Multiple-input multiple-output (MIMO) communication systems
System model

- Info-bits flow from transmitter to receiver
- Transmitter has $N_T$ transmit antennas and $N_R$ receive antennas
- Receiver processes received signals
- Symbols $a_n(k)$ are transmitted
- Received symbols $r_n(k)$

Mathematical representation:

- $N_T$ transmit antennas: $a_1(k), \ldots, a_{N_T}(k)$
- $N_R$ receive antennas: $r_1(k), \ldots, r_{N_R}(k)$

$k$: time index ($k = 1, \ldots, K$)

(coded) data symbols $a_n(k)$, $E[|a_n(k)|^2] = E_s$
System model

\[ r_m(k) = \sum_{n=1}^{N_R} h_{m,n} a_n(k) + w_m(k) \quad (m = 1, \ldots, N_R; k = 1, \ldots, K) \]

\[ w_m(k) : \text{complex i.i.d. AWGN, } \mathbb{E}[|w_m(k)|^2] = N_0 \]

\[ h_{m,n} : \text{complex Gaussian zero-mean i.i.d. channel gains (Rayleigh fading), } \mathbb{E}[|h_{m,n}|^2] = 1 \]

\[
\begin{pmatrix}
    r_1(k) \\
    \vdots \\
    r_{N_R}(k)
\end{pmatrix}
= 
\begin{pmatrix}
    h_{1,1} & \cdots & h_{1,N_T} \\
    \vdots & \ddots & \vdots \\
    h_{N_R,1} & \cdots & h_{N_R,N_T}
\end{pmatrix} 
\begin{pmatrix}
    a_1(k) \\
    \vdots \\
    a_{N_T}(k)
\end{pmatrix} 
+ 
\begin{pmatrix}
    w_1(k) \\
    \vdots \\
    w_{N_T}(k)
\end{pmatrix}
\]

\[ r(k) = H \cdot a(k) + w(k) \]

\[
\begin{pmatrix}
    r(1), \ldots, r(k), \ldots, r(K)
\end{pmatrix}
= 
H \cdot \begin{pmatrix}
    a(1), \ldots, a(k), \ldots, a(K)
\end{pmatrix}
+ 
\begin{pmatrix}
    w(1), \ldots, w(k), \ldots, w(K)
\end{pmatrix}
\]

\[ R = H \cdot A + W \]
**System model**

- $R_b$: information bitrate
- $R_s$: symbol rate *per transmit antenna*
- $E_b$: energy per infobit
- $E_s$: energy per symbol

\[
E_s = E_b r_{\text{MIMO}} \log_2(M) \quad R_b = R_s N_T r_{\text{MIMO}} \log_2(M)
\]

\[
R_b = R_s N_T r_{\text{MIMO}} \log_2(M)
\]

\[
r_{\text{MIMO}} = \frac{\text{#infobits}}{\text{#coded bits}}
\]

\[
0 < r_{\text{MIMO}} \leq 1
\]
System model

Spectrum transmitted bandpass signal

$B_{RF} \geq R_s$ (necessary condition to avoid ISI between successive symbols)

All $N_T$ transmitted signals are simultaneously in the same frequency interval

Bandwidth efficiency:

$$\frac{R_b}{R_s} = N_T \cdot r_{MIMO} \cdot \log_2(M) \text{ (bit/s/Hz)}$$
ML detection

Log-likelihood function $\ln(p(R | A, H))$:

$$-N_0 \ln p(R | A, H) = \sum_{m=1}^{N_r} \sum_{k=1}^{K} \left| r_m(k) - \sum_{n=1}^{N_r} h_{m,n} a_n(k) \right|^2$$

$$= \| R - H \cdot A \|^2 = \text{Tr}[(R - H \cdot A)^H (R - H \cdot A)]$$

ML detection: $\hat{A} = \arg \min_A \| R - H \cdot A \|^2$ (minimization over all symbol matrices that obey the encoding rule)

Frobenius norm of $X$: $\| X \|^2 = \sum_{i,j} |x_{i,j}|^2 = \text{Tr}[X^H X] = \text{Tr}[X \cdot X^H]$

