Magnitude of the Distance Estimation Bias in Received Signal Strength Visible Light Positioning

Nobby Stevens, *Member, IEEE*, and Heidi Steendam, *Senior Member, IEEE*

***Abstract*— In several works, statistical precision of distance estimators in visible light positioning was studied by evaluating the Cramér–Rao lower bound, assuming that the estimator is unbiased. Here, it is demonstrated that the maximum likelihood estimator of the distance is not unbiased. Consequently, the inverse of the Fisher information value is no longer a strict lower bound. We evaluate this bias for several illumination levels and it is found that the impact is very small. For many practical situations, the inverse of the Fisher information can serve as the approximation for the mean-squared error of the maximum likelihood estimator of the distance.**

***Index Terms*— Visible light positioning, maximum likelihood estimator, Fisher information value.**

I. Introduction

ACCURATE indoor positioning at a fair price is a long standing challenge. Different techniques based on radio frequent (RF) signals with different spectral characteristics have been investigated during the last decades [1]. For indoor premises, an artificial illumination infrastructure is generally deployed. The traditional light bulbs are increasingly replaced by light emitting diodes (LEDs) due to the favorable properties of increased energy efficiency, higher life time and flexible color rendering capabilities. For new buildings, the deployment of LEDs for the illumination is not questioned anymore due to the aforementioned advantages. On top of that, LEDs have the interesting property of a high electrical-optical bandwidth that can be deployed in a novel type of wireless information exchange without compromising the original illumination functionality, namely visible light communication (VLC) [2]–[5]. VLC is considered as an emerging technology that will be integrated in future 5G network solutions [6].

Due to the spatial confinement of light cones for illumination, a novel technique called visible light positioning (VLP) to perform indoor localization has gained interest the last couple of years [7]. One approach for VLP is the usage of received signal strength (RSS) for ranging, where the deployment of a single photo diode and a non-synchronized network of LEDs enables indoor positioning [8]. Several authors have investigated the precision of RSS-based VLP [9]–[12].

Manuscript received June 21, 2018; revised August 14, 2018; accepted August 19, 2018. Date of publication August 24, 2018; date of current version November 12, 2018. Part of this work was executed within LEDsTrack, a research project bringing together academic researchers and industry partners. The LEDsTrack project was co-financed by imec (iMinds) and received project support from Flanders Innovation & Entrepreneurship. The second author gratefully acknowledges the funding from the Belgian Research Councils FWO and FNRS through the EOS grant 30452698. The associate editor coordinating the review of this letter and approving it for publication was M. Safari. *(Corresponding author: Nobby Stevens.)*

N. Stevens is with ESAT-DRAMCO, KU Leuven, 3000 Leuven, Belgium (e-mail: nobby.stevens@kuleuven.be).

H. Steendam is with TELIN-DIGCOM, Ghent University, 9000 Gent, Belgium (e-mail: heidi.steendam@ugent.be).

Digital Object Identifier 10.1109/LCOMM.2018.2867026

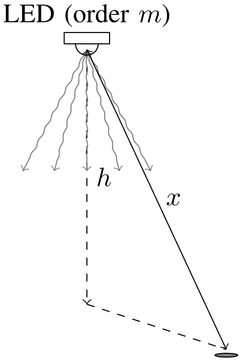

Fig. 1. The LED transmits the signal $s(t)$ during T_s, while the photodiode receives the noise-polluted signal $r(t)$.

As a consequence of the non-linearity of the received signal strength with regard to the distance between LED and receiving photo diode, the distance estimation becomes biased. Up until now, the biased nature of the distance estimate and its impact on the mean-squared error was neglected. In Section II of this work, the statistical framework dealing with the model is elaborated for the distance maximum likelihood estimator. As a closed-form analytical solution is not feasible, we assess the bias impact through numerical evaluation for configurations with representative parameters in Section III. It is shown that the distance estimation is indeed not unbiased. Though, we demonstrate in this work that the impact of this bias is very small compared with the root mean-squared error under normal illumination conditions.

