Compressed Sensing Using Adaptive Sparse Measurements

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Abstract

Compressed sensing (CS) using sparse measurement matrices and iterative message-passing reconstruction algorithms have been recently investigated as a low-complexity alternative to traditional CS methods. In this paper, we investigate the adaptive version of well-known Sudocodes scheme, where the sparse measurement matrix is progressively created based on the outcomes of previous measurements. Inspired by resemblance with rateless coding, we provide a detailed analysis of the adaptive Sudocodes approach in combination with the verification-based LM1 reconstruction. The results show that the adaptivity is a promising feature for reducing complexity and improving performance of CS methods based on sparse measurement matrices.

1 Introduction

Recovery of strictly sparse signals using sparse measurement matrices and iterative message-passing reconstruction algorithms was first discussed by Sarvotham et al [1]. The reconstruction algorithm applied in their Sudocodes scheme was later identified to be a version of the verification decoding algorithm (LM2) proposed for the decoding of Low-Density Parity Check (LDPC) codes [2]. Inspired by the iterative Belief-Propagation (BP) decoding of LDPC codes, the same authors extended the Sudocodes scheme into a more general framework called Compressed Sensing via Belief Propagation (CSBP) [3], that trigerred significant interest in CS methods using sparse measurement matrices by both compressed sensing and sparse-graph coding community.

CSBP is a powerful solution that introduces factor graph modeling and BP reconstruction to recover sparse or approximately sparse signals from noisy measurements. However, CSBP is rather complex both from the analysis and implementation perspective, since the messages exchanged across factor graph represent continuous probability distributions. As a simplification of CSBP amenable to fast implementation and rigorous analysis, verification-based decoding algorithms known as LM1 and LM2 algorithm [2] for noise-free CS recovery of strictly sparse signals have been investigated [4,5]. A related class of reconstruction algorithms are so called Interval-Passing Algorithms (IPA) proposed in [6] and recently analyzed using coding-theoretic tools [7].

The approaches mentioned above apply non-adaptive design of measurement matrices. Recently, benefits of CS methods with *adaptive* measurement matrices have been demonstrated (see [8–12] and the references therein). In an adaptive scenario, the measurements are constructed sequentially making use of the feedback from the observations, such that the sensing energy can be focused on the suspected non-zero components. Most of the reported adaptive CS approaches focus on improving the performance of the support recovery in the presence of noise (reducing the required signal to noise ratio), either by using adaptive Gaussian random matrices [9], collections of independent structured random matrices [10] or repeated bisection of the signal support [11,12]. Much less work has been done on exploring the benefits of adaptive designs for the reduction of the number of measurements. The results in [8] indicate great potentials in this respect. To the best of our knowledge, adaptive designs for the CS schemes based on message passing, such as Sudocodes or generalizations thereof have not been reported yet. In this paper, we introduce and analyze an adaptive version of Sudocodes scheme with LM1 recovery. We provide in-depth analysis of this scheme using tools borrowed from analysis of rateless codes. Our results demonstrate that the adaptivity represents a promising feature for further improvement of CS methods based on sparse measurement matrices.

2 Adaptive CS System Model

We observe a setup where the goal is to recover a strictly sparse signal $\mathbf{x} = (x_1, x_2, \ldots, x_N) \in \mathbb{R}^N$ containing $K \ll N$ non-zero components (the remaining $N - K$ being zero) from a set of measurements $\mathbf{y} = (y_1, y_2, \ldots, y_M) \in \mathbb{R}^M$ where $K < M \ll N$. The CS system model comprises the adaptive measurement subsystem, communication channel and the reconstruction subsystem.

