ARTIFICIAL INTELLIGENCE (E016350B)
GHENT UNIVERSITY
AY 2024/2025

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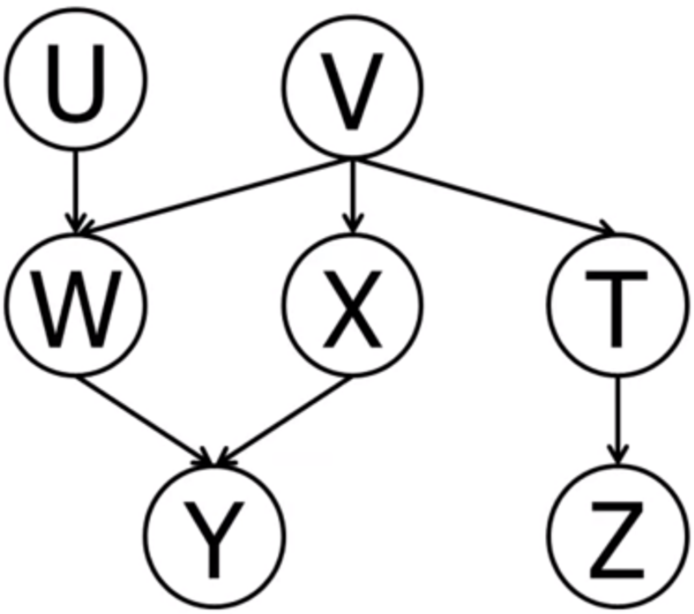
Solutions: Inference in Bayesian networks

1. Consider the Bayes' net given below. Remember that $X \perp\!\!\!\perp Y$ reads as "X is independent of Y given nothing", and $X \perp\!\!\!\perp Y \mid \{Z, W\}$ reads as "X is independent of Y given Z and W". For each expression, indicate whether it is guaranteed to be true or not.



1) $V \perp\!\!\!\perp Z$
2) $V \perp\!\!\!\perp Z \mid T$
3) $U \perp\!\!\!\perp V$
4) $U \perp\!\!\!\perp V \mid W$
5) $U \perp\!\!\!\perp V \mid X$
6) $U \perp\!\!\!\perp V \mid Y$
7) $U \perp\!\!\!\perp V \mid Z$
8) $W \perp\!\!\!\perp X$
9) $X \perp\!\!\!\perp T \mid V$
10) $X \perp\!\!\!\perp W \mid U$
11) $Y \perp\!\!\!\perp Z$
12) $Y \perp\!\!\!\perp Z \mid T$
13) $Y \perp\!\!\!\perp Z \mid X$
14) $Y \perp\!\!\!\perp Z \mid V$
15) $W \perp\!\!\!\perp Z \mid V$
16) $U \perp\!\!\!\perp Z$
17) $U \perp\!\!\!\perp Z \mid Y$

Solution: We will solve this exercise by using the **d-separation** algorithm (see also *L.7.2. Graphical Models and Inference*):

1. Shade all observed nodes $\{Z_1, \ldots, Z_k\}$ in the graph.
2. Enumerate all undirected paths from X to Y.
3. For each path:
   (a) Decompose the path into triples (segments of 3 nodes).
   (b) If all triples are active, this path is active and d-connects X to Y.
4. If no path d-connects X and Y, then X and Y are d-separated, so they are conditionally independent given $\{Z_1, \ldots, Z_k\}$.

We have only two options: guaranteed to be true and not guaranteed to be true.

1) Not guaranteed	7) Guaranteed	13) Not guaranteed
2) Guaranteed	8) Not guaranteed	14) Guaranteed
3) Guaranteed	9) Guaranteed	15) Guaranteed
4) Not guaranteed	10) Not guaranteed	16) Guaranteed
5) Guaranteed	11) Not guaranteed	17) Not guaranteed
6) Not guaranteed	12) Guaranteed	

2. Consider Bayesian networks from Figure below, where S = *Smoking*, L = *LungCancer*, C = *Cough*, B = *BiopsyTest* (*BiopsyTest* is positive only if the result of the test is positive for cancer). All variables are Boolean and the test population consists of 60-years old people who are not smokers or who have smoked for the last 40 years.

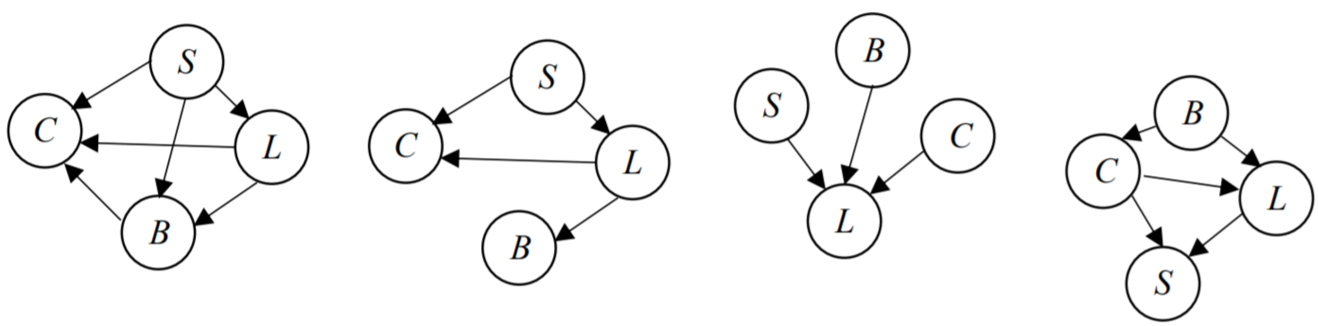


Figure 1: Examples of Bayesian networks.