II. Analysis of the Distance Estimation Accuracy

Consider Fig. 1, where a ceiling attached LED with Lambertian order m transmits an optical intensity modulated signal $s(t)$ during a time interval with duration T_s. The receiver is found in a plane at a vertical distance h below the LED, where h is known at the receiver side. The photo diode has a receiver area A_R and responsivity R_p. The resulting current $r(t)$ at the receiver can be described by [13]

$$r(t) = \alpha R_p s(t) + w(t), \tag{1}$$

where $w(t)$ is a sample of a white gaussian noise process with zero mean and spectral power density N_0. The channel gain α is described by

$$\alpha = \frac{(m+1)}{2\pi} \frac{h^{m+1}}{x^{m+3}} A_R. \tag{2}$$

The goal is that the receiver, based on $r(t)$, determines the Euclidean distance x between the receiver and the transmitting LED. When three or more of these values originating from different LEDs are recovered, indoor positioning becomes feasible.

In order to maximize the signal-to-noise ratio, matched filtering is applied [14]. This leads to the output of the matched filter r:

$$r = \int_{T_s} r(t)s(t)\mathrm{d}t = \alpha R_p \mathcal{E} + w, \tag{3}$$

with

$$\mathcal{E} = \int_{T_s} s^2(t)\mathrm{d}t. \tag{4}$$

Due to the properties of the additive white gaussian noise process, we have that $\mathbb{E}\{w\} = 0$ and $\mathrm{Var}\{w\} = \mathcal{E}N_0$. At a specific location, N i.i.d. measurements of r, noted as $r[n]$, are executed:

$$r[n] = \alpha R_p \mathcal{E} + w[n] = \beta + w[n], \tag{5}$$

with $n = 0, \ldots, N-1$ and

$$\beta \triangleq \frac{(m+1)}{2\pi} h^{m+1} A_R R_p \mathcal{E} \frac{1}{x^{m+3}} = \gamma \frac{1}{x^{m+3}}. \tag{6}$$

Suppose that γ, which includes the contributions of the transmitted waveform $s(t)$ and its power level, is known at the receiver side. The likelihood function of the N samples with the unknown but non-random parameter β is described as:

$$p(\mathbf{r};\beta) = \frac{e^{-\frac{1}{2\mathcal{E}N_0}\sum_{n=0}^{N-1}(r[n]-\beta)^2}}{[2\pi\mathcal{E}N_0]^{N/2}}, \tag{7}$$

leading to

$$\frac{\partial \ln p(\mathbf{r};\beta)}{\partial\beta} = \frac{N}{N_0\mathcal{E}}\left[\frac{1}{N}\sum_{n=0}^{N-1} r[n] - \beta\right]. \tag{8}$$

The maximum likelihood estimate $\hat{\beta}_{ml}(\mathbf{r})$ is the solution of the likelihood equation

$$\left.\frac{\partial \ln p(\mathbf{r};\beta)}{\partial\beta}\right|_{\beta=\hat{\beta}_{ml}(\mathbf{r})} = 0. \tag{9}$$

This readily leads to the value of $\hat{\beta}_{ml}(\mathbf{r})$:

$$\hat{\beta}_{ml}(\mathbf{r}) = \frac{1}{N}\sum_{n=0}^{N-1} r[n]. \tag{10}$$

It is straightforward to see that $\hat{\beta}_{ml}(\mathbf{r})$ is an unbiased estimator since $\mathbb{E}\{r[n]\} = \beta$. Considering the linear form of (8), it can also be concluded that $\hat{\beta}_{ml}(\mathbf{r})$ is an efficient estimator with a mean-squared error equal to the inverse of the Fisher information term $J_F(\beta)$ (11):

$$\mathrm{MSE}\left\{\hat{\beta}_{ml}(\mathbf{r})\right\} = J_F^{-1}(\beta) = \frac{\mathcal{E}N_0}{N}. \tag{11}$$