The adaptive measurement subsystem sequentially samples the signal $\mathbf{x}$ using a sequence of measurement vectors $\phi_i, 1 \leq i \leq M$, in order to produce a stream of measurements $\mathbf{y}$, where ϕ_i is a sparse vector of length N and $y_i = \phi_i \cdot \mathbf{x}^T$. The number of measurements M is not fixed in advance and may be arbitrarily extended. Measurement vectors are adaptively designed, i.e., $\phi_l = f(\{\phi_i\}_{i<l}, \{y_i\}_{i<l})$. The measurement vector ϕ_i is defined by the degree d_i: the number of non-zero elements where, unless otherwise stated, the positions of d_i non-zero values are selected uniformly at random among the elements of $\mathbf{x}$. Overall, the measurement process can be described as $\mathbf{y} = \mathbf{\Phi} \cdot \mathbf{x}^T$, where the measurement matrix $\mathbf{\Phi}$ is a sparse $M \times N$ matrix.

The measurements y_i are sequentially communicated to the reconstruction subsystem via a communication channel. We consider an ideal noise-free channel that conveys real numbers without errors/erasures, leaving considerations of imperfect channels and quantization for future work.

For the reconstruction subsystem, we consider iterative verification-based LM1 and LM2 algorithms [2, 4]. Both algorithms exchange messages across a measurement graph, which is a bipartite graph consisting of N coefficient nodes that correspond to signal coefficients $\mathbf{x}$ and M measurement nodes that correspond to a sequence of measurements $\mathbf{y}$. Edges of the graph connect each measurement node y_i to its neighbor set $N(y_i)$ of coefficients determined by non-zero positions of the measurement vector ϕ_i. Thus the degree $d_i = |N(y_i)|$ is the number of edges incident to y_i. Edges in the graph are in one-to-one correspondence with non-zero entries of the measurement matrix $\mathbf{\Phi}$. If the non-zero entries of $\mathbf{\Phi}$ are arbitrary reals, then the graph is weighted and the edge weights correspond to real entries of $\mathbf{\Phi}$. For simplicity, we assume binary matrix $\mathbf{\Phi}$, thus weights are not needed. Below, we briefly describe LM1 and LM2 algorithms noting that we assume they operate progressively with the arrival of each new measurement until the signal is fully recovered (i.e., all coefficients are verified).

LM1: The LM1 operates iteratively over the measurement graph as follows: 1) If $y_i = 0$ then $\forall x_j \in N(y_i) : x_j = 0$; Verify all $x_j : x_j \in N(y_i)$. 2) If $(y_i \neq 0) \wedge (|N(y_i)| = 1)$ then $x_j = y_i$ for the node $x_j \in N(y_i)$; Verify x_j. 3) Remove verified coefficient nodes and their incident edges from the graph; Subtract out verified values from remaining measurements. 4) Repeat until the LM1 successfully recovers the signal or does not progress in two consecutive iterations.

LM2: Besides the above LM1 rules, LM2 adds the additional one: If $(N(y_i) \cap N(y_j) = \{x_k\}) \wedge (y_i = y_j)$ then $x_k = y_i = y_j$ and $\forall x_l \in \{N(y_i) \cup N(y_i) \setminus x_k\} : x_l = 0$; Verify all $x_l \in \{N(y_i) \cup N(y_i) \setminus x_k\}$.

Note that, in the above setup, if ϕ_i's are created non-adaptively using fixed degree $d_i = L$, we obtain the (first stage of the) Sudocodes scheme. Furthermore, if $\mathbf{x} \in \mathbb{F}_{2^q}^N$, $K = N$ (i.e., the signal is non-sparse), ϕ_i's are created independently (non-adaptively) using degree's d_i drawn from a given degree distribution $\Omega(x)$, the channel is 2^q-ary erasure channel, and we apply LM1 decoder, we obtain standard rateless (LT) coding scenario [13].

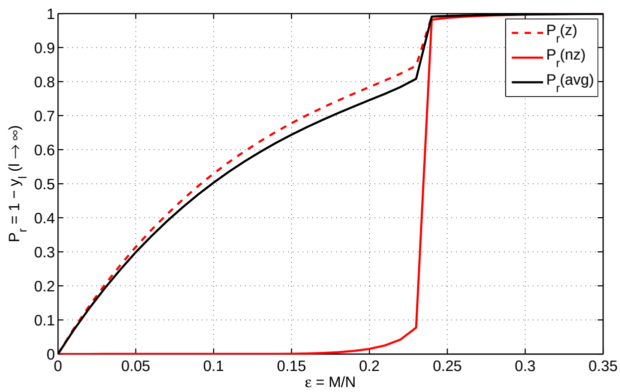

Figure 1: Asymptotic performance of Sudocodes scheme.