Trace of square matrix $M$: $\text{Tr}[M] = \sum_i m_{i,i}$

Properties: $\text{Tr}[P \cdot Q] = \text{Tr}[Q \cdot P]$, $\text{Tr}[M] = \text{Tr}[M^T]$ 

$X^H$ denotes Hermitian transpose (= complex conjugate transpose $(X^*)^T$)
**Bit error rate (BER)**

\[
\text{BER} = \frac{E[\# \text{ infobit errors}]}{\# \text{ infobits transmitted}}
\]

symbol matrix \( \mathbf{A} \) represents \( N_T K_r\text{MIMO} \log_2(M) \) infobits

\[
\text{BER} = \frac{\sum N^{(i,0)} \Pr[\hat{\mathbf{A}} = \mathbf{A}^{(i)}, \mathbf{A} = \mathbf{A}^{(0)}]}{N_T K \log_2(M) r_{\text{MIMO}}}
\]

\( N^{(i,0)}_{\text{info}} \) : \# infobits in which \( \mathbf{A}^{(i)} \) and \( \mathbf{A}^{(i)} \) are different

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**Diagram:**

- Infobits \( r_{\text{MIMO}} \)
- Encoder
- Coded bits
- Mapper
- \( N_T K \) coded symbols
- S/P
- #1
- #n
- #N_T

---

Advanced Modulation and Coding : MIMO Communication Systems
Bit error rate (BER)

\[
\Pr[\hat{A} = A^{(i)}, A = A^{(0)}] = \Pr[\hat{A} = A^{(i)} | A = A^{(0)}] \Pr[A = A^{(0)}] 
\]

\[
\Pr[A = A^{(0)}] = 2^{-N_T K \log_2 (M)} r_{\text{MIMO}} = M^{-N_T K \cdot r_{\text{MIMO}}}
\]

\[
\Pr[\hat{A} = A^{(i)} | A = A^{(0)}] = E_H \left[ \Pr[\hat{A} \neq A^{(0)} | A = A^{(0)}, H] \right] \quad \text{(averaging over statistics of } H) 
\]

\[
\Pr[\hat{A} = A^{(i)} | A = A^{(0)}, H] \leq PEP^{(i,0)}(H) 
\]

\[
PEP^{(i,0)}(H) = \Pr[\| R - H \cdot A^{(i)} \|^2 < \| R - H \cdot A^{(0)} \|^2 | A = A^{(0)}, H] 
\]

PEP^{(i,0)}(H) : pairwise error probability, conditioned on H

\[
PEP^{(i,0)} = E_H[PEP^{(i,0)}(H)] : \text{pairwise error probability, averaged over } H 
\]
Bit error rate (BER)

\[
\text{BER} \leq \frac{\sum_{A_0} \left( \Pr[A = A^{(0)}] \sum_{i \neq 0} N_{\text{info}}^{(i,0)} P\text{EP}^{(i,0)} \right)}{N_T K \log_2(M) r_{\text{MIMO}}}
\]

Define error matrix : \( \Delta^{(i,0)} = A^{(0)} - A^{(i)} \)

\[
P\text{EP}^{(i,0)}(H) = Q\left( \sqrt{\frac{\|H \cdot \Delta^{(i,0)}\|^2}{2N_0}} \right) \leq \frac{1}{2} \exp \left( -\frac{\|H \cdot \Delta^{(i,0)}\|^2}{4N_0} \right)
\]

Averaging over Rayleigh fading channel statistics :

\[
P\text{EP}^{(i,0)} = E_H[P\text{EP}^{(i,0)}(H)] \leq \frac{1}{2} \left( \det \left[ I_{N_T} + \frac{\Delta^{(i,0)} \cdot (\Delta^{(i,0)})^H}{4N_0} \right] \right)^{-N_R} = \frac{1}{2} \left( \prod_{n=1}^{N_T} \left( 1 + \frac{\lambda_n^{(i,0)}}{4N_0} \right) \right)^{-N_R}
\]

\( \{\lambda_n^{(i,0)}, n = 0, ..., N_T\} \) set of eigenvalues of \( N_T \times N_T \) matrix \( \Delta^{(i,0)} \cdot (\Delta^{(i,0)})^H \)