Based on previous observations, one can write the probability density distribution for $\hat{\beta}_{ml}$ as in (12).

$$p\left(\hat{\beta}_{ml};\beta\right) = \frac{N^{1/2} e^{-\frac{N}{2\mathcal{E}N_0}\left(\hat{\beta}_{ml}-\beta\right)^2}}{[2\pi\mathcal{E}N_0]^{1/2}} \tag{12}$$

In order to perform indoor positioning based on trilateration, the distance x is of particular interest. Taking into account that the estimator $\hat{\beta}_{ml}(\mathbf{r})$ is unbiased and efficient, we would like to derive the properties of an estimator for x, i.e. $\hat{x}(\mathbf{r})$. The maximum likelihood estimator commutes over linear as well as non-linear transformations, indicating

$$\hat{x}_{ml}(\mathbf{r}) = \left[\frac{\gamma}{\hat{\beta}_{ml}(\mathbf{r})}\right]^{\frac{1}{m+3}} = \gamma^{\frac{1}{m+3}}\left[\frac{1}{N}\sum_{n=0}^{N-1} r[n]\right]^{\frac{-1}{m+3}}. \tag{13}$$

However, the property of unbiasedness only commutes over linear transformations [15]. Hence, noting that the relationship between β and x is non-linear, we can conclude that the maximum likelihood estimator (13) will not be unbiased. Due to this fact, it is not correct to state that the mean-squared error of the estimator $\hat{x}_{ml}(\mathbf{r})$ is larger than or equal to the inverse of the Fisher information value $J_F(x)$, given by

$$J_F(x) = -\mathbb{E}\left\{\frac{\partial^2 \ln p(\mathbf{r};x)}{\partial x^2}\right\} = \frac{N\mathcal{E}}{N_0}\left(\frac{(m+3)\alpha R_p}{x}\right)^2. \tag{14}$$

This aspect was not covered by other authors elaborating on the statistical properties of distance estimation in RSS-based VLP solutions [9]–[12]. However, a bound similar to the Cramér-Rao bound can be derived for biased estimates. Suppose the average of the estimate is written as

$$\mathbb{E}\{\hat{x}_{ml}(\mathbf{r})\} = x + b(x). \tag{15}$$

It can be demonstrated (see Appendix) that the Cramér-Rao bound for biased estimators (CRB-B) is written as

$$\begin{aligned}\mathrm{MSE}\{\hat{x}_{ml}(\mathbf{r})\} &= \mathbb{E}\left\{(\hat{x}_{ml}(\mathbf{r}) - x)^2\right\} \\ &\geq J_F^{-1}(x)\left(1 + \frac{\mathrm{d}b(x)}{\mathrm{d}x}\right)^2\end{aligned} \tag{16}$$

Knowledge of $b(x)$ and its derivative to x is thus crucial to evaluate the precision and accuracy of the distance estimation. In the following section, these values are determined for a number of representative configurations.

III. Numerical Assessment of the Bias and Mean Square Error

Considering (12) and (13) and applying the rules to transform a probability distribution from one variable to another, it is straightforward to obtain the probability distribution on $\hat{x}_{ml}$, given a certain distance x, in (17).

$$p(\hat{x}_{ml};x) = \frac{\gamma(m+3)}{\hat{x}_{ml}^{(m+4)}} \frac{N^{1/2} e^{-\frac{N\gamma^2}{2\mathcal{E}N_0}\left(\frac{1}{\hat{x}_{ml}^{(m+3)}} - \frac{1}{x^{(m+3)}}\right)^2}}{[2\pi\mathcal{E}N_0]^{1/2}} \tag{17}$$

This leads to

$$b(x) = \int_0^{+\infty} \hat{x}_{ml}\, p(\hat{x}_{ml};x)\,\mathrm{d}\hat{x}_{ml} - x \tag{18}$$

and

$$\mathbb{E}\left\{(\hat{x}_{ml} - x)^2\right\} = \int_0^{+\infty} (\hat{x}_{ml} - x)^2\, p(\hat{x}_{ml};x)\,\mathrm{d}\hat{x}_{ml}. \tag{19}$$