3 Asymptotic Analysis

Using sparse-graph coding methodology, the measurement graph may be described using coefficient and measurement node degree distributions $\Lambda(x) = \sum_{i=1}^{d_{max}^{(c)}} \Lambda_i \cdot x^i$ and $\Omega(x) = \sum_{i=1}^{d_{max}^{(m)}} \Omega_i \cdot x^i$, where Λ_i and Ω_i are the fraction of coefficient and measurement nodes of degree i, respectively, while $d_{max}^{(c)}$ and $d_{max}^{(m)}$ are maximum coefficient and measurement node degrees. It is also useful to define so called edge-oriented degree distributions $\lambda(x) = \sum_{i=1}^{d_{max}^{(c)}} \lambda_i \cdot x^{i-1} = \Lambda'(x)/\Lambda'(1)$ and $\omega(x) = \sum_{i=1}^{d_{max}^{(m)}} \omega_i \cdot x^{i-1} = \Omega'(x)/\Omega'(1)$ [14]. For a given measurement graph, one can (asymptotically) analyze probabilities of reconstruction of signal coefficients using well established tools from coding theory. For LM1 reconstruction algorithm, the iterative "graph-peeling" process is the same as the rateless decoding over erasure channels, with the only difference being signal sparsity and existence of zero-valued measurements. Below, we provide and-or-tree analysis of non-adaptive Sudocodes CS scheme in combination with LM1 recovery:

Lemma 3.1 *Let $p_l^{(z)}$ and $p_l^{(\bar{z})}$ denote the probabilities that a zero and non zero signal coefficient, respectively, is not recovered after l iterations of LM1 recovery. Then*

$$p_l^{(\bar{z})} = \lambda\Big(1 - \sum_{i=1}^{d_{max}^{(m)}} \omega_i \cdot \sum_{j=0}^{i-1} \binom{i-1}{j} (\alpha \bar{p}_{l-1}^{(\bar{z})})^j \cdot (\bar{\alpha} \bar{p}_{l-1}^{(z)})^{i-1-j}\Big) \tag{1}$$

$$p_l^{(z)} = \lambda\Big(1 - \sum_{i=1}^{d_{max}^{(m)}} \omega_i \cdot \sum_{j=0}^{i-1} \binom{i-1}{j} (\alpha \bar{p}_{l-1}^{(\bar{z})})^j \cdot \bar{\alpha}^{i-1-j}\Big),$$

where $\alpha = K/N$ is sparsity-factor, we use compact notation $\bar{x} = 1 - x$, and the recursion is initialized at $p_0^{(z)} = p_0^{(\bar{z})} = 1$. Finally, $p_l = \alpha p_l^{(\bar{z})} + \bar{\alpha} p_l^{(z)}$ is the average probability that a signal coefficient is unrecovered after l iterations of LM1.

Proof: The proof follows directly from and-or-tree analysis [14] by exhaustively analyzing all cases when zero and non-zero valued coefficient nodes are recovered*.

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*The above Lemma is exact and improves over an approximate version provided in [15](Lemma 2), which does not exhaustively cover all recovery scenarios for zero-valued nodes.

Example 3.1 *Fig. 1 shows asymptotic recovery probabilities (as $N \to \infty$) obtained from* Lemma 1 *for Sudocodes scheme that applies $\Omega(x) = x^{20}$ after LM1 recovery of input message of sparsity-factor $\alpha = 0.05$ for zero and non-zero input symbols and the average value.*

In CS, similarly as in rateless coding, the measurement subsystem can control only $\Omega(x)$, while $\Lambda(x)$ depends on how signal coefficients are sampled to participate in each measurement. If this selection is uniformly at random, as in the case of Sudocodes (as well as rateless codes), then $\Lambda(x)$ asymptotically tends to the Poisson distribution, and $\lambda(x) = e^{\mu\epsilon(x-1)}$, where $\mu = \Omega'(1)$ is the average measurement node degree, and $\epsilon = M/N$ is known as the reception overhead in rateless coding, while in CS, ϵ is the product of the oversampling ratio $\theta = M/K$ and the sparsity factor $\alpha = K/N$.

