Performance Analysis of Space-Time Coding With Imperfect Channel Estimation

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***Abstract*—We analyze the error performance of a space-time coding system using N transmit and M receive antennas with imperfect channel estimation in flat Rayleigh fading. A least-squares estimate of the channel matrix is obtained by using a sequence of pilot code vectors. The estimate is found to be perturbed by an $M \times N$ perturbation matrix with zero-mean circular Gaussian entries. Using the characteristic function of the decision variable, we derive a closed-form expression for the pairwise error probability (PEP). From the same expression, the PEP in case of perfect channel estimation is also obtained. Numerical results show the degradation in performance due to imperfect channel estimation that can be compensated by increasing the number of receive antennas.**

***Index Terms*—Imperfect channel estimation, pairwise error probability, pilot code vectors, Rayleigh fading, space-time coding.**

I. Introduction

SPACE-TIME coding [1], [2], which uses the advantage of transmitter diversity, is an effective technique to improve the performance of wireless communication systems. In space-time coding, different signals are simultaneously transmitted from different transmit antennas. The signal which is received is the superposition of the different transmitted signals, and the detection process needs estimates of the channel parameters [3]. Performance analysis methods often consider perfect channel estimates under the assumption that the estimation errors are negligible. Although perfect estimates are desirable, channel estimation methods used in practice give rise to imperfections [4], [5]. It is therefore of considerable relevance to study the effect of imperfect channel estimation on performance. One convenient method of estimation of channel parameters in a space-time coding system is by using pilot code sequences. Owing to the presence of additive noise in the received signal, we apply in this paper a least-squares based channel estimation technique to a space-time coding system using multiple transmit and receive antennas in flat Rayleigh fading and analyze its error performance. The estimate of the channel matrix is obtained from a sequence of pilot code vectors. Using the characteristic function (cf.) of the decision variable, we derive a closed-form expression for the pairwise error probability (PEP). From the same expression, the PEP in case of perfect channel estimation is also obtained. We apply our result to an example of a system using the Alamouti code [2] with binary phase-shift keying (BPSK).

The paper is organized as follows. Section II gives the basic model for a wireless communication system employing space-time coding. We then present the channel estimation technique in Section III. Section IV discusses the performance criterion with imperfect channel estimation. Section V analyzes the error performance in terms of the PEP. As an example, the performance of a system using the Alamouti code with two transmit antennas and BPSK for imperfect as well as perfect channel estimation is presented in Section VI. Section VII gives some concluding remarks.

II. Model

Consider a communication system that employs space-time coding [1] with N transmit and M receive antennas. At time index l, the space-time encoder encodes the information symbol $s(l)$ into N code symbols $c_1(l), c_2(l), \ldots, c_N(l)$, which are transmitted by the N antennas at the same time.

The complex baseband signal received at time index l by the ith antenna after matched filtering is given by [1], [6]

$$r_i(l) = \sqrt{2E_s} \sum_{j=1}^{N} h_{ij}(l) c_j(l) + n_i(l), \quad i = 1, \ldots, M \quad (1)$$

where $2E_s$ is the average energy of the baseband signal constellation, h_{ij} the complex fading channel gain from the $j\text{th}$ transmit antenna to the $i\text{th}$ receive antenna, and $n_i(l)$ the additive white Gaussian noise with power spectral density $2N_0$. Since the noise is white, the noise samples for different time indexes, denoted as $\{n_i(l)\}$, are independent and identically distributed (i.i.d.) zero-mean complex circular Gaussian random variables, each having a $\mathcal{CN}(0, 2N_0)$ distribution. We also assume that $n_i(l)$ and $n_k(l)$ are independent for $i \neq k, 1 \leq i, k \leq M$.

The $N \times 1$ code vector transmitted from the N antennas at time index l is denoted as

$$\mathbf{c}_l = [c_1(l) \quad c_2(l), \ldots, c_N(l)]^T \quad (2)$$

where $(\cdot)^T$ denotes transpose, and the corresponding $M \times 1$ channel vector from the $j\text{th}$ transmit antenna to the M receive antennas as

$$\mathbf{h}_j(l) = [h_{1j}(l) \quad h_{2j}(l), \ldots, h_{Mj}(l)]^T. \quad (3)$$

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In addition, let

$$\mathbf{n}(l) = [n_1(l) \quad n_2(l), \ldots, n_M(l)]^T \tag{4}$$

be the $M \times 1$ noise vector at the receive antennas. Note that the noise vectors for different time indexes, represented as $\{\mathbf{n}(l)\}$, are i.i.d. complex circular Gaussian random vectors, each $\mathbf{n}(l)$ having a $\mathcal{CN}(\mathbf{0}_{M\times 1}, 2N_0\mathbf{I}_M)$ distribution, where $\mathbf{I}_M$ denotes the $M \times M$ identity matrix.

The $M \times N$ *channel matrix* $\mathbf{H}(l)$, which is independent of the noise, is given by $\mathbf{H}(l) = [\mathbf{h}_1(l), \ \mathbf{h}_2(l), \ldots, \mathbf{h}_N(l)]$, and the $M \times 1$ received signal vector $\mathbf{r}(l)$ by $\mathbf{r}(l) = [r_1(l), \ r_2(l), \ldots, r_M(l)]^T$. We can therefore rewrite (1) in matrix form as [1]

$$\mathbf{r}(l) = \sqrt{2E_s}\mathbf{H}(l)\mathbf{c}_l + \mathbf{n}(l). \tag{5}$$

Owing to flat fading, the channel matrix $\mathbf{H}(l)$ is assumed to be constant over the time indexes which span the pilot transmission phase followed by the encoded data transmission phase. Hence, we denote the channel matrix as

$$\mathbf{H} = [\mathbf{h}_1 \quad \mathbf{h}_2, \ldots, \mathbf{h}_N]. \tag{6}$$

Further, we consider a Rayleigh fading channel, with entries of the channel matrix $\mathbf{H}$ being i.i.d. zero-mean complex circular Gaussian random variables each having a $\mathcal{CN}(0, \Omega)$ distribution.

III. Channel Estimation

We use a sequence of L_p pilot code vectors $\mathbf{c}_{p_1}, \ldots, \mathbf{c}_{p_{L_p}}$ which form the $N \times L_p$ pilot code matrix $\mathbf{C}_p$ given by

$$\mathbf{C}_p = [\mathbf{c}_{p_1} \quad \mathbf{c}_{p_2}, \ldots, \mathbf{c}_{p_{L_p}}]. \tag{7}$$

When the pilot code matrix is transmitted, we receive

$$\mathbf{r}_p(l) = \sqrt{2E_s}\mathbf{H}\mathbf{c}_{p_l} + \mathbf{n}_p(l), \quad l = 1, \ldots, L_p \tag{8}$$

where $\mathbf{n}_p(1), \ldots, \mathbf{n}_p(L_p)$ are i.i.d. complex circular Gaussian random vectors, each having a $\mathcal{CN}(\mathbf{0}_{M\times 1}, 2N_0\mathbf{I}_M)$ distribution.