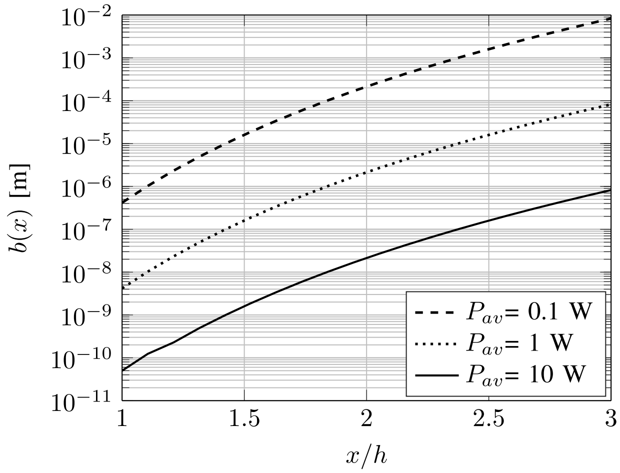

Fig. 2. Value of the bias $b(x)$ in case of the maximum likelihood estimator $\hat{x}_{ml}(\mathbf{r})$. P_{av} is the average emitted optical power.

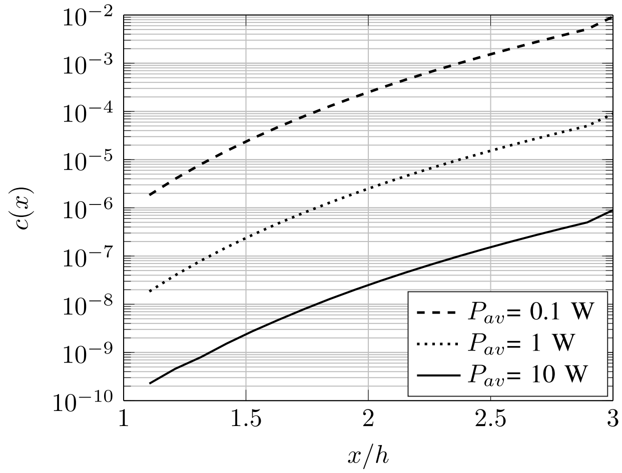

Fig. 3. Value of $c(x)$ in case of the maximum likelihood estimator $\hat{x}_{ml}(\mathbf{r})$. P_{av} is the average emitted optical power.

In what follows, we consider the following representative values for RSS-based VLP solutions, i.e. a receiver area $A_R = 1$ cm^2, a Lambertian order m of the transmitting LED equal to 1, and a responsivity R_p of the receiving photo diode of 0.4 A/W, where the vertical distance h equals 3 m. For the noise spectral density, the model proposed by Kahn and Barry [16] and used by several authors [9], [13], is applied. Considering the receiver surface A_R of 1 cm^2, this leads to a value for N_0 of 2.67 10^{-22} A^2/Hz. Let us further suppose that the LED is transmitting the average optical power P_{av} during the symbol time T_s of 1 ms, where $s(t)$ is a dc-shifted cosine wave:

$$s(t) = P_{av}\left[1 - \cos\left(\frac{2\pi}{T_s}t\right)\right]. \tag{20}$$

We set the number of samples at the matched filter output N equal to 10. This results in a latency of 10 ms to gather the information to be processed. Evaluating (4), we obtain the value $\mathcal{E}$ as in (21) when the optical power signal $s(t)$ equals (20).

$$\mathcal{E} = \frac{3T_s}{2}P_{av}^2. \tag{21}$$

The distance x is varied from h to $3h$. The integrations of (18) and (19) were evaluated numerically with Python by means of the quad function of the scipy.integrate library.