(these eigenvalues are real-valued non-negative)
To be further considered

Single-input single-output (SISO) systems: $N_T = N_R = 1$

Single-input multiple-output (SIMO) systems: $N_T = 1, N_R \geq 1$

Multiple-input multiple-output (MIMO) systems: $N_T \geq 1, N_R \geq 1$
  - Spatial multiplexing: ML detection, ZF detection
  - Space-time coding: trellis coding, delay diversity, block coding
Single-input single-output (SISO) systems
SISO : observation model

Observation model: \( r(k) = h \cdot a(k) + w(k) \quad k = 1, ..., K \)

Bandwidth efficiency: \( \frac{R_b}{R_s} = \frac{r_{SISO}}{r_{SISO} \log_2(M)} \) (bit/s/Hz)
**SISO : ML detection**

\[
\{\hat{a}(k)\} = \arg\min_{\{a(k)\}} \sum_k |r(k) - h \cdot a(k)|^2 = \arg\min_{\{a(k)\}} \sum_k |u(k) - a(k)|^2
\]

\[
u(k) = \frac{r(k)h^*}{|h|^2} = \frac{r(k)}{h}
\]

ML detector looks for codeword that is closest (in Euclidean distance) to \(\{u(k)\}\)

\[
\text{# codewords} = 2^{K \cdot \log_2(M \cdot r_{SISO})} = M^{K \cdot r_{SISO}}
\]

In general: decoding complexity exponential in \(K\)

(trellis codes, convolutional codes:
 complexity *linear* in \(K\) by using Viterbi algorithm)
SISO : PEP computation

Conditional PEP:

$$PEP^{(i,0)}(h) \leq \frac{1}{2} \exp \left( -\frac{|h|^2 d_{E,(i,0)}^2}{4N_0} \right)$$

$$d_{E,(i,0)}^2 = \sum_{k=1}^{K} |a^{(0)}(k) - a^{(i)}(k)|^2$$

squared Euclidean distance between codewords \( \{a^{(0)}(k)\} \) and \( \{a^{(i)}(k)\} \)

Average PEP, AWGN \(|h| = 1\):

$$PEP^{(i,0)} \leq \frac{1}{2} \exp \left( -\frac{d_{E,(i,0)}^2}{4N_0} \right)$$

decreases exponentially with \( E_b/N_0 \)

Average PEP, Rayleigh fading:

$$PEP^{(i,0)} \leq \frac{1}{2} \left( 1 + \frac{d_{E,(i,0)}^2}{4N_0} \right)^{-1}$$

inversely proportional to \( E_b/N_0 \)

(occasionally, \(|h|\) becomes very small \( \Rightarrow \) large PEP)
SISO : numerical results

BER fading channel >> BER AWGN channel
SISO : numerical results

coding gain on fading channel $<<$ coding gain on AWGN channel
Single-input multiple-output (SIMO) systems
SIMO : observation model

Same transmitter as for SISO; $N_R$ antennas at receiver

$R_b$  $r_{SIMO}$  $r_{SIMO}$  $R_s = \frac{R_b}{r_{SIMO} \log_2(M)}$

Observation model : $r_m(k) = h_m a(k) + w_m(k)$  $m = 1, ..., N_R; k = 1, ..., K$

Bandwidth efficiency : $R_b/R_s = r_{SIMO} \cdot \log_2(M)$  (bit/s/Hz)
SIMO : ML detection

\[
\{\hat{a}(k)\} = \arg \min_{\{a(k)\}} \sum_k \sum_{m=1}^{N_R} |r_m(k) - h_m a(k)|^2 = \arg \min_{\{a(k)\}} \sum_k |u(k) - a(k)|^2
\]

\[
u(k) = \frac{\sum_{m=1}^{N_R} h^*_m r_m(k)}{\sum_{m=1}^{N_R} |h_m|^2}
\]

