Multivariate asymptotics for a class of two-dimensional queueing models

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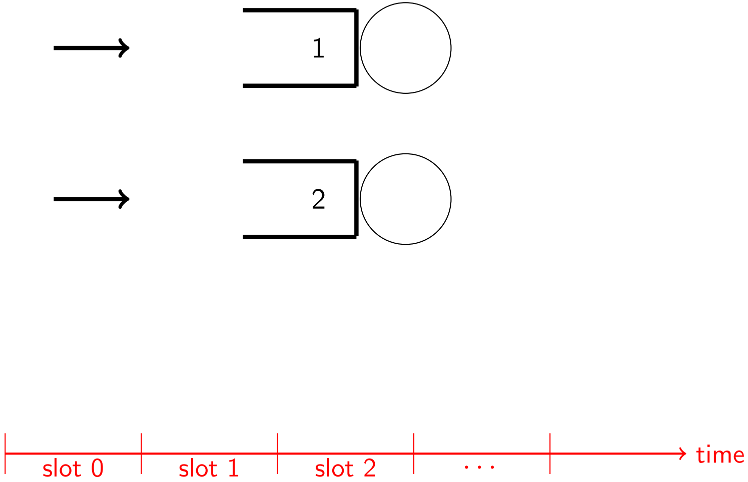
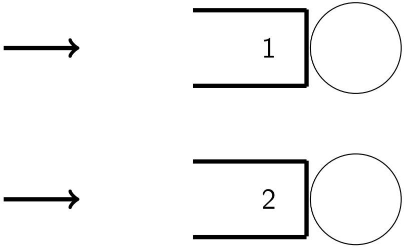
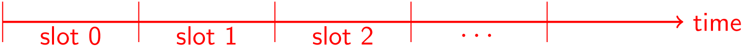
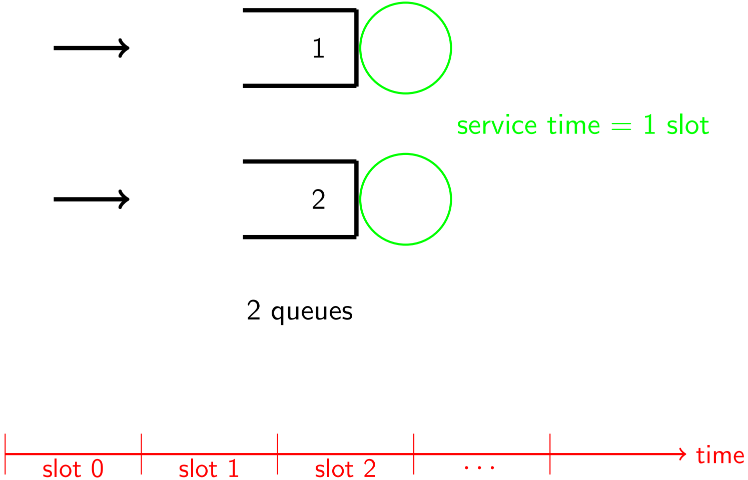
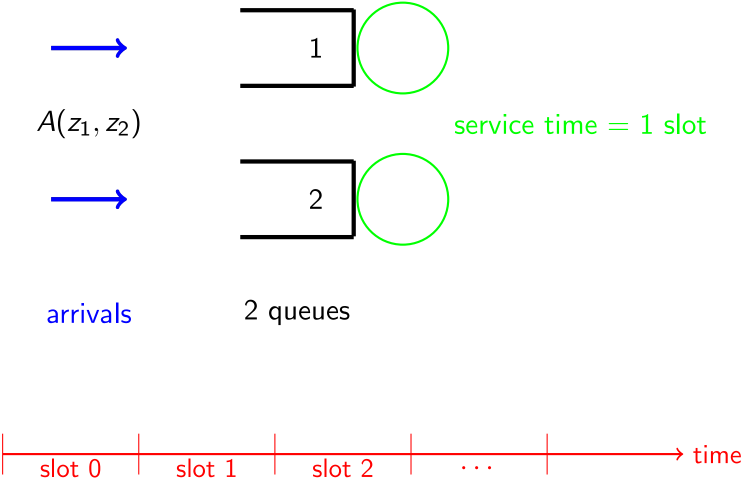

