GHENT UNIVERSITY

FACULTY OF ENGINEERING AND ARCHITECTURE

Image Reconstruction Tutorial

Part 1: Sparse optimization and learning approaches

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Turning images into value through statistical parameter estimation

Ghent, Belgium, 20 September 2019

Outline

1. Model-based iterative reconstruction algorithms
   - Sparse optimization
   - Solution strategies: greedy methods vs. convex optimization
   - Optimization methods in sparse image reconstruction
2. Structured sparsity
   - Wavelet-tree sparsity
   - Markov Random Field (MRF) priors
3. Machine learning in image reconstruction
   - Main ideas and current trends

A fairly general formulation

Reconstruct a signal (image) $\mathbf{x} \in X$ from data $\mathbf{y} \in Y$ where

$$\mathbf{y} = \mathcal{T}(\mathbf{x}) + \mathbf{n}$$

X and Y are Hilbert spaces, $\mathcal{T} : X \mapsto Y$ is the forward operator and $\mathbf{n}$ is noise.
A common model-driven approach is to minimize the negative log-likelihood $\mathcal{L}$:

$$\min_{\mathbf{x} \in X} \mathcal{L}(\mathcal{T}(\mathbf{x}), \mathbf{y})$$

Typically, ill-posed and leads to over-fitting. Variational regularization, also called model-based iterative reconstruction seeks to minimize a regularized objective function

$$\min_{\mathbf{x} \in X} \mathcal{L}(\mathcal{T}(\mathbf{x}), \mathbf{y}) + \tau \phi(\mathbf{x})$$

$\phi : X \mapsto \mathbb{R} \cup \{-\infty, \infty\}$ is a regularization functional. $\tau \geq 0$ governs the influence of the a priori knowledge against the need to fit the data.

Linear inverse problems

Many image reconstruction problems can be formulated as a linear inverse problem. A noisy indirect observation $\mathbf{y}$, of the original image $\mathbf{x}$ is then

$$\mathbf{y} = \mathbf{A}\mathbf{x} + \mathbf{n}$$

Matrix $\mathbf{A}$ is the forward operator. $\mathbf{x} \in \mathbb{R}^n$; $\mathbf{y}, \mathbf{n} \in \mathbb{R}^m$ (or $\mathbf{x} \in \mathbb{C}^n$; $\mathbf{y}, \mathbf{n} \in \mathbb{C}^m$). Here, image pixels are stacked into vectors (raster scanning). In general, $m \neq n$.

Some examples

- CT: $\mathbf{A}$ is the system matrix modeling the X-ray transformation
- MRI: $\mathbf{A}$ is (partially sampled) Fourier operator
- OCT: $\mathbf{A}$ is the first Born approximation for the scattering
- Compressed sensing: $\mathbf{A}$ is a measurement matrix (dense or sparse)

Linear inverse problems

For the linear inverse problem $\mathbf{y} = \mathbf{A}\mathbf{x} + \mathbf{n}$, model-based reconstruction seeks to solve:

$$\min_{\mathbf{x}} \frac{1}{2}\|\mathbf{A}\mathbf{x} - \mathbf{y}\|_2^2 + \tau\phi(\mathbf{x}) \qquad \text{(\textbf{Tikhonov} formulation)}$$

Alternatively,

$$\min_{\mathbf{x}} \phi(\mathbf{x}) \quad \text{subject to} \quad \|\mathbf{A}\mathbf{x} - \mathbf{y}\|_2^2 \leq \epsilon \qquad \text{(\textbf{Morozov} formulation)}$$

$$\min_{\mathbf{x}} \|\mathbf{A}\mathbf{x} - \mathbf{y}\|_2^2 \quad \text{subject to} \quad \phi(\mathbf{x}) \leq \delta \qquad \text{(\textbf{Ivanov} formulation)}$$

Under mild conditions, these are all **equivalent** [Figueiredo and Wright, 2013], and which one is more convenient is problem-dependent.

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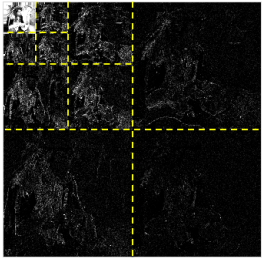
Sparse optimization

- A common assumption: $\mathbf{x}$ is sparse in a well-chosen transform domain.
- We refer to a **wavelet** representation meaning any wavelet-like multiscale representation, including curvelets and shearlets..



$$\mathbf{x} = \boldsymbol{\Psi}\boldsymbol{\theta},\ \boldsymbol{\theta} \in \mathbb{R}^d,\ \boldsymbol{\Psi} \in \mathbb{R}^{n \times d}$$

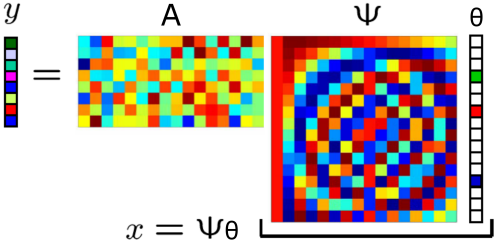
The columns of $\boldsymbol{\Psi}$ are the elements of a **wavelet frame** (an orthogonal basis or an overcomplete dictionary)

- The main results hold for **learned dictionaries**, trained on a set of representative examples to yield optimally sparse representation for a particular class of images.

