaginary part corresponds to the k^{th} parity bit (p_k) alternately originating from encoder 1 (enc1) and encoder 2 (enc2). We assume a phase offset θ of 0.2 rad.

The Viterbi and Viterbi (V&V) estimator

With the traditional Viterbi and Viterbi (V&V) estimator, phase estimates are obtained as [8]

$$q = \frac{1}{M} \arg\left\{ \sum_{k=-K}^{K} |r_k|^X e^{jM \arg\{r_k\}} \right\} \quad (7)$$

We take X=2, since it is indicated in [8] that for QPSK signals, the monomial $|r_k|^2$ is nearly optimal.

The decision-directed (DD) estimator

Decision-directed (DD) phase estimation can be carried out iteratively as shown in (8)

$$q^{(i)} = \arg\left\{ \sum_{k=-K}^{K} \left(d_k^{(i-1)}\right)^* r_k \right\} \quad (8)$$

where $d_k^{(i-1)}$ is the decision regarding the coded symbol a_k based upon a previous phase estimate $\theta^{(i-1)}$. Hence, the phase estimate is refined at each iteration. Depending on the way the decisions $d_k^{(i)}$ are obtained, we distinguish between *hard* DD (hDD) estimation, *tentative* DD (tDD) estimation, and *soft* DD (sDD) estimation. In classical hDD estimation the symbol decision $d_k^{(i)}$ result from applying the rotated sample $r_k\exp(-j\theta^{(i)})$ to a hard symbol-by-symbol decision device that does not take the code properties into account: $d_k^{(i)}$ is the constellation point that is closest to $r_k\exp(-j\theta^{(i)})$. A tDD estimator does take the code properties into account by extracting hard decisions from the decoder during the decoding process of the sequence $\{r_k\exp(-j\theta^{(i)})\}$. When the quantities $d_k^{(i)}$ in (8) are replaced with

$$d_{sk}^{(i)} = \sum_{n=0}^{M-1} \Pr\left[a_k = a_n \mid \mathbf{r}, q^{(i)}\right] a_n \quad (9)$$

with $\Pr\left[a_k = a_n \mid \mathbf{r}, q^{(i)}\right]$ denoting the symbol APPs, the resulting phase estimate is referred to as sDD [1], and $d_{sk}^{(i)}$ are called soft decisions.

Taking into account that the turbo decoder is an iterative process itself, and that a standard turbo decoder provides soft information regarding the information bits only, there are different strategies to implement tDD and sDD estimators. We will consider the following strategies:

- **tDD** : In order to reduce the computational complexity, only one turbo decoding operation is performed for each synchronizer iteration. The hard decisions of the information bits (i_k) are based on the value of the APPs, provided by the turbo decoder. The hard decision of the parity bits (p_k) is obtained as in the hDD case.
- **sDD(i)**: Only 1 turbo iteration is performed at each iteration of the sDD phase estimator. The APPs of the coded symbols are computed as the product of the APPs of the corresponding information bit i_k (APP provided by the turbo decoder) and parity bit p_k (APP computed as if parity bit is uncoded).
- **sDD(ii)**: Only 1 turbo iteration is performed at each iteration of the sDD phase estimator. The APPs of the *coded symbols* (instead of the information bits only) are computed by a slightly modified turbo decoder (we assume i_k and p_k statistically independent if p_k originates from enc2, which is a good approximation for large interleavers; when p_k originates from enc1, the correct dependence of i_k and p_k is taken into account).

Zhang and Burr (Z&B) estimator

In [9], Zhang & Burr propose an iterative ad hoc phase estimator (Z&B) related to maximum likelihood estimation

$$q^{(i)} = \arg\max_{q^{(i)}} \sum_{k=-K}^{K} \ln\left(2\frac{E_s}{N_0}\mathrm{Re}\left(\left(d_{sk}^{(i-1)}\right)^* \mathrm{e}^{-\mathrm{j}q^{(i)}} r_k\right)\right) \quad (10)$$

where $d_{sk}^{(i)}$ are soft decisions given by (9) and computed as in sDD(ii). The secant method enables to find the value that maximizes (10).

Numerical results and discussion

The phase estimators described above can be divided into two categories: '*code-aware*' (tDD, sDD, Z&B) and '*code-unaware*' (V&V, hDD) estimators. The first category exploits code properties in the phase estimation process, while the second category does not. The MSE of '*code-unaware*' phase detectors for coded transmission is essentially the same as for uncoded transmission. $CRB_{uncoded}$ therefore lower bounds the MSE resulting from any 'code-unaware' phase synchronizer, regardless of the underlying encoding rule. As the CRB for a powerful code, evaluated at the normal operating SNR, is considerably smaller than $CRB_{uncoded}$, 'code-aware' phase estimators are potentially more accurate than 'code-unaware' phase estimators. Moreover, the ratio $CRB_{uncoded}/CRB$ indicates to what extent synchronizer performance can be improved by making clever use of the code structure.