1
2
slot 0
slot 1
slot 2
· · ·
time



2 queues




1
2
service time = 1 slot
2 queues
slot 0
slot 1
slot 2
· · ·
time


$A(z_1, z_2)$
1
2
service time = 1 slot
arrivals
2 queues
slot 0
slot 1
slot 2
$\cdots$
time

Motivation:

1. Very simple system to model many natural applications
2. **Joint** analysis of the steady-state queue contents is challenging

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Joint probabilities? Correlation coefficient?

Generating functions

$$P(z_1, z_2) := \mathrm{E}\left[z_1^{N_1} z_2^{N_2}\right]$$

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Functional equation for $P(z_1, z_2)$:

$$[z_1 z_2 - A(z_1, z_2)]P(z_1, z_2) = A(z_1, z_2)L(z_1, z_2)\ ,$$

with

$$L(z_1, z_2) = (z_2-1)P(z_1, 0)+(z_1-1)P(0, z_2)+(z_1-1)(z_2-1)P(0, 0)$$

Objective

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- Questions of "asymptotic correlation"
- $\lim_{n\to+\infty} \mathrm{E}[z^{N_2} | N_1 = n] = ?$

Preliminaries

Two queues are Geo^X/D/1 queues when viewed in isolation:

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- We assume the two queues have steady-state distributions
- We know $\Pr[N_i = \cdot]$, $i = 1, 2$
- In particular: we know the tail distribution

$$\tau_i^k \Pr[N_i = k] \to C_i \qquad (k \to \infty)$$

- τ_i is the *dominant singularity* of $\mathrm{E}[z^{N_i}]$

Main result

Assume that

$$\frac{d}{dz}\frac{A(\tau_1, z)}{A(\tau_1, 1)}\bigg|_{z=1} < 1\ .$$

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where

$$A_2^*(z) := \frac{A(\tau_1, z)}{A(\tau_1, 1)} \qquad \text{and} \qquad \lambda_2^* := \left.\frac{dA_2^*(z)}{dz}\right|_{z=1}$$

Step-by-step overview of the proof

1. Show that τ_1 is the dominant singularity of $P(z, 0)$ (!)

2. Show that τ_1 is a simple pole and compute the residue of $P(z, 0)$ at $z = \tau_1$

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$$P(x, z) = \sum_{n=0}^{\infty} P_n(z) x^n$$

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reminder

$$[xz - A(x, z)]P(x, z) = A(x, z)L(x, z)$$

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7. Normalize:

$$\mathrm{E}[z^{N_2} \mid N_1 = n] = \frac{P_n(z)}{\Pr[N_1 = n]} \to \frac{B(z)}{C_1} \quad (n \to \infty)$$

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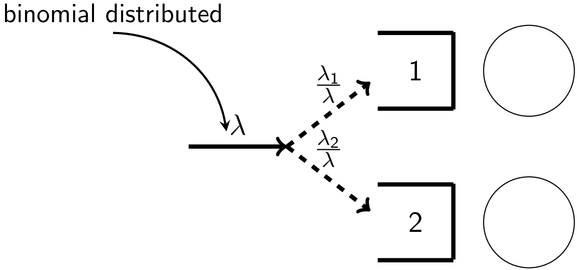
Discussion

$$\frac{B(z)}{C_1} = \frac{[1 - \lambda_2^*](z-1)A_2^*(z)}{z - A_2^*(z)}, \quad A_2^*(z) := \frac{A(\tau_1, z)}{A(\tau_1, 1)}$$

- PGF of the queue content in a Geo^X/D/1 queue!
- Stability condition of this queueing system is

$$\lambda_2^* < 1 \Leftrightarrow \frac{d}{dz}\frac{A(\tau_1, z)}{A(\tau_1, 1)}\bigg|_{z=1} < 1$$