ML detector looks for codeword that is closest to \{u(k)\}

Same decoding complexity as for SISO
SIMO : PEP computation

Conditional PEP : $$\text{PEP}^{(i,0)}(h) \leq \frac{1}{2} \exp \left( - \frac{N_R d_{E,(i,0)}^2 (|h|^2)_{\text{avg}}}{4N_0} \right)$$

$$\text{(|h|^2)_{\text{avg}} = \frac{1}{N_R} \sum_{m=1}^{N_R} |h_m|^2}$$

PEP, AWGN ($|h_m| = 1$) : $$\text{PEP}^{(0,i)} \leq \frac{1}{2} \exp \left( - \frac{N_R d_{E,(i,0)}^2}{4N_0} \right)$$

decreases exponentially with $E_b/N_0$

total received energy per symbol increases by factor $N_R$ as compared to SISO
⇒ same PEP as with SISO, but with $E_b$ replaced by $N_R E_b$
(“array gain” of $10 \log(N_R)$ dB)

PEP, Rayleigh fading : $$\text{PEP}^{(i,0)} \leq \frac{1}{2} \left( 1 + \frac{d_{E,(i,0)}^2}{4N_0} \right)^{-N_R}$$

inversely proportional to $(E_b/N_0)^{N_R}$

PEP is smaller than in SISO setup with $E_b$ replaced by $N_R E_b$
- “array gain” of $10 \log(N_R)$ dB
- additional “diversity gain” (diversity order $N_R$) :
  probability that $(|h|^2)_{\text{avg}}$ is small, decreases with $N_R$
SISO: statistical properties of $<|h|^2>$
SISO: statistical properties of $\langle |h|^2 \rangle$
SIMO : numerical results

AWGN channel : array gain of $10 \cdot \log(N_R)$ dB
SIMO: numerical results

fading channel: array gain + diversity gain
SIMO : numerical results

fading channel : array gain + diversity gain
SIMO : numerical results

fading channel : array gain + diversity gain

SIMO, N_R = 4
uncoded BPSK
SIMO : numerical results

fading channel : coding gain increases with $N_R$
SIMO : numerical results

fading channel : coding gain increases with $N_R$
SIMO : numerical results

fading channel : coding gain increases with $N_R$
Multiple-input multiple-output (MIMO) systems
**MIMO : observation model**

\[
R_s = \frac{R_b}{N_T \cdot r_{\text{MIMO}} \cdot \log_2(M)}
\]  

(per antenna)

\[
r_m(k) = \sum_{n=1}^{N_T} h_{m,n} a_n(k) + w_m(k) \quad (m = 1, \ldots, N_R; k = 1, \ldots, K)
\]

\[
R = H \cdot A + W \quad \text{at any receive antenna, symbols from different transmit antennas interfere}
\]

Bandwidth efficiency : \( R_b / R_s = N_T \cdot r_{\text{MIMO}} \cdot \log_2(M) \)  
(bit/s/Hz)
MIMO : ML detection

ML detector : \( \hat{A} = \arg\min_A \| R - H \cdot A \|^2 \)

\# different symbol matrices \( A = 2^{N_T \cdot r_{\text{MIMO}} \cdot \log_2 (M) \cdot K} = M^{N_T \cdot r_{\text{MIMO}} \cdot K} \)

\( \Rightarrow \) detector complexity in general increases exponentially with \( N_T r_{\text{MIMO}} \) and K
MIMO : PEP computation

PEP, Rayleigh fading:

\[
\text{PEP}^{(i,0)} \leq \frac{1}{2} \left( \det \left[ I_{N_T} + \frac{\Delta^{(i,0)} \cdot \Delta^{(i,0)H}}{4N_0} \right] \right)^{-N_R} = \frac{1}{2} \left( \prod_{n=1}^{N_T} \left( 1 + \frac{\lambda_n^{(i,0)}}{4N_0} \right) \right)^{-N_R}
\]

# nonzero eigenvalues equals \( \text{rank}(\Delta^{(i,0)}) \), with \( 1 \leq \text{rank}(\Delta^{(i,0)}) \leq N_T \)

\( \text{PEP}^{(i,0)} \) inversely proportional to \( (E_b/N_0)^{\text{rank}(\Delta^{(i,0)})} \)

PEPs that dominate BER are those for which \( \Delta^{(i,0)} = A^{(0)} - A^{(i)} \) has minimum rank.