Let the received pilot signal matrix $\mathbf{R}_p$ and the pilot noise matrix $\mathbf{N}_p$ be given by

$$\mathbf{R}_p = [\mathbf{r}_p(1), \quad \mathbf{r}_p(2), \ldots, \mathbf{r}_p(L_p)] \tag{9}$$

$$\mathbf{N}_p = [\mathbf{n}_p(1), \quad \mathbf{n}_p(2), \ldots, \mathbf{n}_p(L_p)]. \tag{10}$$

Using (9) and (10), we can rewrite (8) as

$$\mathbf{R}_p = \sqrt{2E_s}\mathbf{H}\mathbf{C}_p + \mathbf{N}_p. \tag{11}$$

From (11), we obtain a least-squares estimate of the channel matrix [4], [5] which is given by

$$\hat{\mathbf{H}} = \frac{1}{\sqrt{2E_s}}\mathbf{R}_p\mathbf{C}_p^H\left(\mathbf{C}_p\mathbf{C}_p^H\right)^{-1} \tag{12}$$

where $(\cdot)^H$ denotes the Hermitian (conjugate transpose) operator. Note that *we need to choose* $\mathbf{C}_p$ *such that* $\mathbf{C}_p\mathbf{C}_p^H$ *is invertible, which implies that* $L_p \geq N$. Combining (12) and (11), we get

$$\hat{\mathbf{H}} = \mathbf{H} + \frac{1}{\sqrt{2E_s}}\mathbf{N}_p\mathbf{C}_p^H\left(\mathbf{C}_p\mathbf{C}_p^H\right)^{-1}. \tag{13}$$

Note that the estimate of $\mathbf{H}$ is perturbed by an $M \times N$ *perturbation matrix* with zero-mean circular Gaussian entries, which are uncorrelated if $\mathbf{C}_p$ is chosen such that $\mathbf{C}_p\mathbf{C}_p^H$ is a scaled version of $\mathbf{I}_N$, and correlated otherwise.

IV. Performance Criterion

Let $\mathcal{C}_{N,L}$ denote the set of $N \times L$ code matrices used for transmission. Suppose that the code matrix

$$\mathbf{C} = [\mathbf{c}_1 \quad \mathbf{c}_2, \ldots, \mathbf{c}_L] \tag{14}$$

is transmitted after the channel has been estimated. At the receiver, we choose the code matrix

$$\tilde{\mathbf{C}} = [\tilde{\mathbf{c}}_1 \quad \tilde{\mathbf{c}}_2, \ldots, \tilde{\mathbf{c}}_L] \tag{15}$$

using the minimum distance rule, which results in

$$\tilde{\mathbf{C}} = \arg\left\{\min_{[\mathbf{x}_1,\ldots,\mathbf{x}_L]\in\mathcal{C}_{N,L}} \sum_{l=1}^{L} \|\mathbf{r}(l) - \sqrt{2E_s}\hat{\mathbf{H}}\mathbf{x}_l\|^2\right\} \tag{16}$$

where $\|\cdot\|$ denotes the L_2-norm or Euclidean norm of a vector. Substituting (5) and (13) in (16), we get the PEP, which is given by

$$\begin{aligned} P(\mathbf{C} \to \tilde{\mathbf{C}}) &= \Pr\left\{\sum_{l=1}^{L}\left\|\sqrt{2E_s}\mathbf{H}(\mathbf{c}_l - \tilde{\mathbf{c}}_l)\right.\right. \\ &\quad \left. + \mathbf{n}(l) - \mathbf{N}_p\mathbf{C}_p^H\left(\mathbf{C}_p\mathbf{C}_p^H\right)^{-1}\tilde{\mathbf{c}}_l\right\|^2 \\ &\quad \left. < \sum_{l=1}^{L}\left\|\mathbf{n}(l) - \mathbf{N}_p\mathbf{C}_p^H\left(\mathbf{C}_p\mathbf{C}_p^H\right)^{-1}\mathbf{c}_l\right\|^2\right\} \end{aligned} \tag{17}$$

where the $M \times 1$ complex Gaussian vectors $\mathbf{n}(1), \ldots, \mathbf{n}(L)$ are independent of the $M \times L_p$ complex Gaussian matrix $\mathbf{N}_p$.

When the channel estimate is perfect, we have $\hat{\mathbf{H}} = \mathbf{H}$, and the PEP is given by

$$P_{\text{per}}\,\text{f}(\mathbf{C} \to \tilde{\mathbf{C}}) = \Pr\left\{\sum_{l=1}^{L}\left\|\sqrt{2E_s}\mathbf{H}(\mathbf{c}_l - \tilde{\mathbf{c}}_l) + \mathbf{n}(l)\right\|^2 < \sum_{l=1}^{L}\|\mathbf{n}(l)\|^2\right\}. \tag{18}$$

From the Gaussian distribution of $\mathbf{n}(l)$, it can be easily shown that the conditional PEP, conditioned on the channel matrix $\mathbf{H}$, is given by

$$P_{\text{per}}\,\text{f}(\mathbf{C} \to \tilde{\mathbf{C}} \,|\, \mathbf{H}) = Q\left(\sqrt{\frac{E_s}{2N_0}\sum_{l=1}^{L}\|\mathbf{H}(\mathbf{c}_l - \tilde{\mathbf{c}}_l)\|^2}\right) \tag{19}$$

where $Q(\cdot)$ denotes the Gaussian-Q function. The PEP can be obtained by averaging the conditional PEP over the statistics of $\mathbf{H}$, as in [7].

When the channel has an imperfect estimate given by (13), we can write

$$\hat{\mathbf{H}} = \mathbf{H} + \Delta\mathbf{H} \tag{20}$$

where the perturbation matrix $\Delta\mathbf{H}$ is given by

$$\Delta\mathbf{H} = \frac{1}{\sqrt{2E_s}}\mathbf{N}_p\mathbf{C}_p^H\left(\mathbf{C}_p\mathbf{C}_p^H\right)^{-1}. \tag{21}$$