4 Adaptive Sparse Measurements

In the following, we propose usage of adaptive sparse measurements in Sudocodes scheme and analyze its performance in combination with the LM1 recovery. The adaptive version of Sudocodes relies on two simple modifications to the measurement subsystem:

Modification 1: If the measurement subsystem records a measurement $y_i = 0$, the signal coefficients $x_j \in N(y_i)$ are not considered in following measurements. Consequently, by eliminating known zeros, the size of the problem N decreases and the sparsity-factor $\alpha = K/N$ increases as the measurement process evolves.

Modification 2: As the sparsity-factor α changes, the measurement subsystem selects the optimal degree $d^* = d^*(\alpha)$ of each measurement row-vector ϕ_i that maximizes the expected number of zero-valued signal coefficients that will be recovered by the following measurement.

For a given sparsity-factor $\alpha = K/N$, the optimal degree d^* is obtained as follows. A measurement of degree d is a zero-measurement with probability $P_0 = \binom{N-K}{d}/\binom{N}{d}$. The expected number of zero-valued signal coefficients recovered by a measurement of degree d is $n_0 = P_0 \cdot d$. The optimal strategy selects $d = d^*$ that maximizes n_0:

$$d^* = \arg\max_d \{n_0\} = \arg\max_d \left\{ \frac{\binom{N-K}{d}}{\binom{N}{d}} \cdot d \right\}. \tag{2}$$

For a fixed and small α, using Stirling approximation† of binomial coefficients that holds asymptotically as $N \to \infty$, and taking the logarithm of the argument of maximization (which does not change the optimal d), we obtain:

$$\log(n_0) = d\log(\frac{N-K}{d} - \frac{1}{2}) - d\log(\frac{N}{d} - \frac{1}{2}) + \log d \tag{3}$$

$$= d\log(\frac{2N(1-\alpha)-1}{2N-1}) + \log d \tag{4}$$

$$\approx d\log(1-\alpha) + \log d. \tag{5}$$

Taking partial derivative of the above with respect to d and making it equal to zero, we obtain:

$$d^* \approx (\log\frac{1}{1-\alpha})^{-1}, \tag{6}$$

where the approximation‡ is asymptotically tight for small α and $N \to \infty$. The optimal degree d^* asymptotically depends only on the sparsity-factor α. In addition, d^* calculated using (6) closely matches the one calculated by (2) for finite N,K.

†$\log\binom{n}{k} \approx k \cdot \log(n/k - 0.5) + k - 0.5\log(2\pi k)$.

‡We note that the above result is derived in [16](Lemma 2) in order to optimize the (non-adaptive) Sudocodes scheme. We rederive it here using different approximation that we use in the sequel of the paper.

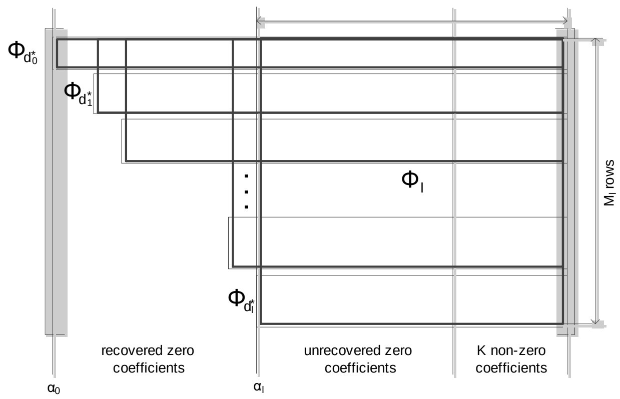

Figure 2: Evolution of $\mathbf{\Phi}$ during measurement process.