\[\Rightarrow \text{diversity order} = N_R \cdot \left( \min_{(i,0)} \text{rank}(\Delta^{(i,0)}) \right)\]
MIMO : spatial multiplexing
Spatial multiplexing

\[ A = \begin{pmatrix} a_1^T \\ \vdots \\ a_{N_T}^T \end{pmatrix} \]

\[ a_n^T = (a_n(1), \cdots, a_n(k), \cdots, a_n(N)) \]

codeword transmitted by antenna \#n

codewords transmitted by different antennas are statistically independent

Aim: to increase bandwidth efficiency by a factor \( N_T \) as compared to SISO/SIMO

Bandwidth efficiency: \( \frac{R_b}{R_s} = N_T \cdot r \cdot \log_2(M) \)
Spatial multiplexing: ML detection

ML detector:
\[
\hat{A} = \arg \min_A \| R - H \cdot A \|^2 
\]

\[
= \arg \min_A \left( \sum_{n,n'=1}^{N_T} (H^H H)_{n,n'} (a_{n}^H a_{n'}) - 2 \text{Re} \left[ \sum_{n=1}^{N_T} a_{n}^H u_n \right] \right) 
\]

\((u_n)^T \) : n-th row of \(H^H R\)

As \(H^H H\) is nondiagonal, codewords on different rows must be detected *jointly* instead of individually.

\# different symbol matrices \(A = 2^{N_T \cdot r \cdot \log_2 (M) \cdot K} = M^{N_T \cdot r \cdot K}\)

⇒ ML detector complexity increases exponentially with \(N_T\) as compared to SISO/SIMO
Spatial multiplexing: PEP computation

\( \Delta^{(i,0)} \) nonzero \( \Rightarrow \) rank(\( \Delta^{(i,0)} \)) \( \geq 1 \)

rank(\( \Delta^{(i,0)} \)) = 1 when all rows of \( \Delta^{(i,0)} \) are proportional to a common row vector \( \delta^T = (\delta(1), \ldots, \delta(k), \ldots, \delta(K)) \).

Example: \( \Delta^{(i,0)} \) has only one nonzero row; this corresponds to a detection error in only one row of \( A \)

\[ \Rightarrow \min_{i \neq 0} \text{rank}(\Delta^{(i,0)}) = 1 \]

Denoting the \( n \)-th row of a rank-1 error matrix \( \Delta^{(i,0)} \) by \( \alpha_n \delta^T \), the bound on the corresponding PEP is

\[
\text{PEP}^{(i,0)} \leq \frac{1}{2} \left( 1 + \frac{|\delta|^2}{4N_0} \sum_n |\alpha_n|^2 \right)^{-N_R}
\]

dominating PEPs are inversely proportional to \((E_b/N_0)^{N_R}\)
\( \Rightarrow \) diversity order \( N_R \) (same as for SIMO)
Spatial multiplexing: zero-forcing detection

Complexity of ML detection is exponential in $N_T$

Simpler sub-optimum detection: zero-forcing (ZF) detection

linear combination of received antenna signals $r_m(k)$ to eliminate mutual interference between symbols $a_n(k)$ from different transmit antennas and to minimize resulting noise variance

Condition for ZF solution to exist: $\text{rank}(H) = N_T$ (this requires $N_R \geq N_T$)

$r(k) = Ha(k) + w(k)$ \quad $\text{rank}(H) = N_T \Rightarrow (H^H H)^{-1}$ exists

$z(k) = (H^H H)^{-1} H^r(k) = a(k) + n(k)$ \quad $n(k) = (H^H H)^{-1} H^w(k)$

$z(k)$: no interference between symbols from different transmit antennas

$E[n(k)n^H(k)] = N_0 (H^H H)^{-1}$: $n_i(k)$ and $n_j(k)$ correlated when $i \neq j$
Spatial multiplexing: zero-forcing detection

Sequences \( \{a_n(k)\} \) are detected independently instead of jointly, ignoring the correlation of the noise on different outputs:

\[
\{\hat{a}_n(k)\} = \text{arg min}_{a_n(k)} \sum_k |z_n(k) - a_n(k)|^2
\]

Complexity of ZF detector is linear (instead of exponential) in \( N_T \)

Performance penalty: diversity equals \( N_R - N_T + 1 \) (instead of \( N_T \))
Spatial multiplexing: numerical results

ML detection: penalty w.r.t. SIMO decreases with increasing $N_R$
Spatial multiplexing: numerical results