Denoting the random vector $\hat{\mathbf{n}}(l)$ as

$$\hat{\mathbf{n}}(l) = \mathbf{n}(l) - \sqrt{2E_s}(\Delta\mathbf{H})\mathbf{c}_l \tag{22}$$

we can express (17) in the form

$$\begin{aligned} &P(\mathbf{C} \to \tilde{\mathbf{C}}) \\ &= \Pr\left\{\sum_{l=1}^{L}\left\|\sqrt{2E_s}\hat{\mathbf{H}}(\mathbf{c}_l - \tilde{\mathbf{c}}_l) + \hat{\mathbf{n}}(l)\right\|^2 \right. \\ &\qquad \left. < \sum_{l=1}^{L}\|\hat{\mathbf{n}}(l)\|^2\right\} \end{aligned} \tag{23}$$

which can be simplified to yield

$$\begin{aligned} &P(\mathbf{C} \to \tilde{\mathbf{C}}) \\ &= \Pr\left\{2\sqrt{2E_s}\text{Re}\left\{\sum_{l=1}^{L}[\hat{\mathbf{H}}(\mathbf{c}_l - \tilde{\mathbf{c}}_l)]^H\hat{\mathbf{n}}(l)\right\} \right. \\ &\qquad \left. < -2E_s\sum_{l=1}^{L}\|\hat{\mathbf{H}}(\mathbf{c}_l - \tilde{\mathbf{c}}_l)\|^2\right\}. \end{aligned} \tag{24}$$

When the channel estimate is perfect, we have $\Delta\mathbf{H} = \mathbf{0}_{M\times N}$, resulting in

$$\hat{\mathbf{H}} = \mathbf{H}, \quad \hat{\mathbf{n}}(l) = \mathbf{n}(l) \tag{25}$$

and (23) reduces to (18). When we substitute (25) in (24), and find the conditional PEP, conditioned on $\mathbf{H}$, we obtain (19), *because* $\mathbf{H}$ *and* $\{\mathbf{n}(l)\}$ are independent. On the other hand, when the channel estimate is imperfect, although $\mathbf{H}, \{\mathbf{n}(l)\}$, and $\Delta\mathbf{H}$ *are independent*, we see from (20) and (22) that *both* $\hat{\mathbf{H}}$ *and* $\{\hat{\mathbf{n}}(l)\}$ *depend on* $\Delta\mathbf{H}$, *and are therefore, not independent*. We can choose appropriate space-time codes such that the marginal statistics of $\hat{\mathbf{H}}$ becomes the same as the statistics of $\mathbf{H}$ and the marginal statistics of $\{\hat{\mathbf{n}}(l)\}$ becomes the same as the statistics of $\{\mathbf{n}(l)\}$. However, owing to the dependence on $\Delta\mathbf{H}$, the joint statistics of $\hat{\mathbf{H}}$ and $\{\hat{\mathbf{n}}(l)\}$ is not the same as the joint statistics of $\mathbf{H}$ and $\{\mathbf{n}(l)\}$. As a result, none of $P(\mathbf{C} \to \tilde{\mathbf{C}} \,|\, \mathbf{H}), P(\mathbf{C} \to \tilde{\mathbf{C}} \,|\, \hat{\mathbf{H}})$, or $P(\mathbf{C} \to \tilde{\mathbf{C}} \,|\, \mathbf{H}, \Delta\mathbf{H})$, will have a form similar to (19). This calls for an alternative analysis of the PEP.

V. Performance Analysis

Instead of obtaining first the conditional PEP when the channel estimate is given by (13) and then averaging it over the statistics of $\mathbf{H}$ to get the PEP, we will derive an expression for the PEP given by (17) from the cf. of the decision variable $\mathcal{D}$ which is defined as

$$\begin{aligned} \mathcal{D} \triangleq &\sum_{l=1}^{L}\left\|\sqrt{2E_s}\mathbf{H}(\mathbf{c}_l - \tilde{\mathbf{c}}_l)\right. \\ &\left. + \mathbf{n}(l) - \mathbf{N}_p\mathbf{C}_p^H\left(\mathbf{C}_p\mathbf{C}_p^H\right)^{-1}\tilde{\mathbf{c}}_l\right\|^2 \\ &- \sum_{l=1}^{L}\left\|\mathbf{n}(l) - \mathbf{N}_p\mathbf{C}_p^H\left(\mathbf{C}_p\mathbf{C}_p^H\right)^{-1}\mathbf{c}_l\right\|^2. \end{aligned} \tag{26}$$

Define $M \times 1$ random vectors $\mathbf{a}_l, \tilde{\mathbf{a}}_l$, and $\mathbf{g}_l$ as

$$\begin{aligned} \mathbf{a}_l &\triangleq \mathbf{n}(l) - \mathbf{N}_p\mathbf{C}_p^H\left(\mathbf{C}_p\mathbf{C}_p^H\right)^{-1}\mathbf{c}_l \\ \tilde{\mathbf{a}}_l &\triangleq \mathbf{n}(l) - \mathbf{N}_p\mathbf{C}_p^H\left(\mathbf{C}_p\mathbf{C}_p^H\right)^{-1}\tilde{\mathbf{c}}_l \\ \mathbf{g}_l &\triangleq \sqrt{2E_s}\mathbf{H}(\mathbf{c}_l - \tilde{\mathbf{c}}_l). \end{aligned} \tag{27}$$

Further, define $L_p \times L$ matrices $\mathbf{B}$ and $\tilde{\mathbf{B}}$ as

$$\begin{aligned} \mathbf{B} &\triangleq \mathbf{C}_p^H\left(\mathbf{C}_p\mathbf{C}_p^H\right)^{-1}\mathbf{C} \\ \tilde{\mathbf{B}} &\triangleq \mathbf{C}_p^H\left(\mathbf{C}_p\mathbf{C}_p^H\right)^{-1}\tilde{\mathbf{C}}. \end{aligned} \tag{28}$$

From (27), we can rewrite the decision variable $\mathcal{D}$ in (26) as

$$\mathcal{D} = \sum_{l=1}^{L}[\|\mathbf{g}_l + \tilde{\mathbf{a}}_l\|^2 - \|\mathbf{a}_l\|^2]. \tag{29}$$

The vectors $\mathbf{g}_1, \ldots, \mathbf{g}_L, \tilde{\mathbf{a}}_1, \ldots, \tilde{\mathbf{a}}_L, \mathbf{a}_1, \ldots, \mathbf{a}_L$ are jointly complex Gaussian. Define the $2ML \times 1$ composite vector $\mathbf{v}$ as

$$\mathbf{v} \triangleq \begin{bmatrix} \mathbf{g}_1 + \tilde{\mathbf{a}}_1 \\ \vdots \\ \mathbf{g}_L + \tilde{\mathbf{a}}_L \\ \mathbf{a}_1 \\ \vdots \\ \mathbf{a}_L \end{bmatrix}. \tag{30}$$