Compressed sensing

Consider $\mathbf{y} = \mathbf{A}\mathbf{x} + \mathbf{n}$, with $\mathbf{y}, \mathbf{n} \in \mathbb{R}^m$, $\mathbf{x} = \mathbf{\Psi}\boldsymbol{\theta}$, $\mathbf{x} \in \mathbb{R}^n$, $\boldsymbol{\theta} \in \mathbb{R}^d$, $m < n$

$$\hat{\boldsymbol{\theta}} = \arg\min_{\boldsymbol{\theta}} \frac{1}{2}\|\mathbf{A}\mathbf{\Psi}\boldsymbol{\theta} - \mathbf{y}\|_2^2 + \tau\phi(\boldsymbol{\theta}), \qquad \hat{\mathbf{x}} = \mathbf{\Psi}\hat{\boldsymbol{\theta}}$$



Commonly: $\min_{\boldsymbol{\theta}} \|\boldsymbol{\theta}\|_0$ s.t. $\|\mathbf{A}\mathbf{\Psi}\boldsymbol{\theta} - \mathbf{y}\|_2^2 \leq \epsilon$ or $\min_{\boldsymbol{\theta}} \frac{1}{2}\|\mathbf{A}\mathbf{\Psi}\boldsymbol{\theta} - \mathbf{y}\|_2^2 + \tau\|\boldsymbol{\theta}\|_1$

[Candès et al., 2006], [Donoho, 2006], [Lustig et al., 2007]

Compressed sensing: recovery guarantees

Consider $\mathbf{y} = \mathbf{A}\mathbf{x} + \mathbf{n}$, with $\mathbf{y}, \mathbf{n} \in \mathbb{R}^m$, $\mathbf{x} = \mathbf{\Psi}\theta$, $\mathbf{x} \in \mathbb{R}^n$, $m < n$

Matrix $\mathbf{\Phi} = \mathbf{A}\mathbf{\Psi}$ has K-restricted isometry property (**K-RIP**) with constant $\epsilon_K < 1$ if $\forall$ K-sparse (having only K non-zero entries) θ:

$$(1 - \epsilon_K)\|\theta\|_2^2 \leq \|\mathbf{\Phi}\theta\|_2^2 \leq (1 + \epsilon_K)\|\theta\|_2^2$$

Suppose matrix $\mathbf{A} \in \mathbb{R}^{m \times n}$ is formed by subsampling a given sampling operator $\bar{\mathbf{A}} \in \mathbb{R}^{n \times n}$. The **mutual coherence** between $\bar{\mathbf{A}}$ and $\mathbf{\Psi}$:

$$\mu(\bar{\mathbf{A}}, \mathbf{\Psi}) = \max_{i,j} |a_i^\top \psi_j|$$

If $m > C\mu^2(\bar{\mathbf{A}}, \mathbf{\Psi})Kn\log(n)$, for some constant $C > 0$, then

$$\min_{\theta} \frac{1}{2}\|\mathbf{A}\mathbf{\Psi}\theta - \mathbf{y}\|_2^2 + \tau\|\theta\|_1$$

recovers $\mathbf{x}$ with high probability, given the K-RIP holds for $\mathbf{\Phi} = \mathbf{A}\mathbf{\Psi}$.

Analysis vs. synthesis formulation

Consider $\mathbf{y} = \mathbf{A}\mathbf{x} + \mathbf{n}$, with $\mathbf{y} \in \mathbb{R}^m$, $\mathbf{x} = \boldsymbol{\Psi}\boldsymbol{\theta}$, $\mathbf{x} \in \mathbb{R}^n$, $\boldsymbol{\theta} \in \mathbb{R}^d$

Synthesis approach:

$$\min_{\boldsymbol{\theta}} \frac{1}{2}\|\mathbf{A}\boldsymbol{\Psi}\boldsymbol{\theta} - \mathbf{y}\|_2^2 + \tau\phi(\boldsymbol{\theta})$$

Analysis approach:

$$\min_{\mathbf{x}} \frac{1}{2}\|\mathbf{A}\mathbf{x} - \mathbf{y}\|_2^2 + \tau\phi(\mathbf{x})$$

Analysis vs. synthesis formulation

Consider $\mathbf{y} = \mathbf{A}\mathbf{x} + \mathbf{n}$, with $\mathbf{y} \in \mathbb{R}^m$, $\mathbf{x} = \boldsymbol{\Psi}\boldsymbol{\theta}$, $\mathbf{x} \in \mathbb{R}^n$, $\boldsymbol{\theta} \in \mathbb{R}^d$

Synthesis approach:

$$\min_{\boldsymbol{\theta}} \frac{1}{2}\|\mathbf{A}\boldsymbol{\Psi}\boldsymbol{\theta} - \mathbf{y}\|_2^2 + \tau\phi(\boldsymbol{\theta})$$

or in a constrained form:

$$\min_{\boldsymbol{\theta}} \phi(\boldsymbol{\theta}) \quad \text{subject to} \quad \|\mathbf{A}\boldsymbol{\Psi}\boldsymbol{\theta} - y\|_2^2 \leq \epsilon$$

Analysis approach:

$$\min_{\mathbf{x}} \frac{1}{2}\|\mathbf{A}\mathbf{x} - \mathbf{y}\|_2^2 + \tau\phi(\mathbf{x})$$

or in a constrained form:

$$\min_{\mathbf{x}} \phi(\mathbf{x}) \quad \text{subject to} \quad \|\mathbf{A}\mathbf{x} - \mathbf{y}\|_2^2 \leq \epsilon$$

Analysis vs. synthesis formulation

Consider $\mathbf{y} = \mathbf{A}\mathbf{x} + \mathbf{n}$, with $\mathbf{y} \in \mathbb{R}^m$, $\mathbf{x} = \boldsymbol{\Psi}\boldsymbol{\theta}$, $\mathbf{x} \in \mathbb{R}^n$, $\boldsymbol{\theta} \in \mathbb{R}^d$

Synthesis approach:

$$\min_{\boldsymbol{\theta}} \frac{1}{2}\|\mathbf{A}\boldsymbol{\Psi}\boldsymbol{\theta} - \mathbf{y}\|_2^2 + \tau\phi(\boldsymbol{\theta})$$

or in a constrained form:

$$\min_{\boldsymbol{\theta}} \phi(\boldsymbol{\theta}) \quad \text{subject to} \quad \|\mathbf{A}\boldsymbol{\Psi}\boldsymbol{\theta} - y\|_2^2 \leq \epsilon$$

Analysis approach that also applies to wavelet regularization

$$\min_{\mathbf{x}} \frac{1}{2}\|\mathbf{A}\mathbf{x} - \mathbf{y}\|_2^2 + \tau\phi(\mathbf{P}\mathbf{x})$$

or in a constrained form:

$$\min_{\mathbf{x}} \phi(\mathbf{P}\mathbf{x}) \quad \text{subject to} \quad \|\mathbf{A}\mathbf{x} - \mathbf{y}\|_2^2 \leq \epsilon$$