ZF detection: same BER as SIMO with $N_R - N_T + 1$ receive antennas
MIMO : space-time coding
Space-time coding

Aim: to achieve higher diversity order than with SIMO
⇒ any $\Delta^{(i,0)}$ must have rank larger than 1
(max. rank is $N_T$, max. diversity order is $N_TN_R$)
⇒ coded symbols must be distributed over different transmit antennas
and different symbol intervals (“space-time” coding)

Property: $r_{\text{MIMO}} \leq 1/N_T$ is necessary condition to have rank $N_T$ for all $\Delta^{(i,0)}$

⇒ at max. diversity, $R_b/R_s \leq \log_2(M)$
$R_b/R_s$ limited by bandwidth efficiency of uncoded SIMO/SISO
Three classes of space-time coding:

- space-time trellis codes (STTrC)
- delay diversity
- space-time block codes (STBC)
STTrC : trellis representation

\[ A = (a(1), \ldots, a(k), \ldots, a(K)) \quad a^T(k) = (a_1(k), \ldots, a_n(k), \ldots, a_{N_T}(k)) \]

\[ b(k) = (b_1(k), \ldots, b_L(k)) \quad \text{L input bits at beginning of k-th symbol interval} \]

\[ S(k) : \text{state at beginning of k-th symbol interval} \]

state transitions : \((S(k+1), a(k))\) determined by \((S(k), b(k))\)

trellis diagram : \((L=1)\)

2^L branches leaving from each state

2^L branches entering each state
STTrC : bandwidth efficiency

Bandwidth efficiency:

\[ r_{\text{MIMO}} = \frac{L}{N T \log_2(M)} \quad \Rightarrow \quad \frac{R_b}{R_s} = L \text{ (bit/s/Hz)} \]

at max. diversity: \[ r_{\text{MIMO}} \leq \frac{1}{N T} \Rightarrow L \leq \log_2(M) \]

\[ \#\text{states} \geq M^{N_T-1} \quad \text{(exponential in } N_T) \]

ML decoding by means of Viterbi algorithm:

decoding complexity linear (instead of exponential) in \( K \)
but still exponential in \( N_T \) (at max. diversity)
STTrC : example

Example for \( N_T = 2 \), 4 states, 4-PSK, \( L = 2 \) infobits per coded symbol pair

Any \( \Delta^{(i,0)} \) has columns \((0, \delta_1)^T, (\delta_2, 0)^T\) with nonzero \( \delta_1 \) and \( \delta_2 \)

\( \Rightarrow \) rank = 2, i.e., max. rank

Max. diversity \((N_T N_R = 2N_R)\),
Max. bandwidth efficiency \((L = \log_2(M))\),
Corresponding to max. diversity
Delay diversity: transmitter

\[
\begin{align*}
\text{info-bits} & \xrightarrow{R_b} \text{encoder} & \xrightarrow{R_b/r} \text{mapper} \\
& \rightarrow \text{delay (n-1)T} & \rightarrow \text{delay (NT-1)T} \\
& \rightarrow \text{diversity order} = NTNR \\
& \text{maximum possible diversity order!}
\end{align*}
\]

Example: \(NT = 3\)

\[
A = \begin{pmatrix}
a(1) & a(2) & a(3) & a(4) & \cdots & a(K-3) & a(K-2) & 0 & 0 \\
0 & a(1) & a(2) & a(3) & \cdots & a(K-4) & a(K-3) & a(K-2) & 0 \\
0 & 0 & a(1) & a(2) & \cdots & a(K-5) & a(K-4) & a(K-3) & a(K-2)
\end{pmatrix}
\]

\[\Rightarrow \text{rank}(\Delta^{(i,0)}) = NT \Rightarrow \text{diversity order} = NTN_R\]
Delay diversity : bandwidth efficiency

Equivalent configuration :

\[
R_b = \frac{R_b}{N_T} \log_2(M) = R_s \Rightarrow \text{same bandwidth efficiency as SIMO/SISO with code rate } r
\]

Max. bandwidth efficiency (at max. diversity) of \( \log_2(M) \) achieved for \( r = 1 \) (uncoded delay diversity)
Delay diversity : trellis representation