From the Gaussian statistics of $\mathbf{n}(1), \ldots, \mathbf{n}(L), \mathbf{N}_p$, and $\mathbf{H}$, we find that $\mathbf{v}$ is a zero-mean complex circular Gaussian random vector having a $\mathcal{CN}(\mathbf{0}_{2ML\times 1}, \mathbf{K}_v)$ distribution, where the covariance matrix $\mathbf{K}_v$ is given by

$$\mathbf{K}_v = 2N_0 \begin{bmatrix} \left(\mathbf{I}_L + \tilde{\mathbf{B}}^T\tilde{\mathbf{B}}^*\right) + \frac{\Omega E_s}{N_0}(\mathbf{C}-\tilde{\mathbf{C}})^T(\mathbf{C}-\tilde{\mathbf{C}})^* & \left(\mathbf{I}_L + \tilde{\mathbf{B}}^T\mathbf{B}^*\right) \\ \left(\mathbf{I}_L + \mathbf{B}^T\tilde{\mathbf{B}}^*\right) & \left(\mathbf{I}_L + \mathbf{B}^T\mathbf{B}^*\right) \end{bmatrix} \otimes \mathbf{I}_M \quad (31)$$

$\otimes$ denoting the Kronecker product, and $(\cdot)^*$ denoting complex conjugation. The decision variable $\mathcal{D}$ in (29) can be written as

$$\mathcal{D} = \mathbf{v}^H \begin{bmatrix} \mathbf{I}_{ML} & \mathbf{0} \\ \mathbf{0} & -\mathbf{I}_{ML} \end{bmatrix} \mathbf{v} \quad (32)$$

which is a Hermitian quadratic form in complex Gaussian random variables. Using the result of [8], we can express the cf. of $\mathcal{D}$ as

$$\Psi_{\mathcal{D}}(\jmath\omega) = \mathbf{E}[e^{\jmath\omega\mathcal{D}}] = \frac{1}{\det\left(\mathbf{I}_{2ML} - \jmath\omega \begin{bmatrix} \mathbf{I}_{ML} & \mathbf{0} \\ \mathbf{0} & -\mathbf{I}_{ML} \end{bmatrix} \mathbf{K}_v\right)}. \quad (33)$$

Substituting (31) in (33), we get

$$\Psi_{\mathcal{D}}(\jmath\omega) = \frac{1}{[\det(\mathbf{I}_{2L} - 2\jmath\omega N_0 \mathbf{A})]^M} \quad (34)$$

where $\mathbf{A}$ is a $2L \times 2L$ matrix given by

$$\mathbf{A} = \begin{bmatrix} \left(\mathbf{I}_L + \tilde{\mathbf{B}}^T\tilde{\mathbf{B}}^*\right) + \frac{\Omega E_s}{N_0}(\mathbf{C}-\tilde{\mathbf{C}})^T(\mathbf{C}-\tilde{\mathbf{C}})^* & \left(\mathbf{I}_L + \tilde{\mathbf{B}}^T\mathbf{B}^*\right) \\ -\left(\mathbf{I}_L + \mathbf{B}^T\tilde{\mathbf{B}}^*\right) & -\left(\mathbf{I}_L + \mathbf{B}^T\mathbf{B}^*\right) \end{bmatrix}. \quad (35)$$

If $\Lambda_1, \ldots, \Lambda_{2L}$ are the $2L$ eigenvalues of $\mathbf{A}$, then the cf. of $\mathcal{D}$ can be simply written as

$$\Psi_{\mathcal{D}}(\jmath\omega) = \frac{1}{\prod_{i=1}^{2L}(1 - 2\jmath\omega N_0 \Lambda_i)^M}. \quad (36)$$

The PEP is given by

$$P(\mathbf{C} \to \tilde{\mathbf{C}}) = \Pr(\mathcal{D} < 0). \quad (37)$$

We now obtain the PEP from the cf. of $\mathcal{D}$ using the inversion theorem [9]. After changing the variable $\jmath\omega$ to $z = 2\jmath\omega N_0$ in (36), we get [9]

$$P(\mathbf{C} \to \tilde{\mathbf{C}}) = -\left[\text{sum of residues of } \frac{\Psi_{\mathcal{D}}\left(\frac{z}{2N_0}\right)}{z} \text{ at poles on left-half } z\text{-plane}\right]. \quad (38)$$

Note from (36) that the poles of $\Psi_{\mathcal{D}}[z/(2N_0)]$ are given by $z = 1/\Lambda_i$, $i = 1, \ldots, 2L$, all of which may not be distinct. Some of these poles will be on the left half z-plane. Let μ_k be a pole of order p_k. The PEP can then be expressed using (38) as

$$P(\mathbf{C} \to \tilde{\mathbf{C}}) = -\sum_{\substack{\text{pole } \mu_k \\ \text{of order } p_k \\ \text{such that} \\ \mathrm{Re}(\mu_k)<0}} \frac{1}{(p_k - 1)!} \frac{d^{p_k-1}}{dz^{p_k-1}} \left[(z-\mu_k)^{p_k} \frac{\Psi_{\mathcal{D}}\left(\frac{z}{2N_0}\right)}{z}\right]\Bigg|_{z=\mu_k}. \quad (39)$$

Since $(z-\mu_k)^{p_k}(\Psi_{\mathcal{D}}(z/(2N_0)))/z$ is a rational function in z, its $(p_k - 1)\text{th}$ derivative can be conveniently obtained using Faa di Bruno's formula [10]. Thus, (39) *provides us with a means of computing the PEP without the use of any integral*.

We now consider the *case when we use orthogonal codes for channel estimation and transmission* [2], [11]. This results in, without loss of generality, the conditions

$$\mathbf{C}_p\mathbf{C}_p^H = \frac{L_p}{L}\mathbf{I}_N, \quad \mathbf{C}\mathbf{C}^H = \tilde{\mathbf{C}}\tilde{\mathbf{C}}^H = \mathbf{I}_N. \quad (40)$$

Note that (40) also implies $L_p \geq N$ and $L \geq N$. In addition, *we consider* $L = N$, which implies the condition

$$\mathbf{C}^H\mathbf{C} = \tilde{\mathbf{C}}^H\tilde{\mathbf{C}} = \mathbf{I}_L. \quad (41)$$

Substituting (40) and (41) in (35), we get

$$\mathbf{A} = \begin{bmatrix} \left(1 + \frac{L}{L_p} + 2\frac{\Omega E_s}{N_0}\right)\mathbf{I}_L - \frac{\Omega E_s}{N_0}\left(\mathbf{C}^T\tilde{\mathbf{C}}^* + \tilde{\mathbf{C}}^T\mathbf{C}^*\right) & \left(\mathbf{I}_L + \frac{L}{L_p}\tilde{\mathbf{C}}^T\mathbf{C}^*\right) \\ -\left(\mathbf{I}_L + \frac{L}{L_p}\mathbf{C}^T\tilde{\mathbf{C}}^*\right) & -\left(1 + \frac{L}{L_p}\right)\mathbf{I}_L \end{bmatrix}. \quad (42)$$