Analysis vs. synthesis formulation

Consider $\mathbf{y} = \mathbf{A}\mathbf{x} + \mathbf{n}$, with $\mathbf{y} \in \mathbb{R}^m$, $\mathbf{x} = \boldsymbol{\Psi}\boldsymbol{\theta}$, $\mathbf{x} \in \mathbb{R}^n$, $\boldsymbol{\theta} \in \mathbb{R}^d$

Synthesis approach:

$$\min_{\boldsymbol{\theta}} \frac{1}{2}\|\mathbf{A}\boldsymbol{\Psi}\boldsymbol{\theta} - \mathbf{y}\|_2^2 + \tau\phi(\boldsymbol{\theta})$$

or in a constrained form:

$$\min_{\boldsymbol{\theta}} \phi(\boldsymbol{\theta}) \quad \text{subject to} \quad \|\mathbf{A}\boldsymbol{\Psi}\boldsymbol{\theta} - y\|_2^2 \leq \epsilon$$

Analysis approach that also applies to wavelet regularization

$$\min_{\mathbf{x}} \frac{1}{2}\|\mathbf{A}\mathbf{x} - \mathbf{y}\|_2^2 + \tau\phi(\mathbf{P}\mathbf{x})$$

or in a constrained form:

$$\min_{\mathbf{x}} \phi(\mathbf{P}\mathbf{x}) \quad \text{subject to} \quad \|\mathbf{A}\mathbf{x} - \mathbf{y}\|_2^2 \leq \epsilon$$

$\mathbf{P}$: a wavelet transform operator or $\mathbf{P} = \mathbf{I}$ (standard analysis)

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Solution strategies: greedy methods vs. convex optimization

Solution strategy is problem-dependent. For non-convex problems like

$$\min_{\mathbf{x}} \|\mathbf{x}\|_0 \quad \text{subject to} \quad \|\mathbf{A}\mathbf{x} - \mathbf{y}\|_2^2 \leq \epsilon$$

Greedy algorithms, e.g.,

- **Matching Pursuit** (**MP**) [Mallat and Zhang, 1993]
- **OMP**[Tropp, 2004], **CoSaMP** [Needell and Tropp, 2009]
- **IHT** [Blumensath and Davies, 2009]

or **convex relaxation** can be applied leading to:

$$\min_{\mathbf{x}} \|\mathbf{x}\|_1 \quad \text{subject to} \quad \|\mathbf{A}\mathbf{x} - \mathbf{y}\|_2^2 \leq \epsilon$$

$$\min_{\mathbf{x}} \frac{1}{2}\|\mathbf{A}\mathbf{x} - \mathbf{y}\|_2^2 + \tau\phi(\mathbf{x})$$

known as **LASSO** [Tibshirani, 1996] or **BPDN** [Chen et al., 2001] problem.

Greedy methods: OMP

OMP algorithm for solving $\min_{\mathbf{x}} \|\mathbf{x}\|_0 \;\; \text{subject to} \;\; \mathbf{A}\mathbf{x} = \mathbf{y}$

Require: $k = 1, \mathbf{r}^{(1)} = \mathbf{y}, \Lambda^{(0)} = \emptyset$

1: **repeat**
2: $\quad \lambda^{(k)} = \arg\max_j |\mathbf{A}_j \cdot \mathbf{r}^{(k)}|$
3: $\quad \Lambda^{(k)} = \Lambda^{(k-1)} \cup \{\lambda^{(k)}\}$
4: $\quad \mathbf{x}^{(k)} = \arg\min_{\mathbf{x}} \|\mathbf{A}_{\Lambda_k}\mathbf{x} - \mathbf{y}\|_2$
5: $\quad \hat{\mathbf{y}}^{(k)} = \mathbf{A}_{\Lambda_k}\mathbf{x}^{(k)}$
6: $\quad \mathbf{r}^{(k+1)} = \mathbf{r}^{(k)} - \hat{\mathbf{y}}^{(k)}$
7: $\quad k = k + 1$
8: **until** stopping criterion satisfied

$\mathbf{A}_j$ is the j-th column of $\mathbf{A}$, and $\mathbf{A}_\Lambda$ a sub-matrix of $\mathbf{A}$ with columns indicated in Λ.

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Proximal operator

Many state-of-the-art image reconstruction algorithms solve problems of the kind

$$\min_{\mathbf{x}} \frac{1}{2}\|\mathbf{A}\mathbf{x} - \mathbf{y}\|_2^2 + \tau\phi(\mathbf{P}\mathbf{x})$$

making use of the proximity operator i.e., the **Moreau proximal mapping** [Combettes and Wajs, 2005]

$$\mathrm{prox}_{\tau\phi}(\mathbf{y}) = \underset{\mathbf{x}}{\mathrm{argmin}} \frac{1}{2}\|\mathbf{x} - \mathbf{y}\|_2^2 + \tau\phi(\mathbf{x})$$

For certain choices of $\phi(\mathbf{x})$, this operator has a closed-form, e.g.,

- $\phi(\mathbf{x}) = \|\mathbf{x}\|_1 \rightarrow \mathrm{prox}_{\tau\ell_1}(\mathbf{y}) = \mathrm{soft}(\mathbf{y}, \tau)$ **component-wise** soft thresholding
- $\phi(\mathbf{x}) = \|\mathbf{x}\|_0 \rightarrow \mathrm{prox}_{\tau\ell_0}(\mathbf{y}) = \mathrm{hard}(\mathbf{y}, \sqrt{2\tau})$ **component-wise** hard thresholding

Another common regularization function is total variation (TV):

- $\phi(\mathbf{x}) = \|\mathbf{x}\|_{TV} \rightarrow \mathrm{prox}_{\tau TV}(\mathbf{y})$ **Chambolle's algorithm** [Chambolle, 2004]

Iterative shrinkage/thresholding (IST)

The standard algorithm for solving

$$\min_{\mathbf{x}} \frac{1}{2}\|\mathbf{A}\mathbf{x}-\mathbf{y}\|_2^2 + \tau\phi(\mathbf{x})$$

is **iterative shrinkage/thresholding (IST)** algorithm [Figueiredo and Nowak, 2003], [Daubechies et al., 2004]:

$$\mathbf{x}^{k+1} = \mathrm{prox}_{\tau\phi}\Big(\mathbf{x}^k - \frac{1}{\gamma}\underbrace{\mathbf{A}^H(\mathbf{A}\mathbf{x}^k-\mathbf{y})}_{\substack{\text{gradient of the data}\\ \text{fidelity term}}}\Big)$$

Its key ingredient is the **proximity operator** $\mathrm{prox}_{\tau\phi}(\mathbf{y}) = \underset{\mathbf{x}}{\mathrm{argmin}}\, \frac{1}{2}\|\mathbf{x}-\mathbf{y}\|_2^2 + \tau\phi(\mathbf{x})$

A general approach in [Daubechies et al., 2004]: $\phi(\mathbf{x})$ is a weighted ℓ_p norm of the coefficients of $\mathbf{x}$ with respect to a wavelet basis.