Uncoded delay diversity \((r = 1)\) can be interpreted as space-time trellis code:

\[
\text{state } S(k) = (a(k-1), a(k-2), \ldots, a(k-N_T+1)) \quad \#\text{states} = M^{N_T-1}
\]

input at time \(k\) : \(a(k)\)
output during \(k\)-th symbol interval : \((a(k), a(k-1), \ldots, a(k-N_T+1))\)

decoding complexity : linear in \(K\), exponential in \(N_T\)
Space-time block codes

ML decoding of space-time code:
minimizing $\|R - HA\|^2$ over all possible codewords $A$

$A$ contains $N_tK$ coded symbols, which corresponds to $N_tK r_{\text{MIMO}} \log_2(M)$ infobits

In general, ML decoding complexity is exponential in $N_tK$

Space-time block codes can be designed to yield maximum diversity
and to reduce ML decoding to *simple symbol-by-symbol decisions*

$\Rightarrow$ decoding complexity linear (instead of exponential) in $N_tK$
STBC : example 1

Example 1 : $N_T = 2$ (Alamouti code)

$$A = \begin{pmatrix} a_1 & -a_2^* \\ a_2 & a_1^* \end{pmatrix} \Rightarrow AA^H = (|a_1|^2 + |a_2|^2)I_2$$

($A$ has orthogonal rows, irrespective of the values of $a_1$ and $a_2$)

ML decoding ($H = (h_1, h_2)$, $R = (r_1, r_2)$, $a_1$ and $a_2$ i.i.d. symbols)

$$(\hat{a}_1, \hat{a}_1) = \arg\min_{a_1, a_2} || R - HA ||^2 = \arg\min_{a_1, a_2} \left( -2 \Re[\text{Tr}[A^HH^HR]] + \text{Tr}[(H^HH)(AA^H)] \right)$$

$$= \arg\min_{a_1, a_2} \left( -2 \Re[a_1^*(h_1^H r_1 + h_2^T r_2^*) + a_2^*(h_2^H r_2 - h_1^T r_2^*)] + (|a_1|^2 + |a_2|^2)(|h_1|^2 + |h_2|^2) \right)$$

no cross-terms involving both $a_1$ and $a_2$ \Rightarrow symbol-by-symbol detection

$$\hat{a}_1 = \arg\min_{a_1} |u_1 - a_1|^2$$

$$\hat{a}_2 = \arg\min_{a_2} |u_2 - a_2|^2$$

$$u_1 = \frac{h_1^H r_1 + h_2^T r_2^*}{|h_1|^2 + |h_2|^2}$$

$$u_2 = \frac{h_2^H r_1 - h_1^T r_2^*}{|h_1|^2 + |h_2|^2}$$
STBC : example 1

\[
A = \begin{pmatrix}
a_1 & -a_2^* \\
a_2 & a_1^*
\end{pmatrix} \Rightarrow \Delta^{(i,0)} = \begin{pmatrix}
a_1^{(0)} - a_1^{(i)} & -(a_2^{(0)} - a_2^{(i)})^* \\
a_2^{(0)} - a_2^{(i)} & (a_1^{(0)} - a_1^{(i)})^*
\end{pmatrix}
\]

orthogonal rows \Rightarrow rank = 2
\quad (= \text{max. rank for } N_T = 2)

Assume M-point constellation
\Rightarrow A \text{ contains } 4 \text{ symbols, i.e. } 4 \cdot \log_2(M) r_{\text{MIMO}} = 2 \cdot \log_2(M) \text{ infobits}
\Rightarrow r_{\text{MIMO}} = 1/2 \ (= 1/N_T) : \text{highest possible rate for max. diversity}

Alamouti space-time block code yields
- maximum diversity (i.e., \( 2N_R \))
- maximum corresponding bandwidth efficiency (i.e., \( R_b/R_s = \log_2(M) \))
- small ML decoding complexity
**STBC : example 2**