To obtain the eigenvalues of $\mathbf{A}$ in (42), we look at the characteristic equation

$$\det(\mathbf{A} - \lambda\mathbf{I}_{2L}) = 0. \quad (43)$$

Let $\nu_1, \ldots, \nu_K$ denote the K distinct real positive eigenvalues of the positive definite Hermitian matrix

$$(\mathbf{C}-\tilde{\mathbf{C}})^T(\mathbf{C}-\tilde{\mathbf{C}})^* = 2\mathbf{I}_L - (\mathbf{C}^T\tilde{\mathbf{C}}^* + \tilde{\mathbf{C}}^T\mathbf{C}^*)$$

such that ν_i has multiplicity q_i for $i = 1, \ldots, K$. Thus, $q_1 + \cdots + q_K = L$. Defining Γ, the *average SNR per diversity branch*, as

$$\Gamma \triangleq \frac{\Omega E_s}{N_0} \quad (44)$$

where Ω is the variance of the complex channel gain h_{ij}, we can simplify the characteristic equation after some algebra, resulting in

$$\prod_{i=1}^{K}\left(\lambda^2-\Gamma\nu_i\lambda-\left(\frac{L}{L_p}+\left[1+\frac{L}{L_p}\right]\Gamma\right)\nu_i\right)^{q_i}=0. \quad (45)$$

Both the roots of

$$\lambda^2-\Gamma\nu_i\lambda-\left(\frac{L}{L_p}+\left[1+\frac{L}{L_p}\right]\Gamma\right)\nu_i=0$$

which we denote as λ_{i1} and λ_{i2}, are real, one being positive and the other negative. Let $\lambda_{i1}>0$ and $\lambda_{i2}<0$. We can rewrite (36) as

$$\Psi_{\mathcal{D}}(\jmath\omega)=\frac{1}{\prod_{i=1}^{K}\left[(1-2\jmath\omega N_0\lambda_{i1})^{Mq_i}(1-2\jmath\omega N_0\lambda_{i2})^{Mq_i}\right]} \quad (46)$$

where

$$\lambda_{i1}=\frac{\Gamma\nu_i+\sqrt{\Gamma^2\nu_i^2+4\left(\frac{L}{L_p}+\left[1+\frac{L}{L_p}\right]\Gamma\right)\nu_i}}{2}$$
$$\lambda_{i2}=\frac{\Gamma\nu_i-\sqrt{\Gamma^2\nu_i^2+4\left(\frac{L}{L_p}+\left[1+\frac{L}{L_p}\right]\Gamma\right)\nu_i}}{2}. \quad (47)$$

The poles of $(\Psi_{\mathcal{D}}(z/(2N_0)))/z$ which are on the left-half z-plane are $\lambda_{i2}^{-1}i=1,\ldots,K$.

Define the functions $G_j(z)$ and $F_j(z)$ as

$$G_j(z)\triangleq\left(z-\lambda_{j2}^{-1}\right)^{Mq_j}\frac{\Psi_{\mathcal{D}}\left(\frac{z}{2N_0}\right)}{z}$$
$$F_j(z)\triangleq\ln G_j(z). \quad (48)$$

From (39), the PEP is given by

$$P(\mathbf{C}\rightarrow\tilde{\mathbf{C}})=-\sum_{j=1}^{K}\frac{1}{(Mq_j-1)!}G_j^{(Mq_j-1)}\left(\lambda_{j2}^{-1}\right) \quad (49)$$

where $G_j^{(Mq_j-1)}(\lambda_{j2}^{-1})$ denotes the $(Mq_j-1)\text{th}$ derivative of $G_j(z)$ evaluated at $z=\lambda_{j2}^{-1}$.

Now the $m\text{th}$ derivative of $F_j(z)$ can be expressed as

$$F_j^{(m)}(z)=\frac{(-1)^m(m-1)!}{z^m}+(m-1)!M\sum_{i=1}^{K}\frac{q_i\lambda_{i1}^m}{(1-z\lambda_{i1})^m}+(m-1)!M\sum_{\substack{i=1\\i\neq j}}^{K}\frac{q_i\lambda_{i2}^m}{(1-z\lambda_{i2})^m}. \quad (50)$$

Using Faa di Bruno's formula [10], the $(Mq_j-1)\text{th}$ derivative of $G_j(z)$ can be expressed as

$$G_j^{(Mq_j-1)}(z)=(Mq_j-1)!G_j(z)\times\sum_{\substack{(l_1,\ldots,l_{Mq_j-1})\\0\le l_1,\ldots,l_{Mq_j-1}\le Mq_j-1\\l_1+2l_2+\cdots+(Mq_j-1)l_{Mq_j-1}=Mq_j-1}}\prod_{m=1}^{Mq_j-1}\frac{1}{l_m!}\times\left(\frac{F_j^{(m)}(z)}{m!}\right)^{l_m} \quad (51)$$

where the summation is over all (Mq_j-1)-tuples $(l_1,\ldots,l_{Mq_j-1})$ of integers in the range $[0,Mq_j-1]$ satisfying $\sum_{m=1}^{Mq_j-1}ml_m=Mq_j-1$. Substituting (51) and (50) in (49), we obtain (52), as found at the bottom of the page,where $\{\lambda_{i1}\},\{\lambda_{i2}\}$ are given by (47). Thus, (52) is a *closed-form expression for the PEP* in terms of the distinct real positive eigenvalues $\nu_1,\ldots,\nu_K$ of the matrix $(\mathbf{C}-\tilde{\mathbf{C}})^T(\mathbf{C}-\tilde{\mathbf{C}})^*$.

To numerically compute the PEP from this expression, we need a precalculated lookup table of enumerations of the indexes $l_1,l_2,\ldots,$ in the composite summation of (52) and the eigenvalues $\nu_1,\ldots,\nu_K$. The lookup table generation, the eigenvalue computation, and the subsequent simple operations which yield the PEP can be easily performed using a mathematical software like MATLAB.

Let the number of enumerations of $(l_1,\ldots,l_{Mq_j-1})$ for which the equality

$$l_1+2l_2+\cdots+(Mq_j-1)l_{Mq_j-1}=Mq_j-1$$

holds when $0\le l_1,\ldots,l_{Mq_j-1}\le Mq_j-1$ be denoted as a function $f(Mq_j)$. For given L,M, and $q_1,\ldots,q_K$, the total number of summations in (52), denoted as $\mathcal{S}_{L,M}$, is then given by

$$\mathcal{S}_{L,M}=f(Mq_1)+\cdots+f(Mq_K). \quad (53)$$

$$P(\mathbf{C}\rightarrow\tilde{\mathbf{C}})=\sum_{j=1}^{K}\frac{(-\lambda_{j2})^{M(2L-q_j)}}{\left[\prod_{i=1}^{K}(\lambda_{i1}-\lambda_{j2})^{Mq_i}\right]\left[\prod_{\substack{i=1\\i\neq j}}^{K}(\lambda_{i2}-\lambda_{j2})^{Mq_i}\right]}\times\sum_{\substack{(l_1,\ldots,l_{Mq_j-1})\\0\le l_1,\ldots,l_{Mq_j-1}\le Mq_j-1\\l_1+2l_2+\cdots+(Mq_j-1)l_{Mq_j-1}=Mq_j-1}}\prod_{m=1}^{Mq_j-1}\frac{1}{l_m!}\left[\frac{1}{m}+\frac{M}{m}\sum_{i=1}^{K}\frac{q_i\lambda_{i1}^m}{(\lambda_{i1}-\lambda_{j2})^m}+\frac{M}{m}\sum_{\substack{i=1\\i\neq j}}^{K}\frac{q_i\lambda_{i2}^m}{(\lambda_{i2}-\lambda_{j2})^m}\right]^{l_m} \quad (52)$$