Iterative shrinkage/thresholding (IST) and extensions

The standard algorithm for solving

$$\min_{\mathbf{x}} \frac{1}{2}\|\mathbf{A}\mathbf{x} - \mathbf{y}\|_2^2 + \tau\phi(\mathbf{x})$$

is **iterative shrinkage/thresholding (IST)** algorithm [Figueiredo and Nowak, 2003], [Daubechies et al., 2004]:

$$\mathbf{x}^{k+1} = \mathrm{prox}_{\tau\phi}\Big(\mathbf{x}^k - \frac{1}{\gamma}\underbrace{\mathbf{A}^H(\mathbf{A}\mathbf{x}^k - \mathbf{y})}_{\substack{\text{gradient of the data}\\ \text{fidelity term}}}\Big)$$

Different accelerated versions:

- **TwIST** [Bioucas-Dias and Figueiredo, 2007]
- **FISTA** [Beck and Teboulle, 2009]
- **SpaRSA** [Wright et al., 2009]

Variable splitting

A very old idea (back to at least [Courant, 1943]): Represent $\min_{\mathbf{x}} f_1(\mathbf{x}) + f_2(\mathbf{G}\mathbf{x})$ as

$$\min_{\mathbf{x}} f_1(\mathbf{x}) + f_2(\mathbf{z}) \quad \text{subject to} \quad \mathbf{G}\mathbf{x} = \mathbf{z}$$

The rationale: it may be easier to solve the constrained problem.

Variable splitting (**VS**) together with the augmented Lagrangian method (**ALM**) and non linear block Gauss-Seidel (**NLBGS**) leads to a form of Alternating Direction Method of Multipliers (**ADMM**). It is this interpretation:

$$(\mathbf{VS} + \mathbf{ALM} + \mathbf{NLBGS}) \rightarrow \mathbf{ADMM}$$

that we give in the next few slides, following [Afonso et al., 2010]

Variable splitting and Augmented Lagrangian Method

A very old idea (back to at least [Courant, 1943]): Represent $\min_{\mathrm{x}} f_1(\mathbf{x}) + f_2(\mathbf{G}\mathbf{x})$ as

$$\min_{\mathrm{x}} f_1(\mathbf{x}) + f_2(\mathbf{z}) \quad \text{subject to} \quad \mathbf{G}\mathbf{x} = \mathbf{z}$$

$$\mathrm{L}_\mu(\mathbf{x}, \mathbf{z}, \boldsymbol{\lambda}) = \underbrace{f_1(\mathbf{x}) + f_2(\mathbf{z}) + \boldsymbol{\lambda}^T(\mathbf{G}\mathbf{x} - \mathbf{z})}_{\text{Lagrangian}} + \underbrace{\frac{\mu}{2}\|\mathbf{G}\mathbf{x} - \mathbf{z}\|_2^2}_{\text{“augmentation”}}$$

Basic augmented Lagrangian method (**ALM**), a.k.a., method of multipliers (**MM**),:

$$\begin{aligned}(\mathbf{x}^{(k)}, \mathbf{z}^{(k)}) &= \operatorname*{argmin}_{(\mathrm{x},\mathrm{z})} \mathrm{L}_\mu(\mathbf{x}, \mathbf{z}, \boldsymbol{\lambda}^{(k-1)}) \\ \boldsymbol{\lambda}^{(k)} &= \boldsymbol{\lambda}^{(k-1)} + \mu(\mathbf{G}\mathbf{x}^{(k)} - \mathbf{z}^{(k)})\end{aligned}$$

Goes back to at least [Hestenes, 1969], [Powell, 1969]

Variable splitting and Augmented Lagrangian Method

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$$\min_{\mathbf{x}} f_1(\mathbf{x}) + f_2(\mathbf{z}) \quad \text{subject to} \quad \mathbf{G}\mathbf{x} = \mathbf{z}$$

$$\mathrm{L}_\mu(\mathbf{x}, \mathbf{z}, \boldsymbol{\lambda}) = \underbrace{f_1(\mathbf{x}) + f_2(\mathbf{z}) + \boldsymbol{\lambda}^T(\mathbf{G}\mathbf{x} - \mathbf{z})}_{\text{Lagrangian}} + \underbrace{\frac{\mu}{2}\|\mathbf{G}\mathbf{x} - \mathbf{z}\|_2^2}_{\text{“augmentation”}}$$

After simple “complete-the-squares” **ALM**/**MM** yields [Afonso et al., 2010]:

$$\begin{aligned}
(\mathbf{x}^{(k)}, \mathbf{z}^{(k)}) &= \underset{(\mathbf{x},\mathbf{z})}{\operatorname{argmin}}\, f_1(\mathbf{x}) + f_2(\mathbf{z}) + \frac{\mu}{2}\|\mathbf{G}\mathbf{x} - \mathbf{z} - \mathbf{d}^{(k-1)}\|_2^2 \\
\mathbf{d}^{(k)} &= \mathbf{d}^{(k-1)} - (\mathbf{G}\mathbf{x}^{(k)} - \mathbf{z}^{(k)})
\end{aligned}$$

ADMM as Variable splitting and ALM

Use variable splitting (**VS**) to represent $\min_{\mathbf{x}} f_1(\mathbf{x}) + f_2(\mathbf{G}\mathbf{x})$ as

$$\min_{\mathbf{x}} f_1(\mathbf{x}) + f_2(\mathbf{z}) \quad \text{subject to} \quad \mathbf{G}\mathbf{x} = \mathbf{z}$$