The MSE resulting from the V&V, hDD, tDD, sDD and Z&B estimators are shown in Fig.2 as a function of E_s/N_0 per coded symbol. The iterative synchronizers are iterated until the phase estimate has negligible bias in the range $E_s/N_0 \geq 0$dB. The initial phase estimate was set to zero. The estimator performance apparently reflects the degree of exploiting the encoding rule during the estimation process. In order of increasing estimation accuracy we observe:

- The V&V and hDD estimator, which are 'code-unaware'. Their performance is lower bounded by $CRB_{uncoded} > CRB$.

- The tDD estimator, which uses hard symbol decisions and therefore loses a great part of the available information.
- The sDD(i) estimator, which exploits the code properties only *partially* as the APPs of the *parity* bits are calculated without taking into account the code trellis.
- The Z&B estimator, whose derivation was based on the simplifying but incorrect assumption that the transmitted data symbols are statistically independent.
- The sDD(ii) estimator, which performs closely to the CRB, in spite of making only one turbo iteration per iteration of the synchronizer.

CONCLUSIONS

This contribution compares the CRB for carrier phase estimation in coded systems against the MCRB, $CRB_{uncoded}$ and the performance of existing phase estimators. Assuming the normal operating SNR of codes with a significant coding gain (say, such that SER < 10^{-3}) we have shown that CRB $\approx$ MCRB << $CRB_{uncoded}$. Furthermore, $CRB_{uncoded}$ has been shown to lower bound the performance of '*code-unaware*' synchronizers, which make no use of the code structure. This implies that in order to approach optimal performance, estimators should make clever use of the code properties in the phase estimation process. The phase estimator presented in [1], which is '*code-aware*', has been shown to operate very closely to the CRB.

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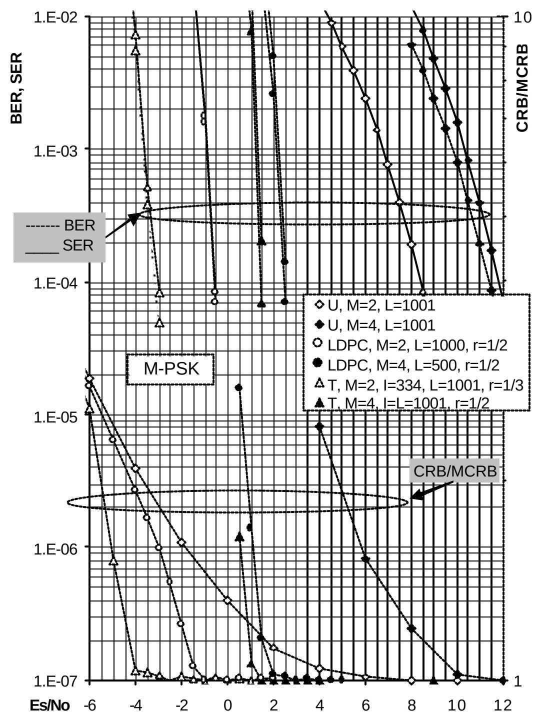

Fig.1: Relation between CRB/MCRB and SER (BER)

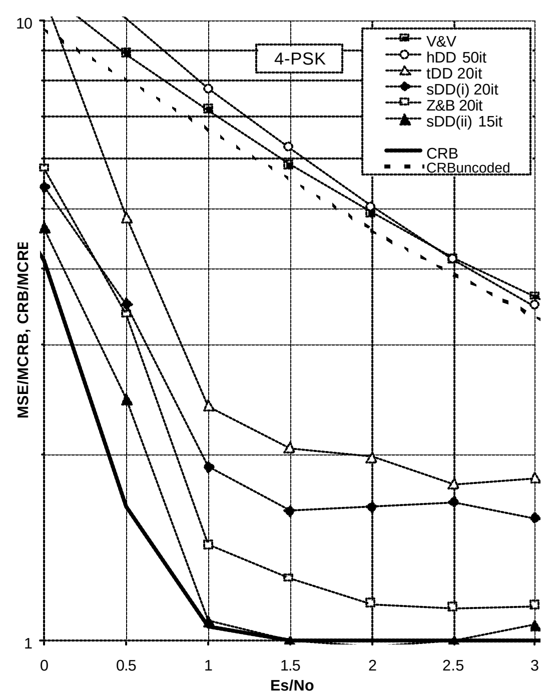

Fig.2: Estimator MSE compared against CRB