TABLE I
NUMBER OF ENUMERATIONS OF $l_1, \ldots, l_{Mq_j - 1}$ IN (52)

Mq_j	$f(Mq_j)$
2	1
3	2
4	3
5	5
6	7
7	11
8	15
9	22
10	30
11	42
12	56
13	77
14	101
15	135
16	176

This number of summations is bounded by two extremes. At the lower bound, we have the case

$$K = L, \quad q_1 = \cdots = q_L = 1$$

while at the upper bound we have the case

$$K = 1, \quad q_1 = L.$$

Therefore, from (53), we can write

$$Lf(M) \leq \mathcal{S}_{L,M} \leq f(ML). \tag{54}$$

Values of $f(Mq_j)$ for $Mq_j = 2, \ldots, 16$ are shown in Table I. Table II shows the lower and upper bounds on the total number $S_{L,M}$ of summations in (52) for some typical values of L and M.

When the channel estimate is perfect, we have $\hat{\mathbf{H}} = \mathbf{H}$ in (13), implying $\mathbf{B} = \tilde{\mathbf{B}} = \mathbf{0}_{L_p \times L}$ in the matrix $\mathbf{A}$ given by (35). This also implies the condition $L_p \gg L$. By putting $L/L_p = 0$ in (45), the characteristic equation of $\mathbf{A}$ can now be written as

$$\prod_{i=1}^{K} (\lambda^2 - \Gamma\nu_i\lambda - \Gamma\nu_i)^{q_i} = 0. \tag{55}$$

Let λ'_{i1} and λ'_{i2} be the roots of $\lambda^2 - \Gamma\nu_i\lambda - \Gamma\nu_i = 0$, where Γ is defined in (44), with $\lambda'_{i1} > 0$ and $\lambda'_{i2} < 0$. Thus

$$\lambda'_{i1}, \lambda'_{i2} = \frac{\Gamma\nu_i \pm \sqrt{\Gamma^2\nu_i^2 + 4\Gamma\nu_i}}{2}. \tag{56}$$

Therefore, the PEP for perfect channel estimation is given by (52) with λ_{i1} replaced by λ'_{i1} and λ_{i2} by λ'_{i2}.

TABLE II
LOWER AND UPPER BOUNDS ON THE TOTAL NUMBER $\mathcal{S}_{L,M}$ OF SUMMATIONS IN (52)

(L, M)	Lower Bound $Lf(M)$	Upper Bound $f(ML)$
(2,2)	2	3
(2,3)	4	7
(2,4)	6	15
(3,2)	3	7
(3,3)	6	22
(3,4)	9	56
(4,2)	4	15
(4,3)	8	56
(4,4)	12	176

VI. AN EXAMPLE

Consider a system using the Alamouti code [2] with $L = N = L_p = 2$. A codeword is given by

$$\mathbf{C} = \begin{bmatrix} s_1 & -s_2^* \\ s_2 & s_1^* \end{bmatrix}. \tag{57}$$

Take the case of BPSK, where $s_1, s_2 \in \{-1/\sqrt{2}, 1/\sqrt{2}\}$. We have four possible code matrices. Assuming all code matrices are equally likely, the matrix $(\mathbf{C} - \tilde{\mathbf{C}})^T(\mathbf{C} - \tilde{\mathbf{C}})^*$ takes the values $2\mathbf{I}_2$ with probability 2/3 and $4\mathbf{I}_2$ with probability 1/3. Thus, we have either $\nu_1 = \nu_2 = 2$ or $\nu_1 = \nu_2 = 4$. Therefore, $K = 1$ and $q_1 = 2$. Denoting $\nu = \nu_1 = \nu_2, \lambda_1 = \lambda_{11}$, and $\lambda_2 = \lambda_{12}$, we get from (52) the PEP

$$\begin{aligned} &P(\mathbf{C} \to \tilde{\mathbf{C}}) \\ &= P(\nu, \Gamma) \\ &= \frac{(-\lambda_2)^{2M}}{(\lambda_1 - \lambda_2)^{2M}} \\ &\quad \times \sum_{\substack{(l_1, \ldots, l_{2M-1}) \\ 0 \leq l_1, \ldots, l_{2M-1} \leq 2M-1 \\ l_1 + 2l_2 + \cdots + (2M-1)l_{2M-1} = 2M-1}} \prod_{m=1}^{2M-1} \frac{1}{l_m!} \\ &\quad \times \left[\frac{1}{m} + \frac{2M}{m} \frac{\lambda_1^m}{(\lambda_1 - \lambda_2)^m}\right]^{l_m} \end{aligned} \tag{58}$$

where, from (47)

$$\lambda_1, \lambda_2 = \frac{\Gamma\nu \pm \sqrt{\Gamma^2\nu^2 + 4(1 + 2\Gamma)\nu}}{2}. \tag{59}$$

We can replace λ_1, λ_2 in (58) by λ'_1, λ'_2, respectively, to get the PEP with perfect channel estimation. Note from (56) that

$$\lambda'_1, \lambda'_2 = \frac{\Gamma\nu \pm \sqrt{\Gamma^2\nu^2 + 4\Gamma\nu}}{2}. \tag{60}$$