ALM/**MM** yields :

$$\begin{aligned}
(\mathbf{x}^{(k)}, \mathbf{z}^{(k)}) &= \underset{(\mathbf{x},\mathbf{z})}{\operatorname{argmin}}\, f_1(\mathbf{x}) + f_2(\mathbf{z}) + \frac{\mu}{2}\|\mathbf{G}\mathbf{x} - \mathbf{z} - \mathbf{d}^{(k-1)}\|_2^2 \qquad (P) \\
\mathbf{d}^{(k)} &= \mathbf{d}^{(k-1)} - (\mathbf{G}\mathbf{x}^{(k)} - \mathbf{z}^{(k)})
\end{aligned}$$

Solve (P) with one step of **NLBGS** $\rightarrow$ "scaled" **ADMM** version [Boyd et al., 2011]:

$$\begin{aligned}
\mathbf{x}^{(k)} &= \underset{\mathbf{x}}{\operatorname{argmin}}\, f_1(\mathbf{x}) + \frac{\mu}{2}\|\mathbf{G}\mathbf{x} - \mathbf{z}^{(k-1)} - \mathbf{d}^{(k-1)}\|_2^2 \\
\mathbf{z}^{(k)} &= \underset{\mathbf{z}}{\operatorname{argmin}}\, f_2(\mathbf{z}) + \frac{\mu}{2}\|\mathbf{G}\mathbf{x}^{(k-1)} - \mathbf{z} - \mathbf{d}^{(k-1)}\|_2^2 \\
\mathbf{d}^{(k)} &= \mathbf{d}^{(k-1)} - (\mathbf{G}\mathbf{x}^{(k)} - \mathbf{z}^{(k)})
\end{aligned}$$

ADMM algorithm

ADMM algorithm for solving: $\min_{\mathbf{x}} f_1(\mathbf{x}) + f_2(\mathbf{G}\mathbf{x})$

Require: $k = 0, \mu > 0, \mathbf{z}^{\{0\}}, \mathbf{d}^{\{0\}}$

1: **repeat**

2: $\quad \mathbf{x}^{(k)} = \underset{\mathbf{x}}{\operatorname{argmin}}\, f_1(\mathbf{x}) + \frac{\mu}{2}\|\mathbf{G}\mathbf{x} - \mathbf{z}^{(k-1)} - \mathbf{d}^{(k-1)}\|_2^2$

3: $\quad \mathbf{z}^{(k)} = \underset{\mathbf{z}}{\operatorname{argmin}}\, f_2(\mathbf{z}) + \frac{\mu}{2}\|\mathbf{G}\mathbf{x}^{(k-1)} - \mathbf{z} - \mathbf{d}^{(k-1)}\|_2^2$

4: $\quad \mathbf{d}^{(k)} = \mathbf{d}^{(k-1)} - (\mathbf{G}\mathbf{x}^{(k)} - \mathbf{z}^{(k)})$

5: $\quad k = k + 1$

6: **until** stopping criterion is satisfied

Equivalent to **split-Bregman** method [Goldstein and Osher, 2009].
Connections with Douglas-Raschford splitting [Setzer, 2009].

ADMM algoritm for linear inverse problems

Instantiate ADMM to our linear inverse problem: $\min_{\mathbf{x}} \|\mathbf{A}\mathbf{x} - \mathbf{y}\|_2^2 + \tau\phi(\mathbf{P}\mathbf{x})$

Require: $k = 0, \mu > 0, \mathbf{z}^{\{0\}}, \mathbf{d}^{\{0\}}$

1: **repeat**

2: $\quad \mathbf{x}^{(k)} = \underset{\mathbf{x}}{\operatorname{argmin}} \|\mathbf{A}\mathbf{x} - \mathbf{y}\|_2^2 + \frac{\mu}{2}\|\mathbf{P}\mathbf{x} - \mathbf{z}^{(k-1)} - \mathbf{d}^{(k-1)}\|_2^2$

3: $\quad \mathbf{z}^{(k)} = \underset{\mathbf{z}}{\operatorname{argmin}}\, \tau\phi(\mathbf{z}) + \frac{\mu}{2}\|\mathbf{P}\mathbf{x}^{(k-1)} - \mathbf{z} - \mathbf{d}^{(k-1)}\|_2^2 = \operatorname{prox}_{\tau\phi/\mu}(\mathbf{P}\mathbf{x}^{(k-1)} - \mathbf{d}^{(k-1)})$

4: $\quad \mathbf{d}^{(k)} = \mathbf{d}^{(k-1)} - (\mathbf{P}\mathbf{x}^{(k)} - \mathbf{z}^{(k)})$

5: $\quad k = k + 1$

6: **until** stopping criterion is satisfied

A variant of ADMM algorithm for more than two functions

Consider $\min_{\mathbf{x}\in\mathbb{R}^n} \sum_{j=1}^{J} g_j(\mathbf{H}_j\mathbf{x})$ and map it into the previous: $\min_{\mathbf{x}} f_1(\mathbf{x}) + f_2(\underbrace{\mathbf{G}\mathbf{x}}_{\mathbf{z}})$

$$f_1(\mathbf{x}) = 0,\ f_2(\mathbf{z}) = \sum_{j=1}^{J} g_j(\mathbf{z}_j),\ \mathbf{G} = \begin{bmatrix}\mathbf{H}_1\\ \vdots \\ \mathbf{H}_J\end{bmatrix} \in \mathbb{R}^{p\times n},\ \mathbf{z}^{(k)} = \begin{bmatrix}\mathbf{z}_1^{(k)}\\ \vdots \\ \mathbf{z}_J^{(k)}\end{bmatrix},\ \mathbf{d}^{(k)} = \begin{bmatrix}\mathbf{d}_1^{(k)}\\ \vdots \\ \mathbf{d}_J^{(k)}\end{bmatrix},$$

$$\mathbf{x}^{(k)} = \left(\sum_{j=1}^{J}((\mathbf{H}_j)^\top\mathbf{H}_j\right)^{-1}\left(\sum_{j=1}^{J}(\mathbf{H}_j)^\top(\mathbf{z}_j^{(k-1)} + \mathbf{d}_j^{k-1})\right)$$

$$\mathbf{z}_1^{(k)} = \mathrm{prox}_{g_1\mu}(\mathbf{H}_1\mathbf{x}^{(k-1)} - \mathbf{d}_1^{(k-1)})$$