Numerical examples: example 1



$$A(z_1, z_2) = \left(1 + \frac{\lambda_1}{8}(z_1 - 1) + \frac{\lambda_2}{8}(z_2 - 1)\right)^8$$

Numerical examples: example 1

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- Negative correlation
- Condition $\lambda_2^* < 1$ is always fulfilled:

$$\lambda_2^* = \frac{\lambda_2}{(1 - \frac{\lambda_1}{8} + \frac{\lambda_1}{8}\tau_1)} < \lambda_2 < 1$$

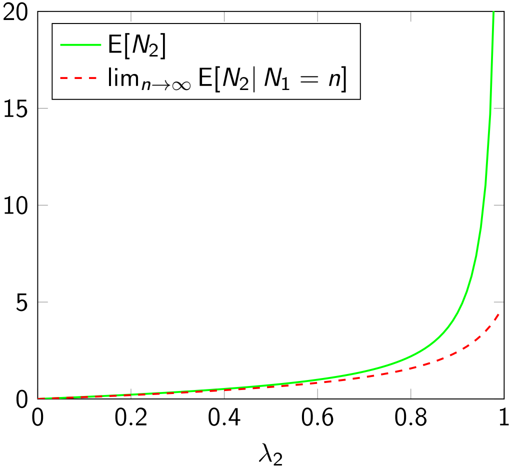
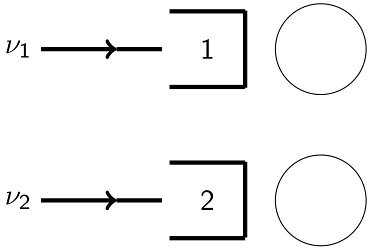


Figure: Average queue content in queue 2 versus λ_2 ($\lambda_1 = 0.4$)

Numerical examples: example 2

Poisson distributed arrivals



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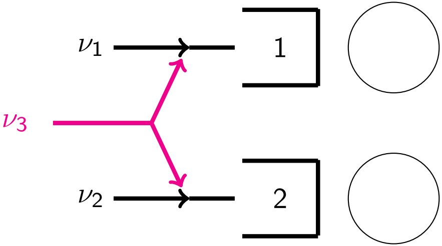

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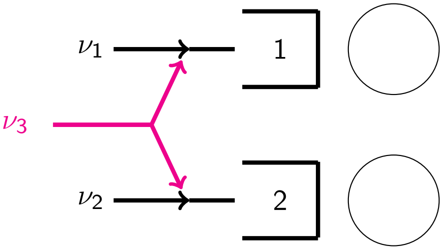

$$A(z_1, z_2) = e^{\nu_1(z_1-1)} e^{\nu_2(z_2-1)} e^{\nu_3(z_1 z_2-1)}$$

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- $\lambda_1 = \nu_1 + \nu_3$, $\lambda_2 = \nu_2 + \nu_3$
- Positive correlation if $\nu_3 > 0$
- Condition $\lambda_2^* < 1$ **not** always fulfilled:

$$\lambda_2^* = \lambda_2 + \underbrace{\nu_3(\tau_1 - 1)}_{>0} \overset{?}{<} 1$$

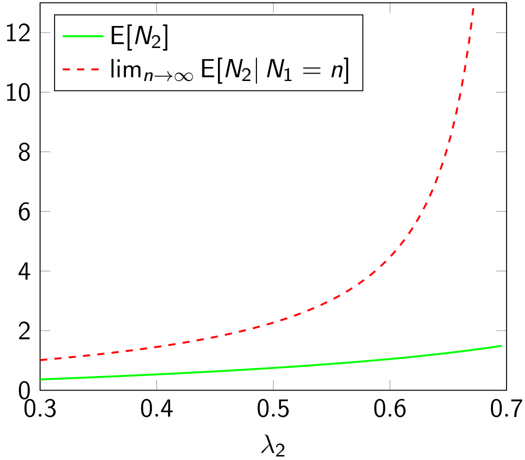

Figure: Mean of the queue content in queue 2 versus λ_2 ($\nu_1 = 0.4$, $\nu_3 = 0.3$)

Final remarks

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Thank you for your attention!