The *average PEP* is given by

$$P(\Gamma) = (2/3)P(2, \Gamma) + (1/3)P(4, \Gamma). \tag{61}$$

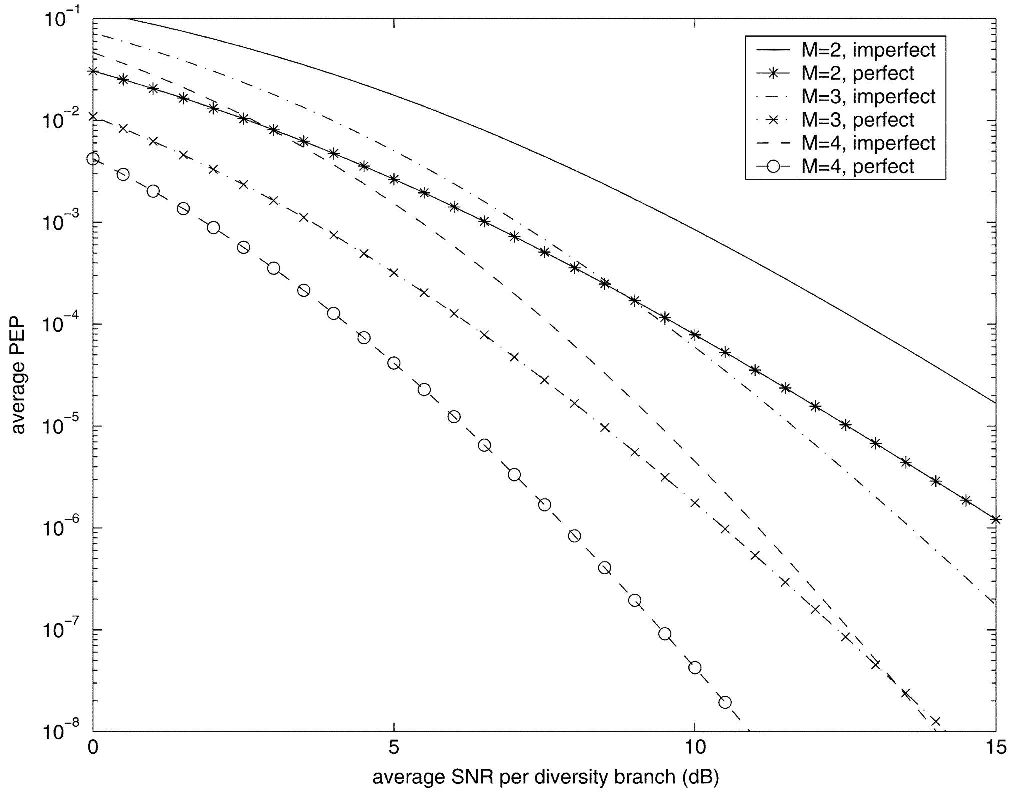

Fig. 1. Average PEP versus Γ for the Alamouti code with BPSK, 2 transmit antennas ($N = 2$), 2 symbol intervals ($L = 2$), 2 pilot code vectors ($L_p = 2$), and number of receive antennas $M = 2, 3, 4$.

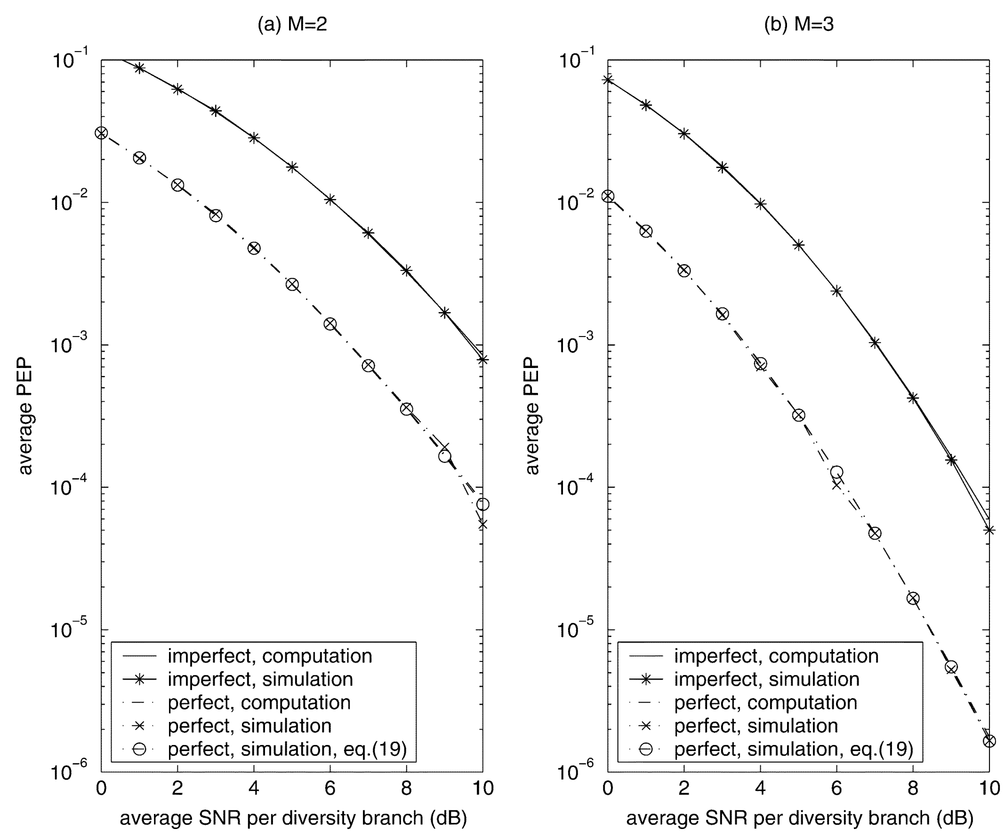

Fig. 2. Comparison of computed and simulated average PEP versus Γ for the Alamouti code with BPSK, 2 transmit antennas ($N = 2$), 2 symbol intervals ($L = 2$), 2 pilot code vectors ($L_p = 2$), and number of receive antennas $M = 2, 3$.

Plots of the average PEP both with imperfect and perfect channel estimation, computed using (58) and (61), are shown in Fig. 1. We find that the degradation in performance due to imperfect channel estimation can be compensated by increasing