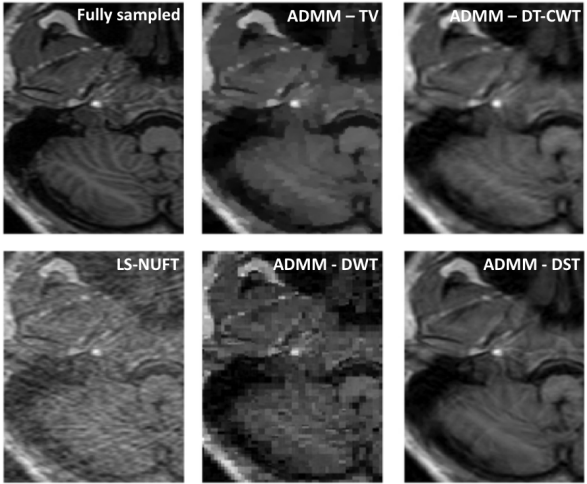
$$\vdots$$

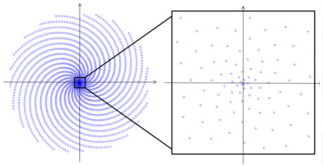
$$\mathbf{z}_J^{(k)} = \mathrm{prox}_{g_J\mu}(\mathbf{H}_J\mathbf{x}^{(k-1)} - \mathbf{d}_J^{(k-1)})$$

$$\mathbf{d}^{(k)} = \mathbf{d}^{(k-1)} - (\mathbf{G}\mathbf{x}^{(k)} - \mathbf{z}^{(k)})$$

C-SALSA [Afonso et al., 2011]

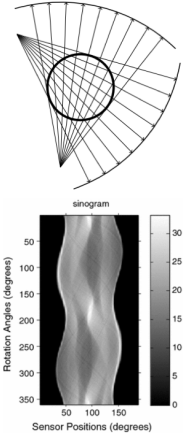
Example: MRI reconstruction with shearlet regularization

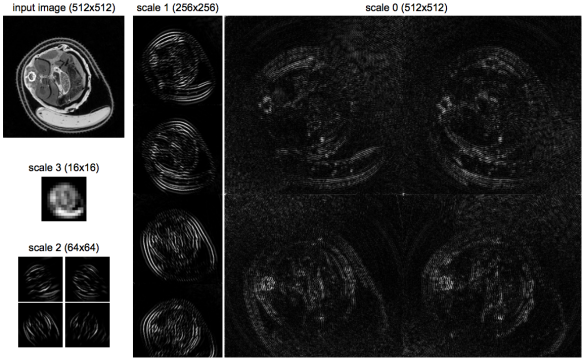




A transversal slice of a FLAIR sequence, resampled along a non-Cartesian trajectory based on an Archimedean spiral (sampling rate 15%). [Aelterman et al., 2011].

Example: CT reconstruction with shearlet regularization

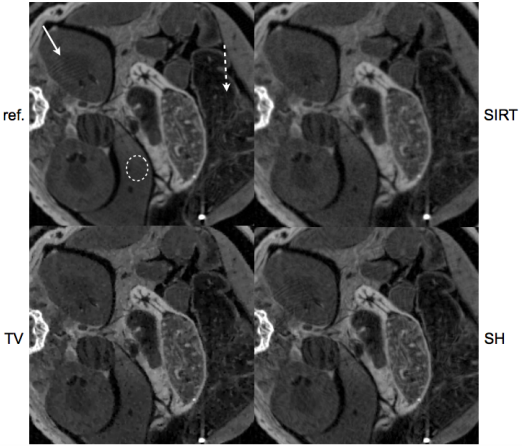




$$\hat{\mathbf{x}} = \arg\min_{\mathbf{x}} \frac{1}{2}\|\mathbf{C}^{-1}(\mathbf{A}\mathbf{x} - \mathbf{y})\|_2^2 + \tau\|\mathbf{P}\mathbf{x}\|_1$$

Matrix $\mathbf{C}$ is a "prewhitener" for the acquisition system [Vandeghinste et al., 2013].

Example: CT reconstruction with shearlet regularization



Top left: reference; **Top right:** SIRT; **Bottom left:** ADMM with TV regularization; **Bottom right:** ADMM with shearlet regularization [Vandeghinste et al., 2013]

Modelling structured sparsity

Two main approaches to modelling structured sparsity in image reconstruction

- in the **acquisition** stage
- in the **reconstruction** stage

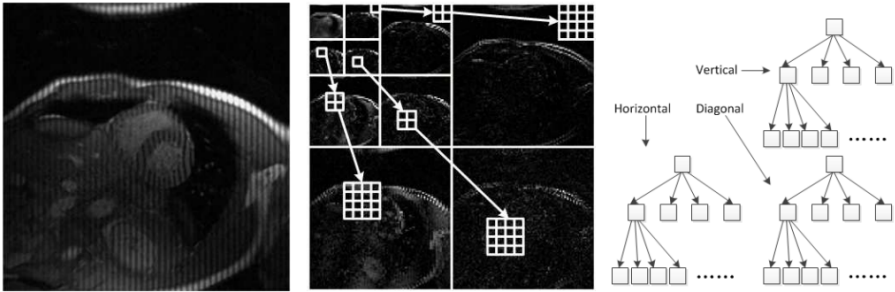
In the following we only focus on the second approach.

For the the improved design of the sampling patterns/sampling trajectories making use of the structured sparsity, see [Roman et al., 2015], [Adcock et al., 2017], [Gozcu, 2018]

Outline

1. Model-based iterative reconstruction algorithms
   - Sparse optimization
   - Solution strategies: greedy methods vs. convex optimization
   - Optimization methods in sparse image reconstruction
2. Structured sparsity
   - Wavelet-tree sparsity
   - Markov Random Field (MRF) priors
3. Machine learning in image reconstruction
   - Main ideas and current trends

Wavelet tree sparsity



[Jacob et al., 2009], [He and Carin, 2009], [Rao et al., 2011].