(a) Which networks are correct, based on the common knowledge about this disease?

(b) Which network has the least amount of parameters? Why?

(c) Write reasonable values for conditional probabilities for the node C in the network (ii).

(d) Using the network (ii), derive a symbolic expression for $\mathbf{P}(B \mid S)$ in terms of conditional probabilities that should be available in the CPT tables (do not use any concrete values for the entries in the CPT tables).

(e) Make a similar derivation for $\mathbf{P}(L \mid B)$, also for the network (ii).

Solution:

(a) Network (ii) is the best representation of the given problem.

(b) For a random binary variable with k parents, the number of all parameters is $2 \cdot 2^k$ and the number of free (independent) parameters is 2^k.
Thus, the number of free parameters for network (ii) is $4 + 2 + 2 + 1 = 9$. For all other networks this number is at least 11 (networks (iii) and (iv)) or larger (15 for network (i)). Network (ii) has the least amount of parameters.

(c) It is enough to give values for the free parameters. So we need four values that correspond to $l, s \in \{True, False\}$ and $c = true$. Thus, we can choose $\mathbf{P}(C \mid L, S) = \langle 0.1, 0.7, 0.6, 0.8 \rangle$ for FF, FT, TF, TT, respectively. FF should be the smallest and TT should be larger than FT and TF.

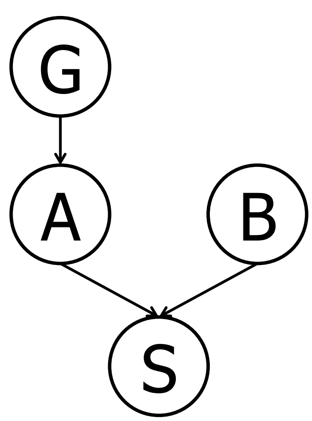
(d) $\mathbf{P}(B \mid S) = \sum_{l} \mathbf{P}(B \mid l)\mathbf{P}(l \mid S)$

(e) $\mathbf{P}(L \mid B) = \alpha\, \mathbf{P}(B \mid L)\mathbf{P}(L) = \alpha\, \mathbf{P}(B \mid L) \sum_{s} \mathbf{P}(L \mid s)P(s)$

3. Suppose that a patient can have a symptom (S) that can be caused by two different diseases (A and B). It is known that the variation of gene G plays a big role in the manifestation of disease A. The Bayes' Net and corresponding conditional probability tables for this situation are shown below. For each part, you may leave your answer as an arithmetic expression.

$\mathbf{P}(G)$	
g	0.1
$\neg g$	0.9

$\mathbf{P}(A \mid G)$		
a	g	1.0
a	$\neg g$	0.1
$\neg a$	g	0.0
$\neg a$	$\neg g$	0.9



$\mathbf{P}(B)$	
b	0.4
$\neg b$	0.6

$\mathbf{P}(S \mid A, B)$			
s	a	b	1.0
s	a	$\neg b$	0.9
s	$\neg a$	b	0.8
s	$\neg a$	$\neg b$	0.1
$\neg s$	a	b	0.0
$\neg s$	a	$\neg b$	0.1
$\neg s$	$\neg a$	b	0.2
$\neg s$	$\neg a$	$\neg b$	0.9

(a) Compute the following entry from the joint distribution:
$P(g, a, b, s) =$

(b) What is the probability that a patient has disease A?
$P(a) =$

(c) What is the probability that a patient has disease A given that they have disease B?
$P(a \mid b) =$

(d) What is the probability that a patient has disease A given that they have symptom S and disease B?
$P(a \mid s, b) =$

(e) What is the probability that a patient has the disease carrying gene variation G given that they have disease A?
$P(g \mid a) =$

(f) What is the probability that a patient has the disease carrying gene variation G given that they have disease B?
$P(g \mid b) =$

Solution:

(a) $P(g, a, b, s) = P(g)P(a \mid g)P(b)P(s \mid b, a) = (0.1)(1.0)(0.4)(1.0) = 0.04.$

(b) $P(a) = P(a \mid g)P(g) + P(a \mid \neg g)P(\neg g) = (1.0)(0.1) + (0.1)(0.9) = 0.19.$

(c) We can easily infer from the graph of the Bayes' net that $A \perp B$ so:

$$P(a \mid b) = P(a) = 0.19.$$

(d) By using similar reasoning as in the previous exercise, we obtain:

$$\begin{aligned} P(a \mid s, b) &= \frac{P(a, b, s)}{P(a, b, s) + P(\neg a, b, s)} = \frac{P(a)P(b)P(s \mid a, b)}{P(a)P(b)P(s|a, b) + P(\neg a)P(b)P(s|\neg a, b)} \\ &= \frac{(0.19)(0.4)(1.0)}{(0.19)(0.4)(1.0) + (0.81)(0.4)(0.8)} = \frac{0.076}{0.076 + 0.2592} \approx 0.2267. \end{aligned}$$

(e)

$$\begin{aligned} P(g \mid a) &= \frac{P(g)P(a \mid g)}{P(g)P(a \mid g) + P(\neg g)P(a \mid \neg g)} \\ &= \frac{(0.1)(1.0)}{(0.1)(1.0) + (0.9)(0.1)} = \frac{0.1}{0.1 + 0.09} = 0.5263 \end{aligned}$$

(f) It can be infered from the graph of Bayes' net that $B \perp G$, so we get:

$$P(g \mid b) = P(g) = 0.1.$$