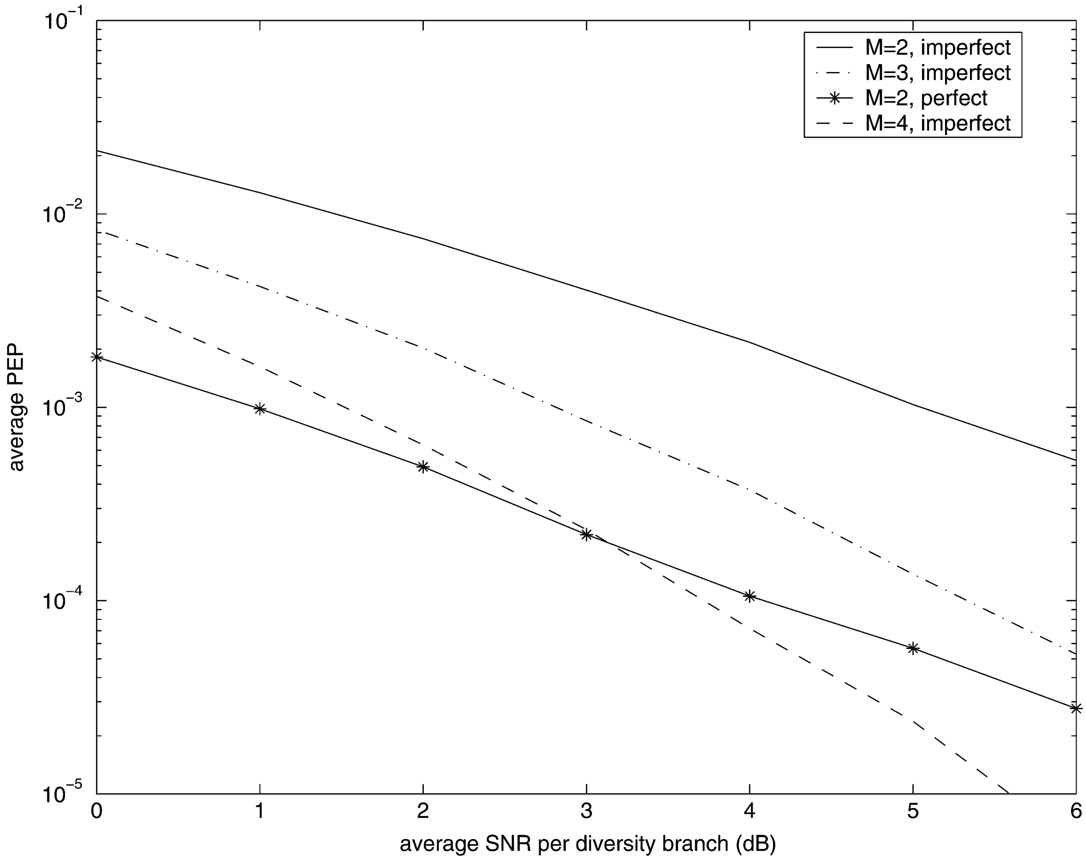

Fig. 3. Average PEP obtained by simulation versus Γ for a nonorthogonal space-time code with BPSK, 3 transmit antennas $(N = 3)$, 3 symbol intervals $(L = 3)$, and 3 pilot code vectors $(L_p = 3)$.

the number M of receive antennas. For example, at $\Gamma \approx 9$ dB, the average PEP with $M = 3$ and imperfect channel estimation is the same as that with $M = 2$ and perfect channel estimation.

Comparison of the average PEP obtained by computation using (58) and (61) with that obtained by simulation is shown in Fig. 2(a) for $M = 2$ and in Fig. 2(b) for $M = 3$. In case of perfect channel estimation, the simulation results obtained by averaging (19) over the statistics of $\mathbf{H}$ are also included. The simulation results are found to match closely with the computed results. In case of imperfect channel estimation, it is not possible to obtain a characterization of the PEP similar to (19).

Plots of the average PEP obtained by simulation for a nonorthogonal space-time code with $L = N = L_p = 3$, is shown in Fig. 3. Here a codeword is given by [12]

$$\mathbf{C} = \begin{bmatrix} s_1 & s_2^* & -s_3^* \\ s_2 & -s_1^* & -s_3^* \\ s_3 & s_1 & s_2^* \end{bmatrix}. \tag{62}$$

We again take the case of BPSK with $s_1, s_2, s_3 \in \{-1, 1\}$. It is found that an increase in the number of receive antennas can compensate for the degradation in performance due to imperfect channel estimation. For example, at $\Gamma \approx 3$ dB, using 4 receive antennas with imperfect channel estimation can achieve the same average PEP as with 2 receive antennas and perfect channel estimation. This behavior is the same as that seen in Fig. 1 with the Alamouti code, which is an orthogonal space-time code.

VII. Conclusion

We have analyzed the error performance of a space-time coding system with imperfect channel estimation in flat Rayleigh fading. Using the cf. of the decision variable, we have derived a closed-form expression for the PEP. The same expression can also be used to obtain the PEP in the case of perfect channel estimation. We have considered an example of a system using the Alamouti code. Numerical results show the degradation in performance due to imperfect channel estimation that can be compensated by increasing the number of receive antennas.

References

[1] V. Tarokh, N. Sheshadri, and A. R. Calderbank, "Space-time codes for high data rate wireless communications: Performance criteria and code construction," *IEEE Trans. Inf. Theory*, vol. 44, pp. 744–765, Mar. 1998.

[2] S. M. Alamouti, "A simple transmitter diversity scheme for wireless communications," *IEEE J. Sel. Areas Commun.*, vol. 16, pp. 1451–1458, Oct. 1998.

[3] H. Arslan and G. E. Bottomley, "Channel estimation in narrowband wireless communication systems," *Wireless Commun. Mobile Comput.*, vol. 1, pp. 201–219, Apr.–Jun. 2001.

[4] Y. G. Li, N. Sheshadri, and S. Ariyavisitakul, "Channel estimation for OFDM systems with transmitter diversity in mobile wireless channels," *IEEE J. Sel. Areas Commun.*, vol. 17, pp. 461–471, Mar. 1999.

[5] Y. G. Li, "Simplified channel estimation for OFDM system with multiple transmit antennas," *IEEE Trans. Wireless Commun.*, vol. 1, pp. 67–75, Jan. 2002.

[6] A. F. Naguib, V. Tarokh, N. Sheshadri, and A. R. Calderbank, "A space-time coding modem for high-data-rate wireless communications," *IEEE J. Sel. Areas Commun.*, vol. 16, pp. 1459–1478, Oct. 1998.

[7] G. Taricco and E. Biglieri, "Exact pairwise error probability of space-time codes," *IEEE Trans. Inf. Theory*, vol. 48, pp. 510–513, Feb. 2002.
[8] G. L. Turin, "The characteristic function of Hermitian quadratic forms in complex normal variables," *Biometrika*, vol. 47, pp. 199–201, 1960.
[9] J. Gil-Pelaez, "Note on the inversion theorem," *Biometrika*, vol. 38, pp. 481–482, 1951.
[10] C.-J. de la Vallee Poussin, *Cours D'Analyze Infinitesimale*, 12th ed, Paris, France: Gauthier-Villars, Libraire Universitaire Louvain, 1959, vol. 1.
[11] V. Tarokh, H. Jafarkhani, and A. R. Calderbank, "Space-time block codes from orthogonal designs," *IEEE Trans. Inf. Theory*, vol. 45, pp. 1456–1467, Jul. 1999.
[12] M. Uysal and C. N. Georghiades, "Non-orthogonal space-time block codes for 3TX antennas," *Electron. Lett.*, vol. 38, no. 25, pp. 1689–1691, Dec. 5, 2002.

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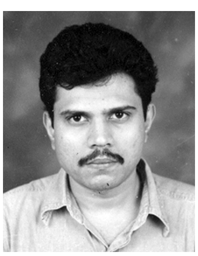

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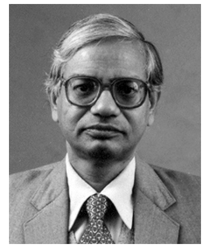

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