Application to MRI [Chen and Huang, 2014].

Outline

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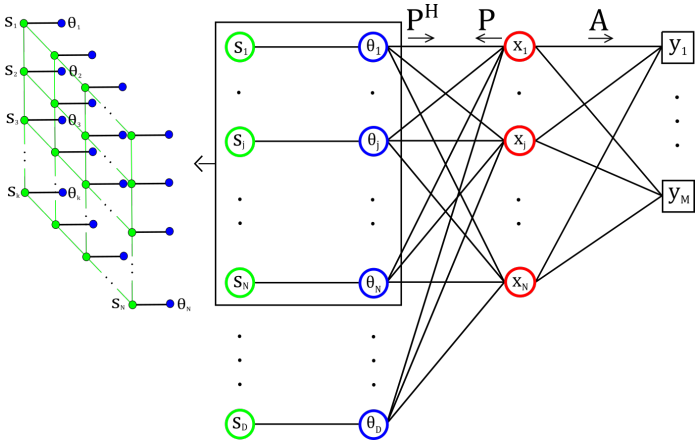
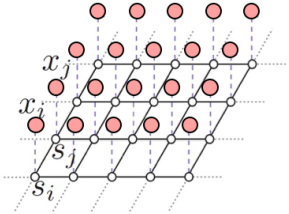
Sparse reconstruction with Markov Random Field priors

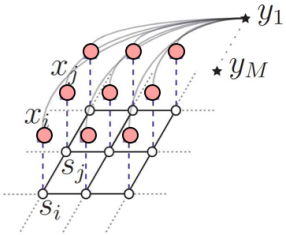


Use Markov Random Field (MRF) as a statistical model for the spatial clustering of important wavelet coefficients [Cevher et al., 2010], [Pižurica et al., 2011]

Sparse reconstruction with Markov Random Field priors

Consider (a little simpler): $\mathbf{y} = \mathbf{A}\mathbf{x} + \mathbf{n}$, with $\mathbf{y}, \mathbf{n} \in \mathbb{R}^m$, $\mathbf{x} \in \mathbb{R}^n$, $s_i \in \{0, 1\}$





$$P(\mathbf{s}) = \frac{1}{Z} \exp\left[-\left(\sum_i \alpha s_i + \sum_{\langle i,j \rangle} \beta s_i s_j\right)\right]$$

$$P(\mathbf{s}, \mathbf{x}, \mathbf{y}) = P(\mathbf{y}|\mathbf{x}) P(\mathbf{x}|\mathbf{s}) P(\mathbf{s})$$

$$[\hat{\mathbf{x}}, \hat{\mathbf{s}}] = \arg\max_{\mathbf{x},\mathbf{s}} \left\{ \sum_{\langle i,j \rangle} \beta s_i s_j + \sum_i [\alpha s_i + \log(p(\mathbf{x}_i|s_i)] - \frac{1}{2\sigma^2} \|\mathbf{y} - \mathbf{A}\mathbf{x}\|_2^2 \right\}$$

Sparse MRI reconstruction with MRF priors

Consider: $\mathbf{y} = \mathbf{A}\mathbf{x} + \mathbf{n}$, with $\mathbf{y}, \mathbf{n} \in \mathbb{C}^m$, $\mathbf{x} \in \mathbb{C}^n$, $\mathbf{x} = \mathbf{\Psi}\boldsymbol{\theta}$, $\boldsymbol{\theta} \in \mathbb{C}^d$, $s_i \in \{0, 1\}$

Let $\Omega_{\mathbf{s}} = \{i \in \mathcal{N} : s_i = 1\}$. Define a **model** for $\boldsymbol{\theta}$ that conforms to the support $\mathbf{s}$:

$$\mathcal{M}_{\mathbf{s}} = \{\boldsymbol{\theta} \in \mathbb{C}^D : \mathrm{supp}(\boldsymbol{\theta}) = \Omega_{\mathbf{s}}\}$$

Our objective is:

$$\min_{\mathbf{x} \in \mathbb{C}^N} \|\mathbf{A}\mathbf{x} - \mathbf{y}\|_2^2 \quad \text{subject to} \quad \mathbf{P}\mathbf{x} \in \mathcal{M}_{\hat{\mathbf{s}}}$$

or equivalently:

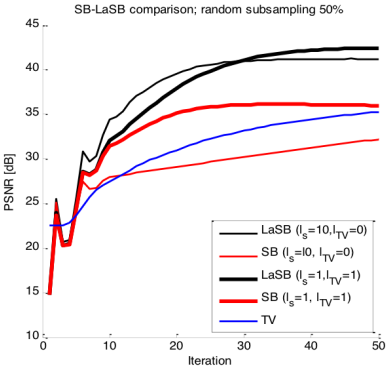
$$\min_{\mathbf{x} \in \mathbb{C}^N} \|\mathbf{A}\mathbf{x} - \mathbf{y}\|_2^2 \quad + \iota_{\Omega_{\mathbf{s}}}(\mathrm{supp}(\mathbf{P}\mathbf{x}))$$

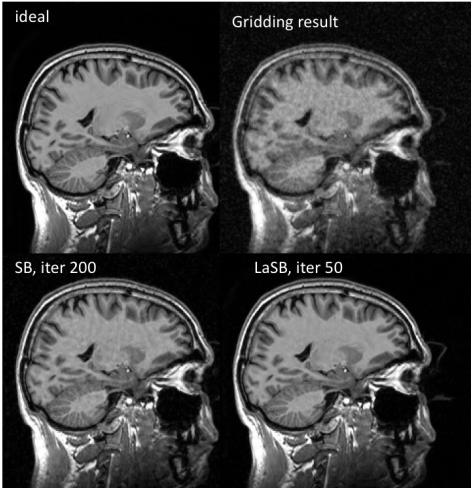
where

$$\iota_{\mathcal{Q}}(\mathbf{q}) = \begin{cases} 0, & \mathbf{q} \in \mathcal{Q} \\ +\infty, & \text{otherwise} \end{cases}$$

LaSB [Pižurica et al., 2011], **GreeLa** [Panić et al., 2016], **LaSAL** [Panić et al., 2017]