Example 2 : \(N_T = 3\)

\[
\mathbf{A} = \begin{pmatrix}
a_1 & -a_2 & -a_3 & -a_4 & a_1^* & -a_2^* & -a_3^* & -a_4^* \\
a_2 & a_1 & a_4 & -a_3 & a_2^* & a_1^* & a_4^* & -a_3^* \\
a_3 & -a_4 & a_1 & a_2 & a_3^* & -a_4^* & a_1^* & a_2^*
\end{pmatrix}
\]

\[
\mathbf{A}\mathbf{A}^H = 2(|a_1|^2 + |a_2|^2 + |a_3|^2 + |a_4|^2)\mathbf{I}_3
\]

rows of \(\mathbf{A}\) are orthogonal \(\Rightarrow\) any \(\Delta^{(i,0)}\) has rank = 3

\(\Rightarrow\) max. diversity (i.e., \(3N_R\))

ML detection of \((a_1, a_2, a_3, a_4)\) : symbol-by-symbol detection (no cross-terms)

\(r_{\text{MIMO}} = 1/6\) \quad \(1/N_T = 1/3\)

Bandwidth efficiency : \(R_b/R_s = 4\cdot\log_2(M)/8 = \log_2(M)/2\)

i.e., only half of the max. possible value for max. diversity

max. bandwidth efficiency (under restriction of max. diversity and symbol-by-symbol detection) cannot be achieved for all values of \(N_T\)
STBC : numerical results

Space-time block codes
BPSK
N_R = 1

BER (bound)

SISO
N_T = 2 (Alamouti)
N_T = 3

Eb/N0 (dB)
STBC: numerical results

Space-time block codes
BPSK
\(N_R = 2\)

BER (bound)

\(E_b/N_0\) (dB)

SISO
SIMO
\(N_T = 2\) (Alamouti)
\(N_T = 3\)
STBC : numerical results

![Graph showing BER vs. Eb/N0 for different systems and STBC configurations](image-url)
STBC : numerical results

Space-time block codes
BPSK
N_R = 4

BER (bound)

SISO
SIMO
N_T = 2 (Alamouti)
N_T = 3

Eb/N0 (dB)
## Summary: SISO/SIMO/MIMO

<table>
<thead>
<tr>
<th></th>
<th>Bandwidth efficiency</th>
<th>Diversity order (ML detection)</th>
<th>ML detector complexity</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>SISO</strong></td>
<td>$r_{\text{SISO}} \log_2(M)$</td>
<td>1</td>
<td>in general: expon. in $K$ (trellis: linear in $K$)</td>
</tr>
<tr>
<td><strong>SIMO</strong></td>
<td>$r_{\text{SIMO}} \log_2(M)$</td>
<td>$N_R$</td>
<td>in general: expon. in $K$ (trellis: linear in $K$)</td>
</tr>
<tr>
<td><strong>MIMO</strong></td>
<td>$N_{Tr_{\text{MIMO}}} \log_2(M)$</td>
<td>$N_R \cdot \left( \max \text{rank}(\Delta_{(i,0)}^{(i,0)}) \right)$</td>
<td>in general: expon. in $N_T K$</td>
</tr>
</tbody>
</table>
## Summary : MIMO

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<th>Diversity order (ML detection)</th>
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<tbody>
<tr>
<td><strong>spatial multiplexing</strong></td>
<td>$N_{Tr} \cdot \log_2(M)$</td>
<td>$N_R$</td>
<td>in general : expon. in $N_TK$ (trellis : linear in $K$, expon. in $N_T$)</td>
</tr>
<tr>
<td><strong>STTrC</strong></td>
<td>$\log_2(M)$ (*)</td>
<td>$N_T N_R$</td>
<td>expon. in $N_T$, linear in $K$</td>
</tr>
<tr>
<td><strong>delay diversity</strong></td>
<td>$r \cdot \log_2(M)$</td>
<td>$N_T N_R$</td>
<td>if uncoded ($r = 1$) : expon. in $N_T$, linear in $K$</td>
</tr>
<tr>
<td><strong>STBC</strong></td>
<td>$\leq \log_2(M)$ (*)</td>
<td>$N_T N_R$</td>
<td>symbol-by-symbol decision</td>
</tr>
</tbody>
</table>

(*) assuming max. diversity $N_T N_R$ is achieved

Remark : spatial multiplexing with ZF detection
- complexity linear in $N_T$
- diversity order is $N_R - N_T + 1$