To analyze the proposed adaptive Sudocodes scheme, we follow the evolution of the of the measurement matrix $\mathbf{\Phi}$ over the measurement process. Starting from the initial sparsity-factor $\alpha_0 = K/N$, the first measurements are performed using the degree calculated from (6) and rounded to the nearest integer $d_0^* = \lfloor d^*(\alpha_0) \rceil$. As the measurement system removes known zero-coefficients during the measurement process, the process passes through the sequence of increasing α-values $\{\alpha_1, \alpha_2, \ldots\}$ at which the optimal d^*-values decrement, resulting in the corresponding set $\{d_1^*, d_2^*, \ldots\}$[§]. The total of m_i measurements are generated using the degree d_i^* during which the sparsity-factor is increased from α_i to α_{i+1}. The evolution of sparsity-factor $\alpha = K/N(m)$ with the number of measurements $m, m \geq 1$, can be described recursively:

$$N(m+1) = N(m) - n_0^* = N(m) - \frac{\binom{N(m)-K}{d^*}}{\binom{N}{d^*}} \cdot d^*. \tag{7}$$

Replacing (6) into (5), we obtain the approximation of the second term in (7), which, after dividing (7) by K results in recursive evolution of $\alpha(m)$:

$$\alpha^{-1}(m+1) = \alpha^{-1}(m) - \frac{1}{K} e^{-(1+\log(\log \frac{1}{1-\alpha(m)}))}. \tag{8}$$

Although m_i can be implicitly obtained from (8) by counting the number of recursions in the interval (α_i, α_{i+1}), it can be approximated by assuming that α_i remains constant during generation of degree-d_i^* measurements:

$$m_i = \frac{K(\alpha_{i+1} - \alpha_i)}{\alpha_i^2 e^{-(1+\log(\log \frac{1}{1-\alpha_i}))}}. \tag{9}$$

We approximate the evolution of $\mathbf{\Phi}$ during the measurement process as follows. Firstly, observe a sequence of matrices $\mathbf{\Phi}_{d_i^*}, i \geq 0$ of dimension $m_i \times N_i$, where $N_i = K/\alpha_i$, that represent groups of measurements of the same degree d_i^*. Then, we define the sequence of measurement matrices $\mathbf{\Phi}_i, i \geq 0$, where $\mathbf{\Phi}_l$ is obtained from the set of $\mathbf{\Phi}_{d_i^*}, 0 \leq i \leq l$ by removing the columns corresponding to all the zero-valued coefficients identified after the first $M_l = \sum_{i=0}^{l} m_i$ measurements. This is shown in Fig. 2, where the columns are sorted so that the rightmost K columns correspond to the non-zero signal coefficient. In addition, starting from the first column, the zero-valued coefficients are sorted in the order they are identified by a sequence of measurements.

[§]The α_i values can be obtained from (6) as $\alpha_i = 1 - e^{-1/d_i^{(th)}}$, where $d_i^{(th)} = \frac{d_{i-1}^* + d_i^*}{2}$.

The degree distributions of $\mathbf{\Phi}_i$ sequence are approximated as follows. In the non-adaptive case, $\Omega(x)$ tends to the Poisson distribution with the average value $\mu\epsilon = \mu M/N = M\mu/N = Mp$, where p is the probability that a coefficient node is selected by a measurement. For the adaptive case, p is a function of α:

$$p(\alpha) = \frac{d(\alpha)}{N(\alpha)} = \frac{\alpha}{K} \cdot \frac{1}{\log \frac{1}{1-\alpha}}. \tag{10}$$

However, by Taylor expansion of $\log(1+x)$ around $x \approx 0$ and neglecting higher-order terms, one obtains $\log(1+x) \approx x$, thus for large interval around $\alpha > 0$, p behaves as $p \approx 1/K$. Thus we approximate $\Omega(x)$ of the matrix $\mathbf{\Phi}_l$ as a Poisson distribution with the mean value $M_l p = M_l/K$. Regarding $\Lambda(x)$ distribution of $\mathbf{\Phi}_l$, it is a mixture of l hypergeometric distributions $\mathcal{H}(N_i, N_l, d_i^*), 1 \leq i \leq l$, weighted by their relative number of measurements m_i/M_l.