In Fig. 2, the bias $b(x) = \mathbb{E}\{\hat{x}_{ml}(\mathbf{r})\} - x$ is shown for different average optical power levels. In order to link these optical power intensities to illumination levels, we considered an LED with a correlated colour temperature (CCT) of 5451 K, corresponding to a cool light. At $x = 3$ m right below the LED (i.e. at $x/h = 1$), this results in illumination levels of 1.2, 12 and 120 lux for respectively P_{av} 0.1, 1 and 10 W. For lighting purposes in e.g., offices, the illumination levels are typically between 250 and 500 lux. Taking into account that multiple LEDs contribute to the total illumination, it is reasonable to state that an LED with an average optical power P_{av} of 10 W can be used in those environments. We can see in Fig. 2 that even for the lowest illumination levels (1.2 lux), the bias in case the receiver is located right under the LED ($x = h$) is very small (below 1 μm). However, this value increases rapidly as a function of x/h. When e.g., $P_{av} = 0.1$ W and $x/h = 3$, the bias equals 1 cm. Nonetheless, for illumination levels that meet the guidelines for indoor environments, the bias remains small.

When considering relationship (16), we see that the $\text{MSE}\{\hat{x}_{ml}(\mathbf{r})\}$ is larger than or equal to $J_F^{-1}(x)\left(1 + \frac{\mathrm{d}b(x)}{\mathrm{d}x}\right)^2$. Unfortunately, this inequality does not tell anything about how much larger the $\text{MSE}\{\hat{x}_{ml}(\mathbf{r})\}$ is. Having determined the value of the bias, its derivative and the $\text{MSE}\{\hat{x}_{ml}(\mathbf{r})\}$, we can assess both sides of the inequality. In order to quantify how much the $\text{MSE}\{\hat{x}_{ml}(\mathbf{r})\}$ is larger than the CRB-B, we introduce a positive parameter $c(x)$ as in (22).

$$\text{MSE}\{\hat{x}_{ml}(\mathbf{r})\} = [1 + c(x)]\, J_F^{-1}(x)\left(1 + \frac{\mathrm{d}b(x)}{\mathrm{d}x}\right)^2 \tag{22}$$

The value of $c(x)$ is shown in Fig. 3 for the same parameter settings as used in Fig. 2. It is clear that the lower theoretical bound is not reached (this would imply that $c(x) = 0$). Nevertheless, the values of $c(x)$ are very small compared to 1, especially right below the LED and for optical power levels that provide sufficient illumination.

Finally, we evaluated the impact of the biasedness of the distance estimator with regard to the overall precision. In general, we can write that

$$\hat{x}_{ml}(\mathbf{r}) = x + b(x) + y(x), \tag{23}$$

where $b(x)$ is the bias (18) and $y(x)$ is the random zero-mean estimation error with

$$\text{MSE}\{\hat{x}_{ml}(\mathbf{r})\} = \text{MSE}\{y(x)\}. \tag{24}$$

It is particularly interesting to compare the value of the bias $b(x)$ (as a measure of the accuracy) and the square root of the variance (as a measure of the precision). The ratio $\sqrt{\text{MSE}\{y(x)\}} = \text{RMSE}\{y(x)\}$ on the bias $b(x)$ is shown in Fig. 4. From this figure, it follows that the ratio is for most cases much larger than one, indicating that the bias value

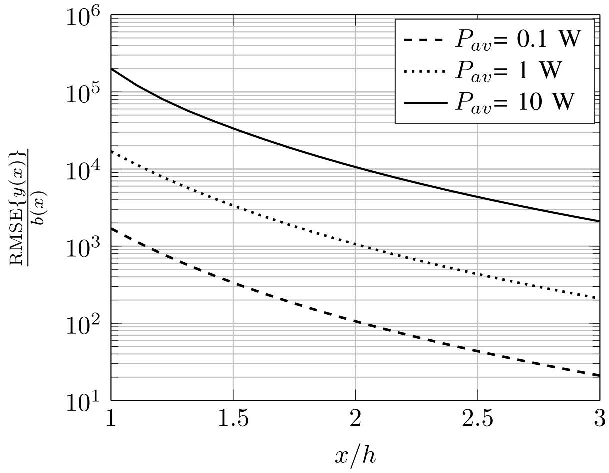

Fig. 4. Evaluation of the impact of the bias value with regard to the overall precision for the maximum likelihood distance estimator. P_{av} is the average emitted optical power.

can be neglected with regard to the precision in the distance assessment for normal illumination levels.