Example: CS-MRI with LaSB - early motivating results for using MRFs

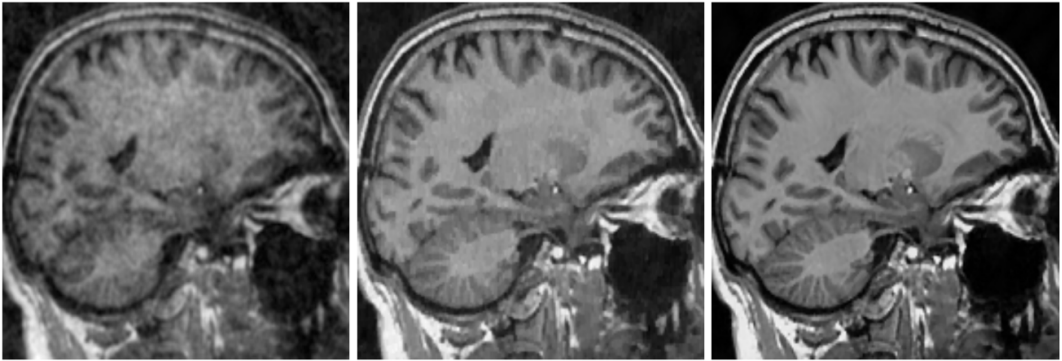




SB (split-Bregman) and **LaSB** implemented with the same shearlet transform.

Example: CS-MRI with MRF priors

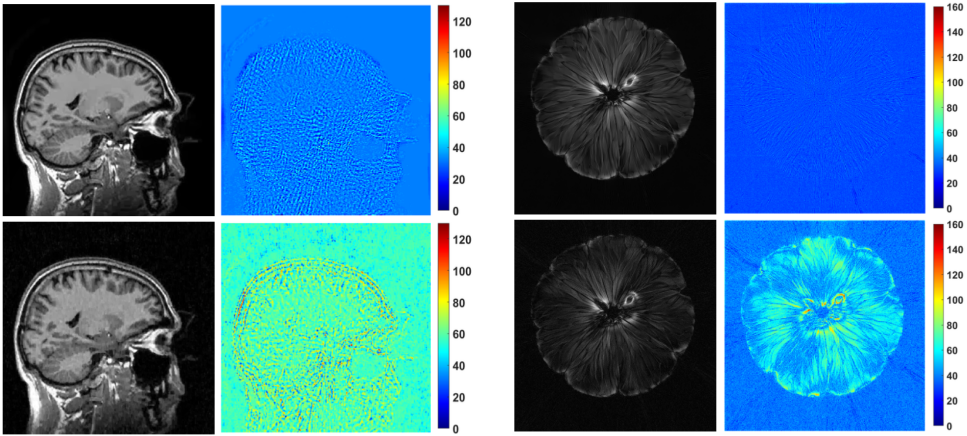
20% measurements, with variable density random sampling



Left: zero fill (PSNR = 19.87 dB)
Middle: **WaTMRI** [Chen and Huang, 2014] (wavelet-tree; PSNR = 28.78 dB)
Right: **LaSAL** [Panić et al., 2017] (MRF-based; PSNR = 33.43 dB)

Example: CS-MRI with MRF priors

Reconstructions from 20% measurements, with radial sampling



Left: reconstructions; **Right**: error images; **Top**: **LaSAL**, **Bottom**: **WaTMRI**

Outline

Machine learning in image reconstruction

Covered in many recent workshops, special sessions and special issues of journals:

IEEE TRANSACTIONS ON MEDICAL IMAGING, VOL. 37, NO. 6, JUNE 2018 1289

Image Reconstruction Is a New Frontier of Machine Learning

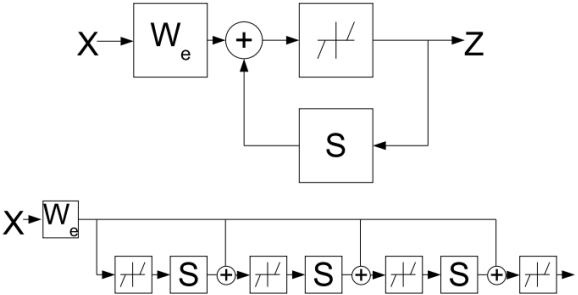
Ge Wang, *Fellow, IEEE*, Jong Chu Ye, *Senior Member, IEEE*, Klaus Mueller, *Senior Member, IEEE*, and Jeffrey A. Fessler, *Fellow, IEEE*

Three main direction have been proposed

- **learned postprocessing** or **learned denoisers**;
- **learn a regularizer** and use it in a classical variational regularization scheme;
- **learning the full reconstruction operator**

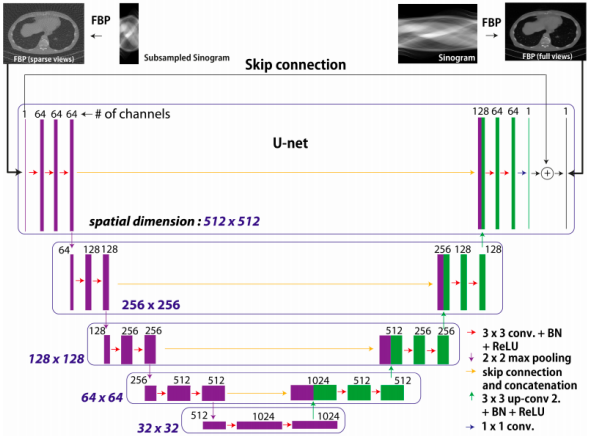
Learning fast approximations of sparse coding

Core idea: time-unfolded version of an iterative reconstruction algorithm, like **IST**, truncated to a fixed number of iterations.
Representatives: **LISTA** [Gregor and LeCun, 2010],[Moreau and Bruna, 2017]



Deep CNN models in image reconstruction

A central question is whether one can combine elements of model and data driven approaches for solving ill-posed inverse problems.



[McCann et al., 2017]

Summary

- Sparse optimization is a fundamental concept in inverse problems like image reconstruction.
- This tutorial covered some basic components of sparse image recovery algorithms, including ADMM-based methods.
- The concept of structured sparsity was underlined with particular attention to using Markov Random Field priors in sparse image recovery.
- A new frontier: machine learning in image reconstruction. Great potential, a huge variability of approaches.

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