Finally, we have all ingredients to track the recovery probability of adaptive Sudocodes scheme by replacing parameters of the sequence of $\mathbf{\Phi}_l$'s into *Lemma* 3.1[¶]. Note from the equations in this section that the entire analysis depends only on initial and final sparsity-factors α_0 and α_l and the number of non-zero valued coefficients K.

5 Numerical and Simulation Results

We illustrate the performance of Sudocodes (SC) and adaptive-SC (aSC) scheme using asymptotic analysis following *Lemma* 3.1 and equations in Sec. 4. For SC scheme and the signal of sparsity factor $\alpha = 0.05$, the optimal degree $d^* = 20$. This is observable from Fig. 3 where we present recovery probability $P_r = 1 - p_{l\to\infty}$ for different values of $d = \{5, 10, 15, 20, 30\}$ as a function of the reception overhead ϵ. The SC performance curves hold asymptotically as $K, N \to \infty$. For finite $K = 50$ ($N = 1000$), in Fig. 3 we provide an example curve obtained by simulations (we note that simulated curves converge to the asymptotic one for large K).

For aSC scheme, we start from $\alpha_0 = 0.05$ and adaptively generate measurements until successful recovery. We present the performance for both asymptotic and simulated case ($K = 50$). Firstly, note that for the simulated case, aSC slightly outperforms SC (we obtain similar performance advantage of aSC over SC for larger values of N). This comes in combination with significantly sparser measurement matrix since, during the measurement process, the initial $d^* = 20$ will gradually drop to as low as $d^* = 1$ (if the recovery process is not terminated earlier). Asymptotic analysis predicts well the simulated curve, however, at large ϵ-values, the performance prediction of the error-floor is conservative due to the approximation of (10) using $p = 1/K$ that holds for small α.

Fig. 3 also shows SC scheme for $d = 30 > d^*$. In this case, the performance deteriorates for intermediate ϵ-values and reaches P_r close to one for larger ϵ compared to the d^* case, however, the larger d values exhibit lower error-floors[‖]. In rateless coding, the error-floor is removed by precoding [17] (which could be also applied in CS by taking a set of intermediate "pre-measurements"), whereas in Sudocodes CS, it is cleared by postprocessing in the second phase that applies small fraction of dense measurements and stronger recovery algorithms [1] [16] [18].

[¶]Note that there are no zero-measurements in $\mathbf{\Phi}_l$. Thus, in eq. (2) of *Lemma* 3.1, we exclude the case $j = 0$ corresponding to a zero-measurement.

[‖]The error-floor follows from non-zero probability that a signal coefficient does not participate in any of the generated measurements [17].

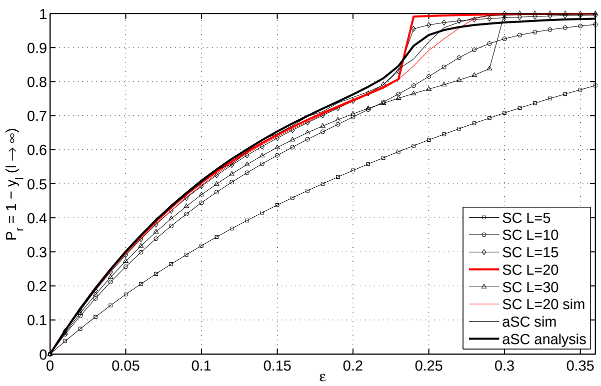

Figure 3: Sudocodes (SC) and adaptive-SC (aSC) performance.

6 Conclusions

We introduced and analyzed adaptive Sudocodes scheme. For LM1 decoder, the adaptation is shown to be simple yet resulting in slightly improved performance and reduced encoding/recovery complexity. For the future work, we will explore adaptation in Sudocodes scheme with LM2 decoder which increases learning opportunities at the measurement subsystem, however, under more complex analysis of LM2 decoding. We will also further investigate connections between rateless coding and CS and how the two concepts could converge in efficient joint source channel coding solutions (as indicated by our system model in this work.)

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