IV. Conclusion

In this work, we demonstrated that the distance estimation in RSS-based VLP is not unbiased. The value of this bias is determined through numerical assessment for a configuration with representative parameters in case of the maximal likelihood estimator for the distance. Different illumination levels were considered. First of all, the bias value is in the order of sub micrometers for realistic illumination levels. It was also found that the maximum likelihood estimator is close to be efficient, thus reaching the CRB-B lower bound. Finally, the precision of the estimation was compared to the bias value, leading to the conclusion that for most practical situations, this bias does not influence significantly the distance estimation.

Appendix

The challenge is to determine the lower bound of the mean-squared error the biased estimator (15). One can write that

$$\mathbb{E}\left\{\hat{x}_{ml}(\mathbf{r})-x\right\} = \int_{\mathbf{r}} \left[\hat{x}_{ml}(\mathbf{r})-x\right] p\left(\mathbf{r};x\right) \mathrm{d}\mathbf{r} = b(x), \quad (25)$$

with $b(x)$ the bias. Further, we have that

$$\frac{\partial}{\partial x}\mathbb{E}\left\{\hat{x}_{ml}(\mathbf{r})-x\right\} = -\int_{\mathbf{r}} p\left(\mathbf{r};x\right) \mathrm{d}\mathbf{r} + \int_{\mathbf{r}} \left[\hat{x}_{ml}(\mathbf{r})-x\right] \frac{\partial p\left(\mathbf{r};x\right)}{\partial x} \mathrm{d}\mathbf{r} = \frac{\mathrm{d}b(x)}{\mathrm{d}x} \quad (26)$$

Using equality (27), one can write (28).

$$\frac{\partial p\left(\mathbf{r};x\right)}{\partial x} = p\left(\mathbf{r};x\right) \frac{\partial \ln p\left(\mathbf{r};x\right)}{\partial x} \quad (27)$$

$$\int_{\mathbf{r}} \left[\hat{x}_{ml}(\mathbf{r})-x\right] p\left(\mathbf{r};x\right) \frac{\partial \ln p\left(\mathbf{r};x\right)}{\partial x} \mathrm{d}\mathbf{r} = 1 + \frac{\mathrm{d}b(x)}{\mathrm{d}x} \quad (28)$$

Finally, application of the Schwartz inequality, leads to

$$\overbrace{\int_{\mathbf{r}} \left[\hat{x}_{ml}(\mathbf{r})-x\right]^2 \; p\left(\mathbf{r};x\right) \mathrm{d}\mathbf{r}}^{\mathrm{MSE}\{\hat{x}_{ml}(\mathbf{r})\}} \overbrace{\int_{\mathbf{r}} \left(\frac{\partial \ln p\left(\mathbf{r};x\right)}{\partial x}\right)^2 p\left(\mathbf{r};x\right) \mathrm{d}\mathbf{r}}^{J_F(x)} \geq \left(1+\frac{\mathrm{d}b(x)}{\mathrm{d}x}\right)^2, \quad (29)$$

resulting in the Cramér-Rao bound for biased estimators (CRB-B) (16):

$$\mathrm{MSE}\{\hat{x}_{ml}(\mathbf{r})\} = \mathbb{E}\left\{\left(\hat{x}_{ml}(\mathbf{r})-x\right)^2\right\} \geq J_F^{-1}(x)\left(1+\frac{\mathrm{d}b(x)}{\mathrm{d}x}\right)^2$$

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